A Study on Circulant Matrices and its Application in Solving Polynomial Equations and Data Smoothing

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Abstract — This article presents a short study on circulant matrices and some of its properties. We discuss the eigenvalues, eigenvectors and diagonalization of circulant matrices in this paper. We also discuss the application of circulant matrices in solving polynomial equations and data smoothing.

Keywords — Circulant, permutation, eigenvalues, fourier, diagonalization, smoothing.

I. Introduction

Circulant matrices has excited many researchers ever since its first occurence in a paper by Catalan in [1] and have been widely used in the analysis of time series by Anderson in [2] and Fuller in [3]. Being a special type of Toeplitz matrix, it has many applications in solutions to differential and integral equations, spline functions and various problems in physics, mathematics, statistics and signal processing. Circulant matrices also find applications in study of error correcting codes and Discrete Fourier Transform (DFT).

Solving polynomial equations is one of the oldest problems in mathematics. The methods to solve polynomials have evolved for over years. Various developments in solving cubic and quartic polynomial equations were made by Tartagla, Ferrari, Cardano and Lagrange. Dan Kalman and James White gave a unified approach to solving polynomials in [4] using circulant matrix with a specified characteristic polynomial.

Smoothing is a type of statistical technique in handling data, in which an approximating function is created to capture the important patterns in a data and leave the noise in the data. There are various methods/algorithms to smooth a data including smoothing using linear transformation.

This article is a short study on circulant matrices, its properties and applications widely based on [6] and [4]. In this article we discuss the basic properties of circulant matrices specifically using permutation matrices. The eigenvalues, eigenvectors and diagonalization of circulant matrices by Fourier matrices and Vandermonde matrices are studied in this paper. We study the application of circulant matrix in solving polynomial equations of degree upto four. We also study the use of circulant matrices in smooting a data.

A. *Preliminaries*

Definition I.1. *[6] A circulant matrix is a square matrix of the form*

$$C = circ(c_1, c_2, \cdots, c_n) \tag{1}$$

$$= \begin{pmatrix} c_1 & c_2 & \cdots & c_n \\ c_n & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & & \vdots \\ c_2 & c_3 & \cdots & c_1 \end{pmatrix}. \tag{2}$$

The whole circulant is evidently determined by the first row. We may also write a circulant in the form

$$C = (c_{jk}) = (c_{k-j+1 \ mod \ n}).$$

It can be observed that the space of all circulant matrices form a subspace of the space of $n \times n$ matrices.

Definition I.2. *[6] Let* $\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}$ *and* $e_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ *be a* $1 \times n$ *unit vector, where 1 is in the* j^{th} *position. A permutation matrix of order* n *is of the form,*

$$P = P_\sigma = \begin{pmatrix} e_{i_1} \\ e_{i_2} \\ \vdots \\ e_{i_n} \end{pmatrix} \tag{3}$$

In other words, $P = (a_{ij})$ where $a_{ij} = \begin{cases} 1, & \text{if } j = \sigma(i) \\ 0, & \text{otherwise} \end{cases}$

We list some properties of permutation matrices which can be easily verified [6]:

1) If $A = (a_{ij})$ then $P_\sigma A = (a_{\sigma(i)j})$
2) $P_\sigma A P_\sigma^T = (a_{\sigma(i)\sigma(j)})$
3) $P_\sigma^T = P_{\sigma^{-1}}$
4) $P_\sigma^T = P_\sigma^{-1}$
5) $P_\sigma P_\tau = P_{\sigma\tau}$

Properties of circulant matrices can be studied using π matrix as defined in [6], a special permutation matrix of the form

$$\pi = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{4}$$

Some properties of π matrix discussed in [6] are as follows:

1) $\pi^n = I$
2) $\pi^T = \pi^* = \pi^{-1}$
3) $\pi A \pi^T = (a_{i+1,j+1})$

From the above properties it is observed that $A = (a_{ij})$ is a circulant matrix if and only if $(a_{ij}) = (a_{\sigma(i)\sigma(j)})$ where $\sigma = (1, 2, \cdots, n)$ is a permutation.

Some properties of circulant matrix:

1) Let A be an n $\times$ n matrix, then A is circulant iff $A\pi = \pi A$
2) A is circulant iff A^* is circulant.
3) If A and B are circulant matrices, then AB is circulant.
4) If A is a circulant matrix, then A^n, where $n \in \mathbb{N}$ is circulant.

We can represent a circulant matrix using π matrix as follows:

$$\text{circ}(c_0, c_1, \cdots, c_n) = c_0 I + c_1 \pi + \cdots + c_n \pi^n \tag{5}$$

That is, as $\pi^k = circ(0, \cdots, 0, 1, 0, \cdots, 0)$, where 1 is in the $k + 1^{th}$ position, we can write

$$\begin{aligned} \text{circ}(c_0, c_1, \cdots, c_n) =& c_0 \text{ circ}(1, 0, \cdots, 0) + c_1 \text{ circ}(0, 1, 0, \cdots, 0) \\ &+ c_2 \text{ circ}(0, 0, 1, \cdots, 0) \cdots + c_n \text{ circ}(0, 0, \cdots, 1). \end{aligned}$$

If $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$ then $p(x)$ is called the representer of the circulant.
If $\gamma = (c_0, c_1, \cdots, c_n)$ then circulant C is given as $C = \text{ circ}(\gamma) = p_\gamma(\pi)$.

II. Diagonalization of Circulant Matrices

Theorem II.1. *[6] The eigenvalues of a circulant matrix* C *are* $p_\gamma(\omega^k)$ *for* $0 \leq k < n$*. Hence det* $C = \prod_{k=0}^{n-1} p_\gamma(\omega^k)$.

Proof. Characteristic polynomial of π,

$$det(\lambda I - \pi) = \begin{vmatrix} \lambda & -1 & 0 & \cdots & 0 \\ 0 & \lambda & -1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ -1 & 0 & 0 & \cdots & \lambda \end{vmatrix} = 0$$

$$\lambda^n - 1 = 0$$

Since, ω^k for $0 \leq k < n$ satisfies the characteristic polynomial of π,

$$det(\lambda I - \pi) = \prod_{k=0}^{n-1} (\lambda - \omega^k).$$

Therefore, eigenvalue of π is ω^k for $0 \leq k < n$. Since $C = p_\gamma(\pi)$, eigenvalue of C is $p_\gamma(\omega^k)$ and hence $det\ \ C = \prod_{k=0}^{n-1} p_\gamma(\omega^k)$. □

Let $x_k = (1, \omega^k, \omega^{2k}, \cdots, \omega^{(n-1)k})^T$ then,

$$\begin{aligned} \pi x_k &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \omega^k \\ \vdots \\ \omega^{(n-1)k} \end{pmatrix} \\ &= (\omega^k, \omega^{2k}, \cdots, \omega^{(n-1)k}, 1)^T \\ &= \omega^k x_k \end{aligned}$$

That is x_k is the eigenvector of π corresponding to eigenvalue ω_k. As $C = p_\gamma(\pi)$, x_k is the eigenvector of C corresponding to eigenvalue $p_\gamma(\omega_k)$ for $0 \leq k < n$.

Definition II.1. *Fourier matrix of order n, is given by $F = (f_{ij})$ where*

$$f_{ij} = \frac{1}{\sqrt{n}} \left(\omega^{(j-1)(i-1)} \right) \tag{6}$$

$$F = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)}. \end{pmatrix}. \tag{7}$$

Theorem II.2. *[6] Let $\Omega = \Omega_n = diag(1, \bar{\omega}, \bar{\omega}^2, \cdots, \bar{\omega}^{n-1})$. Then $\pi = F^* \Omega F$.*

Proof. From the definition of Fourier matrix, the j^{th} row of F^* is

$$\left(\frac{1}{\sqrt{n}}\right) \left(\bar{\omega}^{(j-1)0}, \bar{\omega}^{(j-1)1}, \cdots, \bar{\omega}^{(j-1)(n-1)} \right)$$

Hence the j^{th} row of $F^* \Omega$ is

$$\left(\frac{1}{\sqrt{n}}\right) (\bar{\omega}^{(j-1)r} \bar{\omega}^r) = \left(\frac{1}{\sqrt{n}}\right) (\bar{\omega}^{jr})$$

where $r = 0, 1, \cdots, n-1$.
The k^{th} column of F is

$$\frac{1}{\sqrt{n}} (\omega^{(k-1)r}).$$

Hence the jk^{th} element of $F^*\Omega F$ is

$$\begin{aligned}\frac{1}{n}\sum_{r=0}^{n-1}\omega^{(k-1)r-jr} &= \frac{1}{n}\sum_{r=0}^{n-1}\omega^{(k-1-j)r} \\ &= \begin{cases}1, & \text{if } j = k-1 \text{ mod n} \\ 0, & \text{if } j \neq k-1 \text{ mod n}\end{cases} \\ &= jk^{th} \text{ element of } \pi\end{aligned}$$

Hence $F^*\Omega F = \pi$. □

Theorem II.3. *[6] Every circulant matrix C is diagonalized by Fourier Matrix F.*

Proof. From Theorem 3.2, we get

$$\begin{aligned}C &= p_\gamma(\pi) \\ &= p_\gamma(F^*\Omega F) \\ &= F^* p_\gamma(\Omega) F \\ &= F^* diag(p_\gamma(1), p_\gamma(\bar{\omega}), \cdots, p_\gamma(\bar{\omega}^{n-1}))F.\end{aligned}$$

that is, $C = F^*\Phi F$ where, $\Phi = diag(p_\gamma(1), p_\gamma(\bar{\omega}), \cdots, p_\gamma(\bar{\omega}^{n-1}))$. □

Definition II.2. *Vandermonde Matrix $V(z_0, z_1, \cdots, z_{n-1})$ is a matrix of the form*

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z_0 & z_1 & \cdots & z_{n-1} \\ z_0^2 & z_1^2 & \cdots & z_{n-1}^2 \\ \vdots & \vdots & & \vdots \\ z_0^{n-1} & z_1^{n-1} & \cdots & z_{n-1}^{n-1} \end{pmatrix}$$

From equation (7) we get that

$$V(1, \omega, \omega^2, \cdots, \omega^{n-1}) = n^{1/2}F^* \tag{8}$$

$$V(1, \bar{\omega}, \bar{\omega^2}, \cdots, \bar{\omega^{n-1}}) = n^{1/2}\bar{F}^* = n^{1/2}F \tag{9}$$

Definition II.3. *Let $\phi(x) = x^n - a_{n-1}x^{n-1} - a_{n-2}x^{n-2} - \cdots - a_1x - a_0$ be a monic polynomial of degree n. The Companion Matrix C_ϕ of ϕ is defined by,*

$$C_\phi = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \end{pmatrix}$$

Remark: Characteristic Polynomial of companion matrix is the polynomial itself, that is, if C_ϕ is the companion matrix of ϕ then characteristic polynomial of C_ϕ is $\phi(x)$.

Lemma II.1. *Let $V = V(\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$ designate the Vandermonde formed with $\alpha_0, \alpha_1 \cdots, \alpha_{n-1}$. Let $D = diag(\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$. Then $VD = C_\phi V$.*

Theorem II.4. *If α_i's are distinct, V is nonsingular, which gives us the digonalization, $C_\phi = VDV^{-1}$. Hence for any polynomial $p(z)$,*

$$p(C_\phi) = Vp(D)V^{-1} \tag{10}$$

Proof. Since $det(V) = \prod_{0\leq i<j<n}(\alpha_j - \alpha_i)$, it follows that V is nonsingular iff the α_i's are distinct. This implies by previous theorem that, $C_\phi = VDV^{-1}$ and hence $p(C_\phi) = Vp(D)V^{-1}$. □

Taking $\phi(x) = x^n - 1$, we get $C_\phi = \pi$. The roots of ϕ are ω^j for $j = 0, 1, \cdots, n-1$ and V is a scaled version of F^*. Since all polynomials in $C_\phi = \pi$ are circulants and vice versa, by theorem 4.4, $p(C_\phi) = p(\pi) = Vp(D)V^{-1}$ that is V diagonalizes circulant matrices.

III. Solving polynomial equations using Circulant Matrices

In this section, we study the application of circulant matrix in solving polynomial equations as discussed in [4].

- **Quadratic equations:**
  Consider a general quadratic polynomial : $p(x) = x^2 + \alpha x + \beta$
  and a general 2×2 circulant matrix of the form

$$C = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

  Characteristic polynomial of C is given as $x^2 - 2ax + a^2 - b^2$.
  Now finding the roots of $p(t)$ is equivalent to finding the values of a and b such that the characteristic polynomial of C equals the given polynomial $p(t)$ and thus the roots of $p(t)$ are the eigenvalues of C. [4]
  Here, $a = \frac{-\alpha}{2}$ and $b = \pm\sqrt{\frac{\alpha^2}{4} - 4}$. For convenience we define b with the positive sign. Hence

$$C = \begin{pmatrix} \frac{-\alpha}{2} & \sqrt{\frac{\alpha^2}{4} - 4} \\ \sqrt{\frac{\alpha^2}{4} - 4} & \frac{-\alpha}{2} \end{pmatrix}.$$

  and

$$f(t) = \frac{-\alpha}{2} + t\sqrt{\frac{\alpha^2}{4} - 4}$$

  The roots of the original quadratic equation are now found by applying f to the two square roots of unity:

$$f(1) = \frac{-\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - 4}$$
$$f(-1) = \frac{-\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - 4}$$

- **Cubic equations:**
  Here we use the general result which states that for a general polynomial $p(x) = x^n + \alpha_{n-1}x^{n-1} + \dots$, the substitution $y = x - \frac{\alpha_{n-1}}{n}$, eliminates the term of degree $n - 1$.
  In the context of circulant matrices this is equivalent to making the trace of the circulant matrix zero.
  Hence the objective is to obtain the expressions for the roots of $p(x) = x^3 + \beta x + \gamma$ as the eigenvalues of a traceless circulant matrix. [4]
  Let

$$C = \begin{pmatrix} 0 & b & c \\ c & 0 & b \\ b & c & 0 \end{pmatrix}$$

  Its characteristic polynomial is $x^3 - b^3 - c^3 - 3bcx$.
  This equals $p(x)$ if

$$b^3 + c^3 = -\gamma$$
$$3bc = -\beta$$

  which on solving gives

$$b = \sqrt[3]{\frac{-\gamma + \sqrt{\gamma^2 + \frac{4\beta^2}{27}}}{2}}$$
$$c = \sqrt[3]{\frac{-\gamma - \sqrt{\gamma^2 + \frac{4\beta^2}{27}}}{2}}$$

  Now, the roots of $p(x)$ are the eigenvalues of C with b and c so obtained.

- **Quartic equations:** Using the same idea as in cubic equations, we consider the general quartic polynomial of the form $p(x) = x^4 + \beta x^2 + \gamma x + \delta$ and a traceless circulant matrix

$$C = \begin{pmatrix} 0 & b & c & d \\ d & 0 & b & c \\ c & d & 0 & b \\ b & c & d & 0 \end{pmatrix}.$$

Its characteristic polynomial is $x^4 - (4bd + 2c^2)x^2 - 4c(b^2 + d^2)x + c^4 - b^4 - d^4 - 4bdc^2 + 2b^2d^2$. Equating this with $p(x)$, gives

$$\begin{aligned} bd &= \frac{-2c^2 - \beta}{4} \\ b^2 + d^2 &= \frac{-\gamma}{4c} \\ c^4 - (b^2 + d^2) + 4(bd)^2 - 4bdc^2 &= \delta. \end{aligned}$$

In terms of c we get

$$c^4 - \frac{\gamma^2}{16c^2} + \frac{(\beta + 2c^2)62}{4} + (2c^2 + \beta)c^2 = \delta$$

On simplifying we have

$$c^6 + \frac{\beta}{2}c^4 + \left(\frac{\beta^2}{16} - \frac{\delta}{4}\right)c^2 - \frac{\gamma^2}{64} = 0$$

which is a cubic polynomial equation in c^2, and is solvabe by usual methods, which gives non-zero value of c. Substituting in the given system we get value of b and d.
Now we construct a circulant matrix C such that its eigenvalues are roots of $p(x)$ [4] as

$$C = b\pi + c\pi^2 + d\pi^3 = q(\pi).$$

Then the eigenvalues of C are computed by applying $q(x)$ to the fourth roots of unity as follows,

$$\begin{aligned} q(1) &= b + c + d \\ q(-1) &= -b + c - d \\ q(i) &= -c + i(b - d) \\ q(-i) &= -c - i(b - d). \end{aligned}$$

Hence the roots of $p(x)$ are $q(1), q(-1), q(i)$ and $q(-i)$.

IV. Data Smoothing using Circulant Matrices

Data smoothing is a technique to remove or reduce any volatality, or any other type of noise when a data is compiled. Data smoothing is done using an algorithm to remove noise from a data set. There are various algorithms in which data smoothing can be done. This section studies the smoothing of a data using linear transformation as discussed in [6].
If the smoothed values of the data can be writen as a linear transformation of the observed values, the smoothing operation is known as a linear smoother, and the matrix representing the transformation is known as a smoother matrix or hat matrix. The operation of appplying such a matrix transformation is called convolution. Thus the matrix is also called convolution kernel. In the case of simple series of data points, the convolution kernel is a one-dimensional vector.

A. *Smoothing using Matrix*

[6] Let $z = (z_1, z_2, \cdots, z_n)^T$ be the sequence of data values. To smooth these data values we apply linear transformation A on z as

$$\hat{z} = Az$$

where $\hat{z}$ is the smoothed values of the data z. As proposed by Greville in [7] the matrix A will be called a smoothing if

1) A has an eigenvalue 1.
2) $A^\infty = \lim_{p\to\infty} A^p$ exists.

The space of eigenvectors S corresponding to eigenvalue $\lambda = 1$ of A is called the *smooth vectors*. If $z \in S$, then

$$Az = z$$

implies that z is already smooth and remains unaffected by the operation A.
If z is any vector to be smoothed, we repeatedly apply A on z which gives,

$$A(A^\infty z) = A^\infty z$$

which means $A^\infty z \in S$ and hence any vector is smoothed by the matrix A.
A necessary and sufficient condition for A to be smoothing was proposed by Greville in [7] as

1) A has an eigenvalue $\lambda = 1$.
2) $\lambda = 1$ must be a simple root of the minimal polynomial of A and if $\lambda \neq 1$ is an eigenvalue, then $|\lambda| < 1$.

B. *Representation of Circulant Matrix*

[6] We know that any circulant matrix C can be diagonalized by Fourier matrix F, that is

$$C = F^* DF$$

where $D = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$. Let $D_k = diag(0, 0, \cdots, 0, 1, 0, \cdots, 0)$ where 1 is at the k^{th} position. Then D can be written in terms of D_k as,

$$D = \sum_{k=1}^{n} \lambda_k D_k.$$

Hence,

$$C = \sum_{k=1}^{n} \lambda_k F^* D_k F$$

Let $B_k = F^* D_k F$, which gives

$$C = \sum_{k=1}^{n} \lambda_k B_k \tag{11}$$

and

$$C^p = F^* D^p F = \sum_{k=1}^{n} \lambda_k^p B_k \tag{12}$$

Let $M_k = \text{circ}(1, \omega^k, \omega^{2k} \cdots, \omega^{(n-1)k})$ and

$$p_{M_k}(z) = 1 + \omega^k z + (\omega^k z)^2 + \cdots + (\omega^k z)^{n-1}.$$

Then

$$p_{M_k}(z) = \begin{cases} \frac{(\omega^k z)^n - 1}{\omega^k z - 1} & \text{if } \omega^k z \neq 1 \\ n & \text{if } \omega^k z = 1 \end{cases}$$

and

$$p_{M_k}(\omega^{j-1}) = \begin{cases} \frac{(\omega^{k+j-1})^n - 1}{\omega^{k+j-1} - 1} = 0 & \text{for } j \neq n + 1 - k (\text{mod n}) \\ n & \text{for } j = n + 1 - k (\text{mod n}) \end{cases}$$

Hence the eigenvalues of $\frac{1}{n}M_k$ are the eigenvalues of $B_{n+1-k(\text{mod } n)}$ which gives

$$B_{n+1-k(\text{mod } n)} = \frac{1}{n}\text{circ}(1, \omega^k, \omega^{2k} \cdots, \omega^{(n-1)k}).$$

For $k = 0$,

$$B_1 = \frac{1}{n} \text{ circ}(1, 1, \cdots, 1) \tag{13}$$

Convergence of Sequence of Matrix: A matrix $A_r = (a_{ij}^r)$ for $r = 1, 2, \cdots$ is said to be convergent if and only if $\lim_{r\to\infty} A_r$ exist, that is there exist a matrix $A = (a_{ij})$ such that $\lim_{r\to\infty} A_r = A$ that is

$$\lim_{r\to\infty} A_r = \lim_{r\to\infty} (a_{ij}^r) = (a_{ij})$$

Theorem IV.1. *[6] Let C be a circulant matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. $\lim_{r\to\infty} C^r$ exists if and only if $\lambda_k = 1$ or $|\lambda_k| < 1$ for $k = 1, 2, \cdots, n$.*

Proof. From equation (12) we have $C^r = \sum_{k=1}^n \lambda_k^r B_k$, which implies, $\lim_{r\to\infty} C^r$ exists if and only if $\lim_{r\to\infty} \sum_{k=1}^n \lambda_k^r B_k$ exists.
As $\lim_{r\to\infty} \sum_{k=1}^n \lambda_k^r B_k$ exists if and only if $\lambda_k = 1$ or $|\lambda_k| < 1$ for $k = 1, 2, \cdots, n$, we get $\lim_{r\to\infty} C^r$ exists if and only if $\lambda_k = 1$ or $|\lambda_k| < 1$ for $k = 1, 2, \cdots, n$.

□

Let $G \subseteq \{1, 2, \cdots, n\}$ for which $\lambda_k = 1$. Then

$$C^\infty = \begin{cases} \sum_{k\in G} B_k & \text{if } G \neq \phi \\ 0 & \text{if } G = \phi \end{cases} \tag{14}$$

C. Smoothing using Circulant Matrices

By definition of smoothing operator and the condition for existence of $\lim_{r\to\infty} C^r$ a circulant matrix C is said to be a smoothing operator if and only if

1) $\lambda = 1$ is an eigenvalue of C.
2) For any eigenvalue $\lambda \neq 1$ of C, $|\lambda| < 1$.

From (11), for any vector z we get,

$$Cz = (\sum_{k\in G} B_k)z$$

as in G, $\lambda_k = 1$. Hence $(\sum_{k\in G} B_k)z$ are smooth vectors.

Theorem IV.2. *[6] Let $C = circ(c_1, c_2, \cdots, c_n)$ where $c_k \geq 0$ for all $k = 1, 2, \cdots, n$ and $c_1 + c_2 + \cdots + c_n = 1$. Then C is a smoothing matrix and the set of smooth vectors consists of the constant vectors.*

Proof. We know that, any eigenvalue λ_k of C is given by $\lambda_k = p_\gamma(\omega^{k-1})$. Hence

$$\lambda_1 = c_1 + c_2 + \cdots + c_n = 1$$

and

$$\lambda_2 = c_1 + c_2\omega + \cdots + c_n\omega^{n-1}$$

As λ_2 is a convex combination of $(1, \omega, \omega^2 \cdots, \omega^{n-1})$, it lies inside the unit circle. Similarly, every λ_k is a convex combination of a subset of $(1, \omega, \omega^2 \cdots, \omega^{n-1})$ and hence λ_k also lies inside the unit circle for all $k = 1, 2, \cdots, n$.
Thus $|\lambda_k| \leq 1$ for all $k = 1, 2, \cdots, n$ which implies C is smoothing operator.
Also, here $G = 1$, so the set of smooth vectors consists of $B_1 z$.
From (13),

$$B_1 = \frac{1}{n} \text{ circ}(1, 1, \cdots, 1)$$

which implies the smooth vectors $B_1 z$ are constant vectors. □

V. Conclusion

Circulant matrix is one of the most widely used matrix due to its applications in signal processing and error correction of codes.

In this paper we studied the properties of circulant matrices using permutation. We studied the eigenvalues of circulant matrices in terms of n^{th} roots of unity and diagonalization of circulant matrices using Fourier matrix and Vandermonde matrix.

We studied the application of circulant matrices in solving the quadratic, cubic and quartic polynomials.

We also studied the technique of data smoothing using circulant matrices and observed how data smoothing becomes simpler using circulant matrices by making any sequence of vectors, a smooth vector by making it a constant vector.

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