Some Results On k-Relaxed Mean Graphs

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Abstract *A graph* $\boldsymbol{G = (V, E)}$ *with* $\boldsymbol{p}$ *vertices and* $\boldsymbol{q}$ *edges is said to be a* $\boldsymbol{k}$*-relaxed mean graph if there exists a function* $\boldsymbol{f}$ *from the vertex set of* $\boldsymbol{G}$ *to* $\{\boldsymbol{k-1, k, k+1, k+2, \dots, k+q}\}$ *such that in the induced map* $\boldsymbol{f^*}$ *from the edge set of* $\boldsymbol{G}$ *to* $\{\boldsymbol{k, k+1, k+2, \dots, k+q-1}\}$ *defined by* $\boldsymbol{f^*(e = uv)} = \begin{cases} \frac{f(u)+f(v)}{2} & \text{if } f(u)+f(v) \text{ is even} \\ \frac{f(u)+f(v)+1}{2} & \text{if } f(u)+f(v) \text{ is odd} \end{cases}$ *the resulting edge labels are distinct. In this paper, we prove some results on* $\boldsymbol{k}$*-relaxed mean labelling of some graphs.*

Keywords — $\boldsymbol{k}$*-relaxed mean labeling (*$\boldsymbol{k}$*-RML),*$\boldsymbol{k}$*-relaxed mean graph (*$\boldsymbol{k}$*-RMG)*

AMS Subject Classification: *05C78*

I. INTRODUCTION

In this paper, we consider all graphs are finite, simple and undirected. Terms not defined here are used in the sence of Frank Harary [1]. The symbols $V(G)$ and $E(G)$ will denote the vertex set and the edge set of a graph G.

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (or edges) then the labeling is called a vertex labeling (or an edge labeling). For an excellent survey on graph labeling, we refer to Gallian [3].

Mean labeling was introduced by S. Somasundaram and R. Ponraj [6-7]. Relaxed mean labeling was introduced by V. Maheswari, D.S.T. Ramesh and V. Balaji [4-5]. Further, it was studied by R. Thayalarajan and S. Nanthini. We have established path, cycle, ladder, the graph P_n^2, triangular snake T_n, quadrilateral snake Q_n, alternate triangular snake $A(T_n)$, Alternate quadrilateral snake $A(Q_n)$ as k-relaxed mean labeling of graphs.

II. MAIN RESULTS

Defintion 2.1: A graph $G = (V, E)$ with p vertices and q edges is said to be a k-relaxed mean graph if there exists a function f from the vertex set of G to $\{k-1, k, k+1, k+2, \dots, k+q\}$ such that in the induced map f^* from the edge set of G to $\{k, k+1, k+2, \dots, k+q-1\}$ defined by $f^*(e = uv) = \begin{cases} \frac{f(u)+f(v)}{2} & if\ f(u)+f(v) is\ even \\ \frac{f(u)+f(v)+1}{2} & if\ f(u)+f(v) is\ odd \end{cases}$ the resulting edge labels are distinct. A graph G which admits a k-relaxed mean labeling is called a $\boldsymbol{k}$-relaxed mean graph (k-RMG).

Definition 2.2: A graph is called a path if the degree $d(v)$of every vertex v is ≤ 2 and there are no more than 2 end vertices. An end vertex is a vertex of degree 1.

Definition 2.3: A walk in which no vertex (except the initial of final vertex) appear more than once is called a cycle.

Definition 2.4: The product graph $P_2 \times P_n$ is called a ladder and it is denoted by L_n.

Definition 2.5: The k^{th} power graph G^k of a connected graph G, where $k \geq 1$ is that graph with $V(G^k) = V(G)$ for which $uv \in E(G^k)$ if and only if $1 \leq d_G(u, v) \leq k$. The graph G^2 is referred to as square of G.

Definition 2.6: A triangular snake T_n is obtained from a path $v_1, v_2, v_3 \dots, v_{n+1}$ by joining v_i and v_{i+1} to a new vertices u_i for $1 \leq i \leq n$. That is, every edge of a path is replaced by a triangle C_3.

Definition 2.7: A quadrilateral snake Q_n is obtained from a path $u_1, u_2, u_3, \dots, u_n$ by joining u_i and u_{i+1} to a new vertex v_i for $1 \leq i \leq n-1$. That is, every edge of a path is replaced by a quadrilateral C_4.

Definition 2.8: An alternate triangular snake$A(T_n)$ is obtained from a path $u_1, u_2, u_3, \dots, u_n$ by joining u_i and u_{i+1} (alternately) to a new vertex v_i. That is, every edge of a path is replaced by C_3(alternately).

Definition 2.9: A alternate quadrilateral snake $A(Q_n)$ is obtained from a path $u_1, u_2, u_3, \dots, u_n$ by joining u_i and u_{i+1} (alternate) to a new vertex v_i for $1 \le i \le n-1$. That is, every edge of a path is replaced by C_4(alternate).

Theorem 2.10: The path P_n is a k-relaxed mean graph for all $k \ge 1$ and for all $n \ge 2$.

Proof: Let the vertices of P_n be $\{u_1, u_2, \dots, u_n\}$ and the edges of P_n be $\{e_1, e_2, \dots, e_{n-1}\}$, which are denoted as in Figure 1.1.

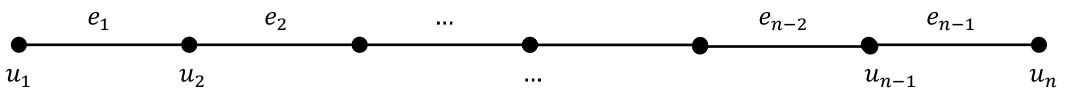

Figure 1.1: Ordinary labeling of P_n

We know that $|V(P_n)| = n$ and $|E(P_n)| = n-1$.

First we label the vertices as follows:

Define $f: V(P_n) \to \{k-1, k, k+1, k+2, \dots, k+q\}$ by

$$f(u_i) = \begin{cases} k+i-2 & ; 1 \le i \le n-1 \\ k+n-1 & ; \quad i = n \end{cases}$$

Then, the induced edge labels of P_n are

$$f^*(e_i) = k+i-1;\ 1 \le i \le n-1$$

Clearly, the induced edge labels are distinct. Hence, the path P_n is a k-relaxed mean graph for all $k \ge 1$ and for all $n \ge 2$.

Illustration 2.11: 4-RML of P_8 is shown in Figure 1.2.

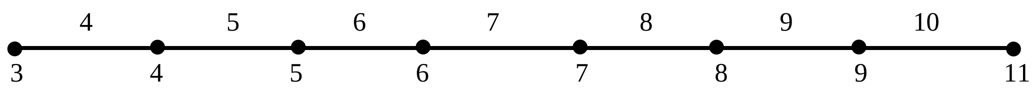

Figure 1.2: 4-RML of P_8

Illustration 2.12: 11-RML of P_9 is shown in Figure 1.3.

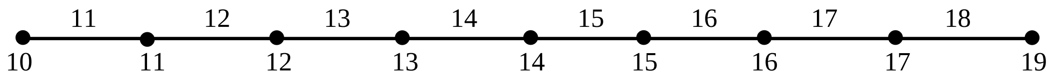

Figure 1.3: 11-RML of P_9

Theorem 2.13: The cycle C_n is a k-relaxed mean graph for all $k \ge 1$ and for all $n \ge 3$.

Proof: Let the vertices of C_n be $\{u_1, u_2, \dots, u_n\}$ and let the edges of C_n be $\{e_1, e_2, \dots, e_n\}$, which are denoted as in Figure 1.4.

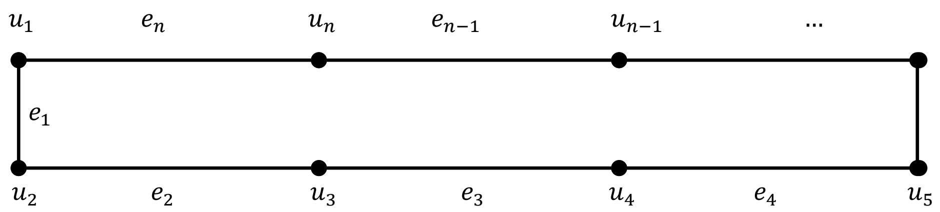

Figure 1.4: ***Ordinary labeling of C_n***

We know that $|V(C_n)| = n$ and $|E(C_n)| = n$.

First we label the vertices as follows:

Define $f: V(C_n) \to \{k-1, k, k+1, k+2, \dots, k+q\}$ by

For n odd , $1 \leq i \leq n$

$$f(u_i) = \begin{cases} k-1 & ; \quad i = 1 \\ k+i-2 & ; \quad 2 \leq i \leq \frac{n-1}{2} \\ k+i-1 & ; \frac{n+1}{2} \leq i \leq n-1 \\ k+n & ; \quad i = n \end{cases}$$

For n even , $1 \leq i \leq n$

$$f(u_i) = \begin{cases} k-1 & ; \quad i = 1 \\ k+i-2 & ; \quad 2 \leq i \leq \frac{n}{2} \\ k+i-1 & ; \frac{n}{2}+1 \leq i \leq n-1 \\ k+n & ; \quad i = n \end{cases}$$

Then, the induced edge labels of C_n are

For n odd , $1 \leq i \leq n$

$$f^*(e_i) = \begin{cases} k & ; \quad 1 \leq i \leq \frac{n-1}{2} \\ k+i & ; \frac{n+1}{2} \leq i \leq n-1 \\ \frac{2k+n-1}{2} & ; \quad i = n \end{cases}$$

For n even , $1 \leq i \leq n$

$$f^*(e_i) = \begin{cases} k+i-1 & ; \quad 2 \leq i \leq \frac{n}{2} \\ k+i & ; \frac{n}{2}+1 \leq i \leq n-1 \\ \frac{2k+n}{2} & ; \quad i = n \end{cases}$$

Clearly, the induced edge labels are distinct. Hence, the cycle C_n is a k-relaxed mean graph for all $k \geq 1$ and for all $n \geq 3$.

Illustration 2.14: 7-RML of C_8 is shown in Figure 1.5.

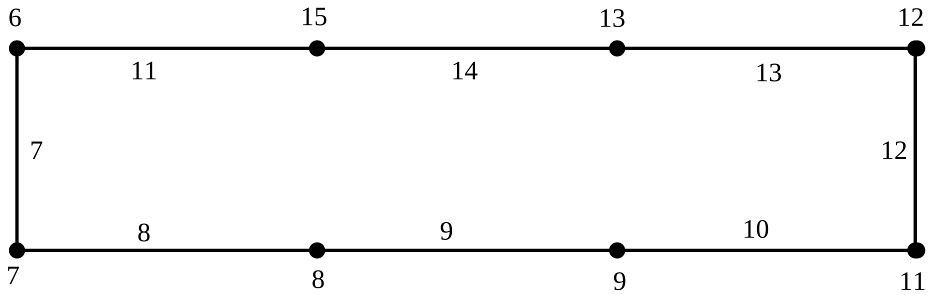

Figure 1.5: 7-*RML of* C_8

Illustration 2.15: 4-SDML of C_{11} is shown in Figure 1.6.

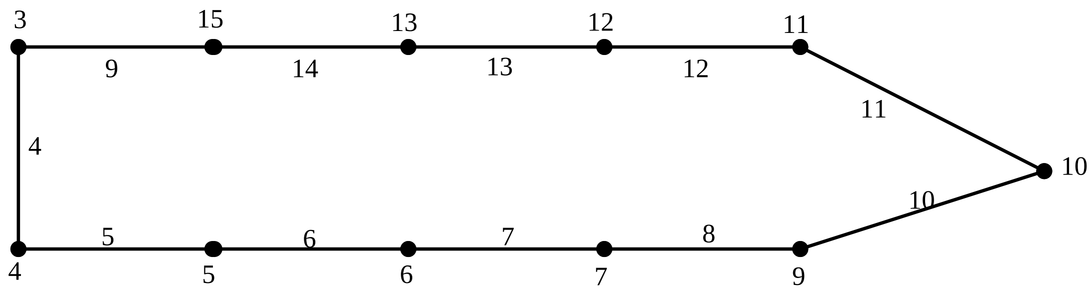

Figure 1.6: 4-*RML of* C_{11}

Theorem 2.16: The ladder L_n is a k-relaxed mean graph for all $k \geq 1$ and for all $n \geq 3$.

Proof: Let the vertices of L_n be $\{u_1, u_2, \dots, u_n\} \cup \{v_1, v_2, \dots, v_n\}$ and the edges of L_n be $\{e_1, e_2, \dots, e_{n-1}\} \cup \{e_1', e_2', \dots, e_{n-1}'\} \cup \{e_1'', e_2'', \dots, e_n''\}$, which are denoted as in Figure 1.7.

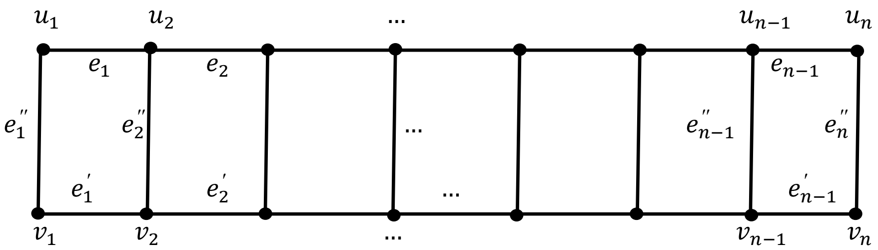

Figure 1.7: ***Ordinary labeling of L_n***

We know that $|V(L_n)| = 2n$ and $|E(L_n)| = 3n - 2$.

First we label the vertices as follows:

Define $f: V(L_n) \to \{k-1, k, k+1, k+2, \dots, k+q\}$ by

$$f(u_i) = \begin{cases} k + 3i - 3 & ; 1 \leq i \leq n-1 \\ k + 3n - 2 & ; \quad i = n \end{cases}$$

$$f(v_i) = k + 3i - 4 \;; 1 \leq i \leq n$$

Then, the induced edge labels of L_n are

$$f^*(e_i) = k + 3i - 1 \;; \; 1 \leq i \leq n-1$$

$$f^*(e_i') = k + 3i - 2 \;; 1 \leq i \leq n-1$$

$$f^*(e_i'') = k + 3i - 3 \;; 1 \leq i \leq n$$

Clearly, the induced edge labels are distinct. Hence, the ladder L_n is a k-relaxed mean graph for all $k \geq 1$ and for all $n \geq 3$.

Illustration 2.17: 4-RML of L_8 is shown in Figure 1.8.

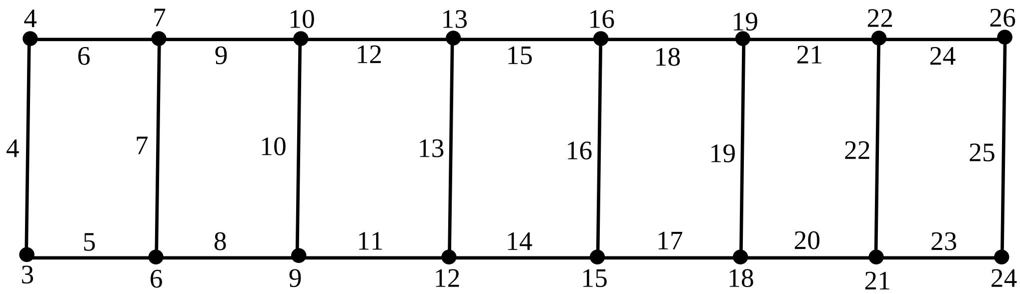

Figure 1.8: 4-*RML of* L_8

Illustration 2.18: 7-RML of L_7 is shown in Figure 1.9.

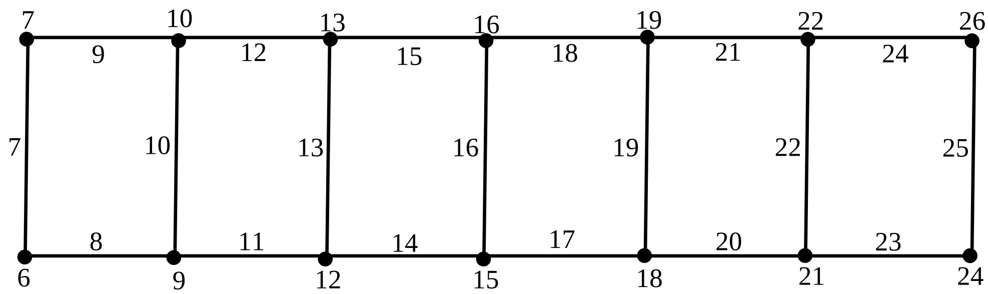

Figure 1.9: 7-*RML of* L_7

Theorem 2.19: The graph P_n^2 is a k-relaxed mean graph for all $k \geq 1$ and for all $n \geq 3$.

Proof: Let the vertices of P_n^2 be $\{u_1, u_2, \ldots, u_n\}$ and let the edges of P_n^2 be $\{e_1, e_2, \ldots, e_{n-1}\} \cup \{e_1^{'}, e_2^{'}, \ldots, e_{n-2}^{'}\}$, which are denoted as in Figure 1.10.

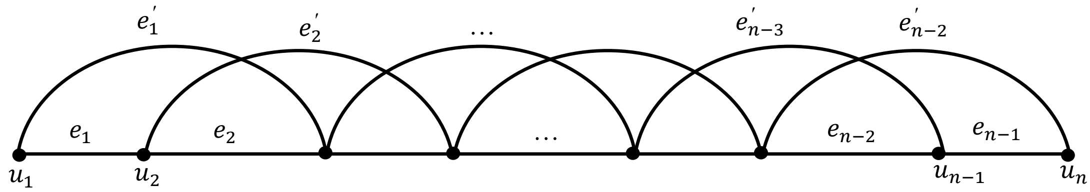

Figure 1.10: *Ordinary labeling of P_n^2*

We know that $|V(P_n^2)| = n$ and $|E(P_n^2)| = 2n - 3$.

First we label the vertices as follows:

Define $f: V(P_n^2) \to \{k-1, k, k+1, k+2, \ldots, k+q\}$ by

$f(u_i) = k + 2i - 3 \ ; 1 \leq i \leq n$

Then, the induced edge labels of P_n^2 are

$f^*(e_i) = k + 2i - 2 \ ; 1 \leq i \leq n - 1$

$f^*(e_i^{'}) = k + 2i - 1 \ ; 1 \leq i \leq n - 2$

Clearly, the induced edge labels are distinct. Hence, the P_n^2 is a k-relaxed mean graph for all $k \geq 1$ and for all $n \geq 3$.

Illustration 2.20: 2-RML of P_8^2 is shown in Figure 1.11.

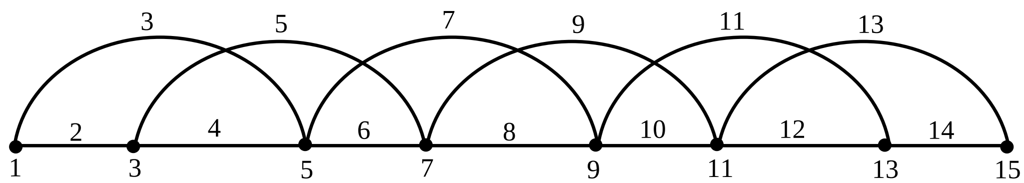

Figure 1.11: 2-*RML of P_8^2*

Illustration 2.21: 5-RML of P_5^2 is shown in Figure 1.12.

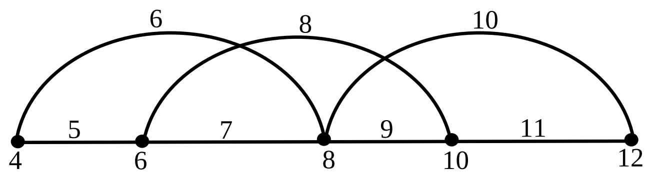

Figure 1.12: 5-*RML of P_5^2*

Theorem 2.22: The triangular snake T_n is a k-relaxed mean graph for all $k \geq 1$ and for all $n \geq 2$.

Proof: Let the vertices of T_n be $\{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_{n+1}\}$ and let the edges of T_n be $\{e_1, e_2, \ldots, e_n\} \cup \{e_1^{'}, e_2^{'}, \ldots, e_n^{'}\} \cup \{e_1^{''}, e_2^{''}, \ldots, e_n^{''}\}$, which are denoted as in Figure 1.13.

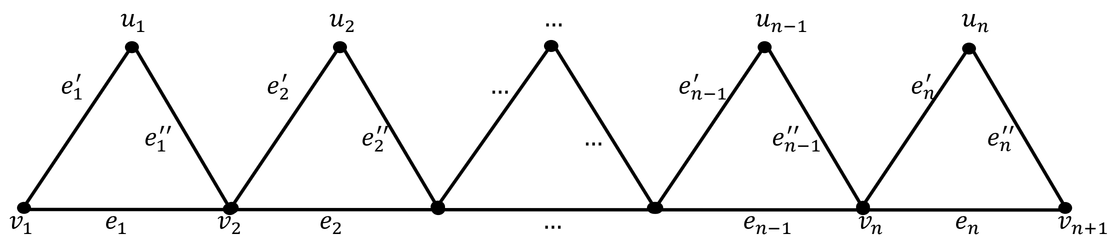

Figure 1.13: *Ordinary labeling of T_n*

We know that $|V(T_n)| = 2n + 1$ and $|E(T_n)| = 3n$.

First we label the vertices as follows:

Define $f: V(T_n) \to \{k-1, k, k+1, k+2, \ldots, k+q\}$ by

$$f(u_i) = k + 3i - 2 \ ; 1 \le i \le n$$

$$f(v_i) = \begin{cases} k + 3i - 4 \ ; 1 \le i \le n \\ k + 3n - 3 \ ; i = n+1 \end{cases}$$

Then, the induced edge labels of T_n are

$$f^*(e_i) = k + 3i - 2 \ ; 1 \le i \le n$$

$$f^*(e_i') = k + 3i - 3 \ ; 1 \le i \le n$$

$$f^*(e_i'') = k + 3i - 1 \ ; 1 \le i \le n$$

Clearly, the induced edge labels are distinct. Hence, the triangular snake T_n is a k-relaxed mean graph for all $k \ge 1$ and for all $n \ge 2$.

Illustration 2.23: 6-RML of T_9 is shown in Figure 1.14.

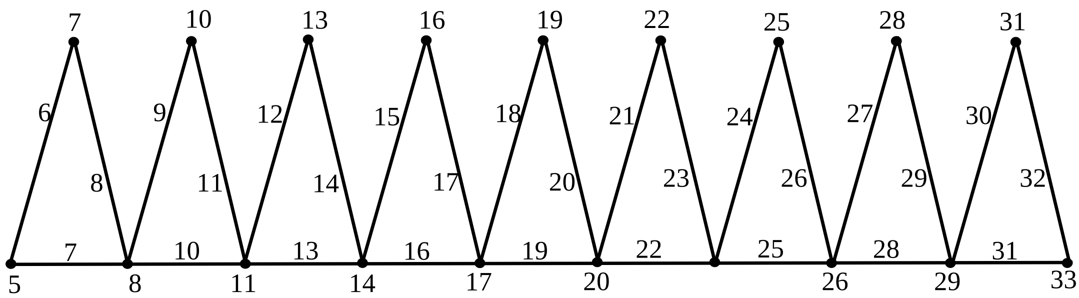

Figure 1.14: 6-*RML of* T_9

Illustration 2.24: 3-RML of T_6 is shown in Figure 1.15.

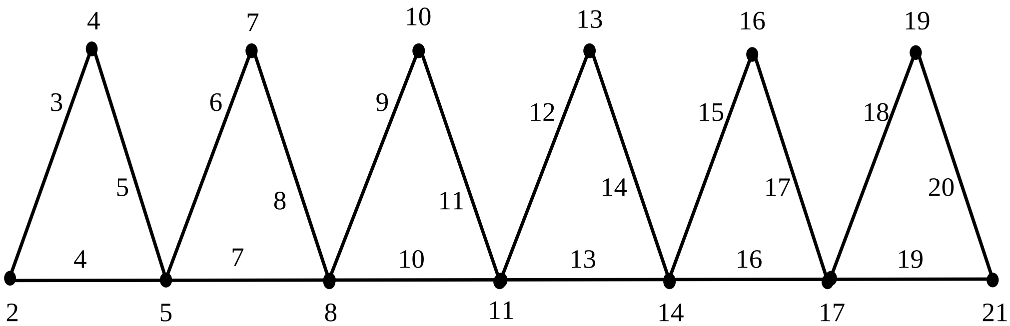

Figure 1.15: 3-*RML of* T_6

Theorem 2.25: The quadrilateral snake Q_n is a k-relaxed mean graph for all $k \ge 1$ and for all $n \ge 3$.

Proof: Let the vertices of Q_n be $\{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_{2n-2}\}$ and let the edges of Q_n be $\{e_1, e_2, \ldots, e_{n-1}\} \cup \{e_1', e_2', \ldots, e_{n-1}'\} \cup \{e_1'', e_2'', \ldots, e_{2n-2}''\}$, which are denoted as in Figure 1.16.

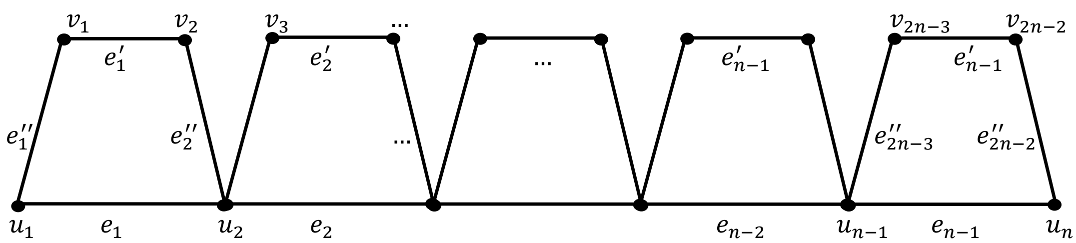

Figure 1.16: *Ordinary labeling of* Q_n

We know that $|V(Q_n)| = 3n - 2$ and $|E(Q_n)| = 4n - 4$.

First we label the vertices as follows:

Define $f: V(Q_n) \to \{k-1, k, k+1, k+2, \dots, k+q\}$ by

$$f(u_i) = \begin{cases} k + 4i - 5 & ; 1 \le i \le n-1 \\ k + 4n - 4 ; & i = n \end{cases}$$

For $1 \le i \le 2n - 2$

$$f(v_i) = \begin{cases} k + 2i - 1 ; & i\ odd \\ k + 2i - 2 ; & i\ even \\ k + 4n - 8 & ; i = 2n - 3 \\ k + 4n - 7 & ; i = 2n - 2 \end{cases}$$

Then, the induced edge labels of Q_n are

$$f^*(e_i) = \begin{cases} k + 4i - 3 & ; 1 \le i \le n-2 \\ k + 4n - 6 ; & i = n - 1 \end{cases}$$

$$f^*(e_i') = \begin{cases} k + 4i - 2 & ; 1 \le i \le n-2 \\ k + 4n - 7 ; & i = n - 1 \end{cases}$$

$$f^*(e_i'') = \begin{cases} k + 2i - 2 ; & i\ odd \\ k + 2i - 1 ; & i\ even \\ k + 4n - 8 & ; i = 2n - 3 \\ k + 4n - 5 & ; i = 2n - 2 \end{cases}$$

Clearly, the induced edge labels are distinct. Hence, the quadrilateral snake Q_n is a k-relaxed mean graph for all $k \ge 1$ and for all $n \ge 3$.

Illustration 2.26: 11-RML of Q_6 is shown in Figure 1.17.

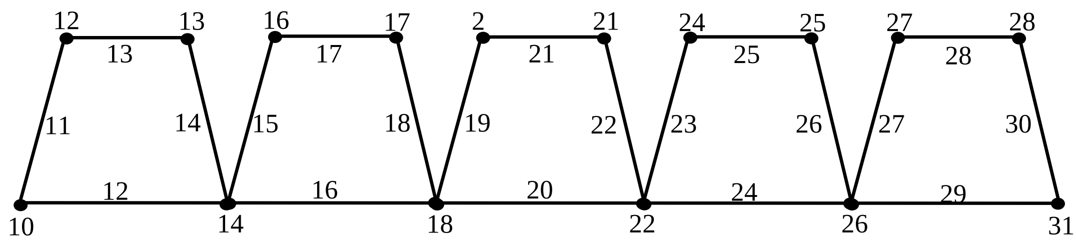

Figure 1.17: 11-*RML of* Q_6

Illustration 2.27: 4-RML of Q_3 is shown in Figure 1.18.

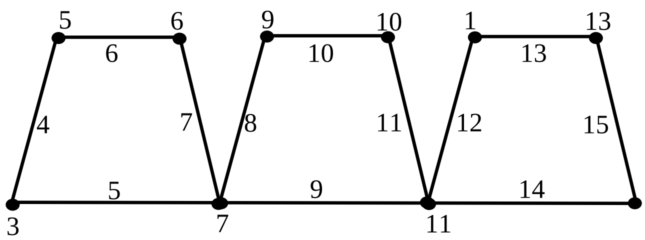

Figure 1.18: 4-*RML of* Q_3

Theorem 2.28: The alternate triangular snake $A(T_n)$ is a k-relaxed mean graph for all $k \ge 1$ and for all $n \ge 2$.

Proof: Let the vertices of $A(T_n)$ be $\{u_1, u_2, \dots, u_n\} \cup \{v_1, v_2, \dots, v_{2n}\}$ and let the edges of $A(T_n)$ be $\{e_1, e_2, \dots, e_{2n-1}\} \cup \{e_1', e_2', \dots, e_{2n}'\}$, which are denoted as in Figure 1.19.

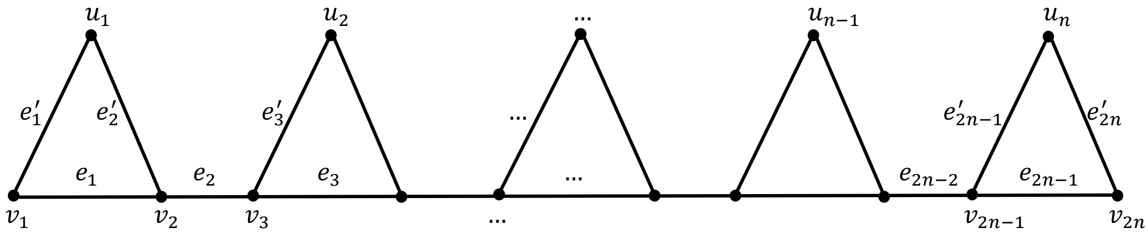

Figure 1.19: ***Ordinary labeling of*** $\boldsymbol{A(T_n)}$

We know that $|V(A(T_n))| = 3n$ and $|E(A(T_n))| = 4n - 1$.

First we label the vertices as follows:

Define $f: V(A(T_n)) \to \{k-1, k, k+1, k+2, \dots, k+q\}$ by

$$f(u_i) = k + 4i - 3 \ ; 1 \le i \le n$$

$$f(v_i) = \begin{cases} \frac{2k+4i-6}{2} & ; i \ odd \\ \frac{2k+4i-4}{2} & ; i \ even \\ \frac{2k+4n-2}{2} & ; i = 2n \end{cases}$$

Then, the induced edge labels of $A(T_n)$ are

$$f^*(e_i) = k + 2i - 1 \ ; 1 \le i \le 2n - 1$$

$$f^*(e_i') = k + 2i - 2 \ ; 1 \le i \le 2n$$

Clearly, the induced edge labels are distinct. Hence, the alternate triangular snake $A(T_n)$ is a k-relaxed mean graph for all $k \ge 1$ and for all $n \ge 2$.

Illustration 2.29: 6-RML of T_5 is shown in Figure 1.20.

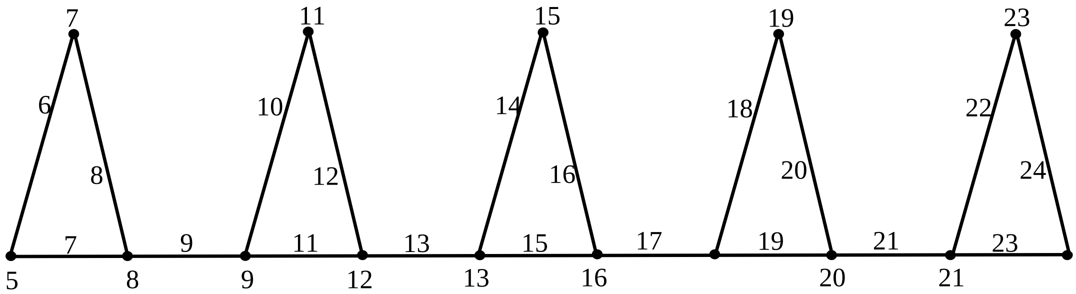

Figure 1.20: 6-*RML of* $\boldsymbol{A(T_5)}$

Illustration 2.30: 3-RML of T_3 is shown in Figure 1.21.

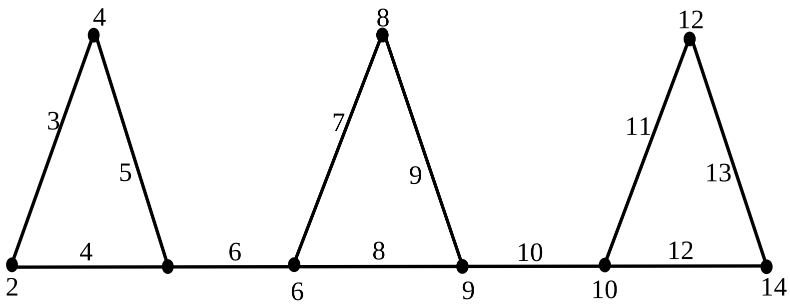

Figure 1.21: 3-*RML of* $\boldsymbol{A(T_3)}$

Theorem 2.31: The alternate quadrilateral snake $A(Q_n)$ is a k-relaxed mean graph for all $k \geq 1$ and for all $n \geq 3$.

Proof: Let the vertices of $A(Q_n)$ be $\{u_1, u_2, \dots, u_{2n}\} \cup \{v_1, v_2, \dots, v_{2n}\}$ and let the edges of $A(Q_n)$ be $\{e_1, e_2, \dots, e_{2n-1}\} \cup \{e_1^{'}, e_2^{'}, \dots, e_n^{'}\} \cup \{e_1^{''}, e_2^{''}, \dots, e_{2n}^{''}\}$, which are denoted as in Figure 1.22.

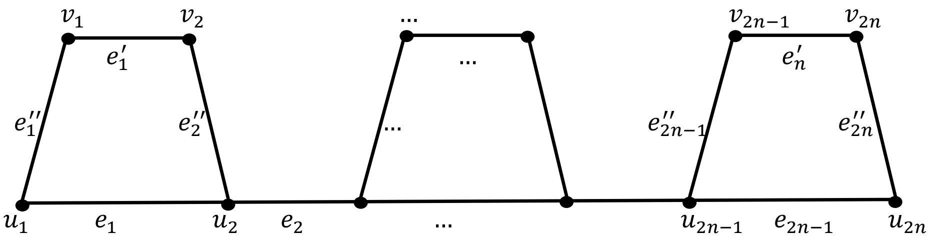

Figure 1.22: ***Ordinary labeling of*** $\boldsymbol{A(Q_n)}$

We know that $|V(A(Q_n))| = 4n$ and $|E(A(Q_n))| = 5n - 1$.

First we label the vertices as follows:

Define $f: V(A(Q_n)) \to \{k-1, k, k+1, k+2, \dots, k+q\}$ by

For $1 \leq i \leq 2n$

$$f(u_i) = \begin{cases} \frac{2k+5i-7}{2} & ; i \text{ odd} \\ \frac{2k+5i-4}{2} & ; i \text{ even} \end{cases}$$

For $1 \leq i \leq 2n$

$$f(v_i) = \begin{cases} \frac{2k+5i-3}{2} & ; i \text{ odd} \\ \frac{2k+5i-6}{2} & ; i \text{ even} \end{cases}$$

Then, the induced edge labels of $A(Q_n)$ are

For $1 \leq i \leq 2n - 1$

$$f^*(e_i) = \begin{cases} \frac{2k+5i-3}{2} & ; i \text{ odd} \\ \frac{2k+5i-2}{2} & ; i \text{ even} \end{cases}$$

$$f^*(e_i^{'}) = k + 5i - 3 \; ; 1 \leq i \leq n$$

For $1 \leq i \leq 2n$

$$f^*(e_i^{''}) = \begin{cases} \frac{2k+5i-5}{2} & ; i \text{ odd} \\ \frac{2k+5i-4}{2} & ; i \text{ even} \end{cases}$$

Clearly, the induced edge labels are distinct. Hence, the alternate quadrilateral snake $A(Q_n)$ is a k-relaxed mean graph for all $k \geq 1$ and for all $n \geq 3$.

Illustration 2.32: 5-RML of $A(Q_3)$ is shown in Figure 1.23.

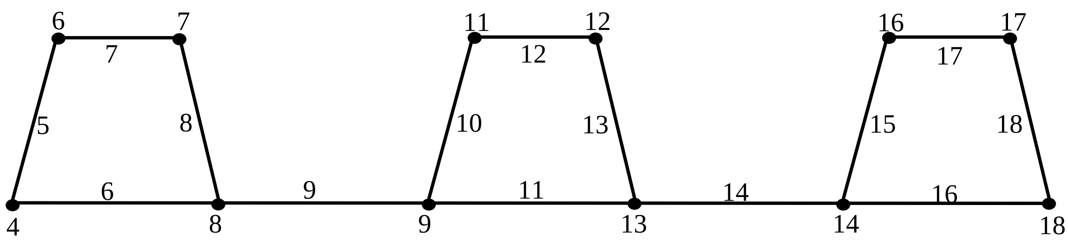

Figure 1.23: 5-*RML of* $\boldsymbol{A(Q_3)}$

Illustration 2.33: 2-RML of $A(Q_n)$ is shown in Figure 1.24.

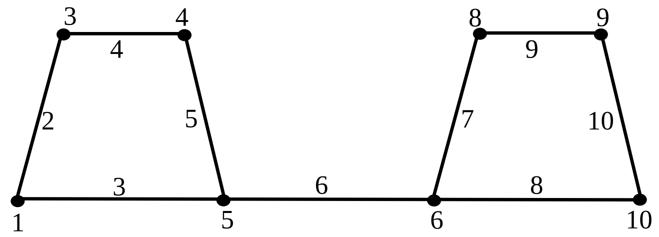

Figure 1.24: 2-*RML of* $\boldsymbol{A(Q_2)}$

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