On Companion BN_1-algebra

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Abstract. *BN-algebra is a non-empty set* $(X, *, 0)$ *satisfies the axioms (B1)* $x * x = 0$, *(B2)* $x * 0 = x$ *and (BN)* $(x * y) * z = (0 * z) * (y * x)$ *for all* $x, y, z \in X$. *BN_1-algebra is a BN-algebra that satisfies axiom (BN1)* $(x * y) * z = (0 * z) * (y * x)$. *The operation* $\odot$ *is called a subcompanion when satisfies* $((x \odot y) * x) * y = 0$ *in BN_1-algebra. The subcompanion operation* $\odot$ *is said to be a companion if* $(z * x) * y = 0$, *then* $z * (x \odot y) = 0$. *In this paper, we develop the formula definition of companion BN_1-algebra. We also investigate the properties of of companion BN_1-algebra, such as being unique or singular and commutative with certain conditions.*

Keywords— *B-algebra, BN-algebra, BN_1-Algebra, Companion.*

I. INTRODUCTION

In 1966 Y. Imai and K. Iseki introduced a new a subclass of Q-Algebra: BCK-algebra. Iseki also [15] constructed a general class of BCK-algebra which is called BCI-algebra. In 1985, the wide class of BCI-algebra, BCH-algebra, was introduced by Ping Hu and Xin Li [20]. Then in 2002, based on the properties of BCI, BCK and BCH-algebra, J. Neggers and H. S. Kim [13] found a new structure algebra: B-algebra. B-algebra is a set of non-empty (*X*, *0*) of type (2.0) which the binary operations $*$ that satisfies the axioms (B1) $x * x = 0$, (B2) $x * 0 = x$ and (B) $(x * y) * z = x * (z * (0 * y))$ [13].

Four years later, in 2006, C.B. Kim and H. S. Kim [2] introduced a new algebra, BG-algebra, which is a generalization of B-algebra. In 2008, they also introduced BM-algebra [3]. In 2013, they introduced a new algebra which is called BN-algebra by taking some properties of B-algebra [4]. BN-algebra is a non-empty set (*X*, 0) that satisfies the axioms (B1) and (B2) and (BN) $(x * y) * z = (0 * z) * (y * x)$ for all $x, y, z \in X$.

Recently, L. D. Naingue and J. P. Vilela [16] developed the companion concept in B-algebra which called companion B-algebra. Based on companion B-algebra, we applies the companion concept in BN_1-algebra and develop the formula definition of companion BN_1-algebra. Furthermore, we also investigate its properties.

II. B-ALGEBRA, BN-ALGEBRA AND BN_1-ALGEBRA

In this section, the definition and some properties of B-Algebra, BN-algebra and BN_1-algebra are given.

Definition 1.1 (B-algebra) A non-empty set $(X, *, 0)$ is called B-algebra when it satisfies the following axioms [13]:

(B1) $x * x = 0$,

(B2) $x * 0 = x$,

(B) $(x * y) * z = x * (z * (0 * y))$.

for all $x, y, z \in X$.

On B-algebra, the companion concept is given by [13]:

Definition 1.2 (Companion B-algebra) Let $(X, *, 0)$ be a B-algebra and binary operation $\odot$. Define the binary operation $\odot$ is called a subcompanion of X when satisfies axiom $((x \odot y) * x) * y = 0$. A subcompanion operation $\odot$ is said to be a companion of X if $(z * x) * y = 0$, then $z * (x \odot y) = 0$ for any $x, y, z \in X$.

C.B Kim and H.S Kim [4] introduced a new structure algebra, BN-algebra, by using some properties of B-algebra in 2013.

Definition 1.3 (BN-algebra) A non-empty set $(X, *, 0)$ with binary operation $*$ is called BN-algebra when it satifies the following axioms [13]:

(B1) $x * x = 0$,

(B2) $x * 0 = x$,

(BN) $(x * y) * z = (0 * z) * (y * x)$, .

for any $x, y, z \in X$.

Example 1.1 Let $X := 0, 1, 2$ be a set with Cayley table as follows:

TABLE 1
TABEL CAYLEY OF A BN-ALGEBRA

$*$	0	1	2
0	0	1	2
1	1	0	1
2	2	1	0

then is X a BN-algebra based on axioms on Definition 1.3 and Table 1.

Theorem 1.1 For any non-empty set $(X, *, 0)$ BN-algebra with a binary operation $*$ satisfying the following additional axioms:

(i) $0 * (0 * x) = 0,$

(ii) $y * x = (0 * x) * (0 * y),$

(iii) $(0 * x) * y = (0 * y) * x,$

(iv) $x * y = 0 \Rightarrow y * x = 0,$

(v) $0 * x = 0 * y \Rightarrow x = y,$

(vi) $(x * z) * (y * z) = (z * y) * (z * x).$

Proof. we can see in [4].

Sub-algebra of BN-algebra is defined as follows [4]:

Definition 1.4 Let $(X, *, 0)$ be a BN-algebra and $\oslash \neq S \subseteq X$. S is called a subalgebra of X if $x * y \in S$ for all $x, y \in S$.

One of the subalgera of BN-algebra is BN_1-algebra. It was introduced by C. B. Kim and H. S. Kim [4] and define by:

Definition 1.5 (BN_1-Algebra) A non-empty set $(X, *, 0)$ of BN- algebra is said to be BN_1-algebra if satisfies (BN1) $x = (x * y) * y$ for any $x, y \in X$.

Example 1.2 Let $X := 0, 1, 2, 3$ be a set with Cayley table as follows:

TABLE 2
TABEL CAYLEY OF A BN_1-ALGEBRA

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then X is a BN_1-algebra based on axioms on definition 1.3, definition 1.5 and and table 2.

Theorem 1.2 If $(X, *, 0)$ be a BN_1-algebra with a binary operation $*$, then for any $x, y, z \in X$, we have

(i) $0 * x = x,$

(ii) $x * y = y * x.$

Proof.

(i) Let $x, y, z \in X$. By using (BN1) $x = (x * y) * y$ and (B1) $x * x = 0$, for $x = y$, we have $(x * y) * y = (x * x) * x = 0 * x = x$.

(ii) By theorem 1.1 (ii) and theorem, 1.2 (i), we have $y * x = (0 * x) * (0 * y) = x * y$.

Theorem 1.3 Let $(X, *, 0)$ be a BN$_1$-algebra with a binary operation $*$. If $x * y = 0$, then $x = y$ for any $x, y \in X$.

Proof. we can see in [4]

III. COMPANION BN$_1$-ALGEBRA

In this section, we develop the formula definition of Companion BN$_1$-algebra. We also investigate its properties such as unique and commutative with certain conditions.

Definition 1.6 (Companion BN$_1$-algebra) Let $(X, *, 0)$ be a BN$_1$-algebra with binary operation $\odot$. Define the binary operation $\odot$ is called a subcompanion of X when satisfies axiom (SC) $((x \odot y) * x) * y = 0$. A subcompanion operation $\odot$ is said to be a companion of X if (C) $(z * x) * y = 0$, then $z * (x \odot y) = 0$ for any $x, y, z \in X$.

Companion BN$_1$-algebra ia unique or singular and can be stated as the following theorem:

Theorem 1.4 Let $(X, *, 0)$ be a BN$_1$-algebra. If $\odot$ is a companion operation in X, then it is unique.

Proof. Assume the binary operations $\odot_1$ and $\odot_2$ are companion operations in X. By using (SC) in $\odot_1$, we have

$$((x \odot_1 y) * x) * y = 0$$

and $z = x \odot_1 y$. By apllying (C) in $\odot_2$ for $(z * x) * y = 0$, we have

$$z * (x \odot_2 y) = 0.$$

Since $z = x \odot_1 y$, so we have

$$(x \odot_1 y) * (x \odot_2 y) = 0.$$

By using theorem 1.3 (ii), we obtain

$$(x \odot_1 y) = (x \odot_2 y)$$
$$\odot_1 = \odot_2.$$

Hence the operation $\odot$ is unique.

Example 1.2 Let $X := 0, 1, 2, 3$ be a BN$_1$-algebra with Cayley table as follows

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\odot$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

By using definition 1.8, then $(X, *, \odot, 0)$ is a companion BN$_1$-algebra.

Theorem 1.5 Let $(X, *, \odot, 0)$ be a companion BN$_1$-algebra and $\bullet$ is a binary operation on X such that satifies,

$$(x * y) * z = x * (y \bullet z)$$

for any $x, y, z \in X$. Then $(X, *, \bullet, 0)$ is a companion BN$_1$-algebra and $\bullet$ is a exactly the operation $\odot$.

Proof. Assume $x, y, z \in X$ and $(x * y) * z = x * (y \bullet z)$. We can prove that $\bullet$ is subcompanion by $((x \bullet y) * x) * y = 0$. From $(x * y) * z = x * (y \bullet z)$, we have

$$((x \bullet y) * x) * y = (x \bullet y) * (x \bullet y).$$

By using (B1), we obtain

$$((x \bullet y) * x) * y = 0$$

statisfying (SC) which profing that $\bullet$ is subcompanion. Then we must to prove that $\bullet$ is a companion. Interchange $x = z,\ y = x$ and $z = y$ in $(x * y) * z = x * (y \bullet z),$ so we get

$$z * (x \bullet y) = (z * x) * y.$$

If $(z * x) * y = 0$, then

$$z * (x \bullet y) = 0$$

satisfying (C) which implies that $\bullet$ is a companion operation. Hence $(X, *, \bullet, 0)$ is a companion BN_1-algebra and $\bullet$ is a exactly the operation $\odot$

Theorem 1.6 If $(X, *, \odot, 0)$ be a companion BN_1-algebra, then

(i) $x \odot 0 = x,$

(ii) $0 \odot x = x,$

for any $x, y \in X$.

Proof.

(i) Let $x, y \in X$ and $y = 0$. By using (SC), (B1) and theorem 1.3, we have

$$((x \odot y) * x) * y = 0$$
$$((x \odot 0) * x) * 0 = 0$$
$$(x \odot 0) * x = 0$$
$$x \odot 0 = x$$

(ii) By applying theorem 1.3 (ii) in theorem 1.6 i), we get

$$x \odot 0 = x$$
$$0 \odot 0 = x.$$

Theorem 1.7 Let $(X, *, \odot, 0)$ be a companion BN_1-algebra. If $(x * y) * z = (x * y) * z,$ then

(i) $x \odot y = x * y,$

(ii) $x = (x \odot y) * y,$

(iii) $\odot$is comutative,

for any $x, y, z \in X$.

Proof.

(i) Let $x, y, z \in X$ and $(x * y) * z = (x * y) * z.$. By using (SC) and theorem 1.3, we have

$$((x \odot y) * x) * y = 0$$
$$((x \odot 0) * (x * y) = 0$$
$$(x \odot 0) * x = 0$$
$$x \odot y = x * y$$

(iv) From (BN1) and (i), we get

$$x = (x * y) * y$$
$$x = (x \odot y) * y.$$

(ii) By applying (i) and theorem 1.2 (ii), we obtain $x \odot y = x * y = y * x = y \odot x$.

IV. CONCLUSIONS

The companion concept can be applied to BN-algebra and it is called Companion BN_1-algebra. Let $(X,*,0)$ be a BN_1-algebra and the binary operation $\odot$ is called a subcompanion when satisfies $((x \odot y) * x) * y = 0$. A subcompanion operation $\odot$ is said to be a companion if $(z * x) * y = 0$, then $z * (x \odot y) = 0$ for any $x, y, z \in X$. Companion BN_1-algebra has several properties. First, it is single or unique. If there is another binary operation, $\bullet$, on Companion BN_1-algebra that fulfils $(x * y) * z = x * (y \bullet z)$, then the operation is a companion. Companion BN_1-algebra is also commutative if $(x * y) * z = (x * y) * z$ for any $x, y, z \in X$.

REFERENCES

[1] H. A. S. Abujabal dan N. O. Al-Shehri , On Left Derivations Of BCIalgebras, Soochow Journal Of Mathematics, 33(2007), 435–444.
[2] H. A. S. Abujabal dan N. O. Al-Shehri, Some Results On Derivations Of BCI-algebras, PK ISSN 0022-2941 CODEN JNSMAC, 46 (2006), 13–19.
[3] [N. O. Al-shehrie, Derivation of B-algebras, JKAU: Sci, 22 (2010), 71–82.
[4] N. O. Al-shehrie, Derivations of MV-algebras, International Journal of Mathematics and Mathematical Sciences, (2010), 1–7.
[5] D. S. Dummit dan R. M. Foote, Abstract Algebra, 3rd Edition, New Jersey: Prentice Hall, Inc, 2003.
[6] J. S. Durbin, Modern Algebra: An Introduction, 3rd Edition, New Jersey: John Wiley & Sons, 1992.
[7] G. Dymek dan A. Walendziak, (Fuzzy) Ideals of BN-Algebras, Scientific World Journal,(2015), 1–9.
[8] J. C. Endam dan J. P. VilelaThe, Second Isomorphism Theorem for Balgebras, Applied Mathematical Sciences, 8(2014), 1865–1872.
[9] E. Fitria, S. Gemawati dan A. Hadi, On Derivasi BN-Algebra, preprint, 2019.
[10] E. Fitria, S. Gemawati dan Kartini, Prime Ideals in B-Algebras, International Journal of Algebra, 11(2017), 30–309.
[11] S. Ilbira, A. Firat, dan Y. B. Jun, On symmetric bi-derivations of BCI- algebras, Applied Mathematical Sciences, 5 (2011), 2957-2966.
[12] K. Iseki, On BCI-algebras, Math. Seminar Notes, 8(1980), 125–130.
[13] Y. B. Jun dan X. L. Xin, On derivations of BCI-algebras, Inform. Sci.,159(2004), 167–176.
[14] C. B. Kim and H. S. Kim, On BG-algebras, Demo. Math., 41(2008), 497–505.
[15] C. B. Kim and H. S. Kim, On BM-algebras, Sci.Math. Japonicae 63(2006), 421–427.
[16] C. B. Kim dan H. S. Kim, On BN-algebras, Kyungpook Math, 53 (2013),175–184.
[17] L. D. Naingue dan J. P. Vilela, On Companion B-algebra, EJPAM, 12(2019), 1248-1259.
[18] J. Neggers dan H. S. Kim, On B-algebras, Mate. Vesnik, 54(2002), 21–29.
[19] M. A. Ozturk, Y. Ceven, dan Y. B. Jun, Generalized derivations of BCI- algebras, Honam Mathematical Journal, 31 (2009), 601-609.
[20] Qing Ping Hu and Xin Li, On proper BCH-algebras, Math. Japonica, 30 (1985), 659-661.
[21] C. Prabpayak dan U. Leerawat, On Derivations of BCC-algebras, Kasetsart J. (Nat. Sci.), 43(2009), 398–401.
[22] A. Walendziak, Some Results On BN1-algebras, Scientiae Mathematicae Japonicae 78, 3 (2015), 335–342.