Mathematical Modelling of Transmission Dynamics of Rabies Virus

Oliver C. Eze[1], Godwin E. Mbah[2], Daniel U. Nnaji[3] and Netochukwu E. Onyiaji[4].
[1,2,3,4]*Department of Mathematics, University of Nigeria, Nsukka.*

Abstract*: In this work, we presented an SEIR and SEIV model to describe the transmission dynamics of rabies virus in dogs and humans. The basic reproduction number was computed using next generation method. We computed the disease free and endemic equilibrium points. If the basic reproduction number,* $R_0 < 1$ *the disease-free equilibrium is locally asymptotically stable this means the disease will dies out within a period of time. The endemic equilibrium points were investigated wherever they exist, using the Descartes' rule of signs, the endemic equilibrium points are locally asymptotically stable. We also obtained a control solution for the model which predicts that the best way of eliminating deaths from rabies as projected by the global alliance for rabies control is using more of pre-exposure prophylaxis in both dogs and humans and public education; however, the results show that pre-exposure prophylaxis and post-exposure prophylaxis in humans with use of vaccination in the dog population is beneficial if total elimination of the disease is desirable*.

Keywords: *Differential Equation, Equilibrium points, Homotopy Perturbation, Rabies, and Stability Analysis.*

I. INTRODUCTION

Rabies is a viral disease caused by Lyssaviruses. It induces inflammation of the brain in mammals. In the early stage of infection, the symptoms include fever and tickling at the site of exposure, followed by one or more other symptoms like; wild movement, inability to control parts of the body, loss of consciousness, confusion fear of water and uncontrollable excitement. It often takes one to three months from the day of exposure to the disease before symptoms appear. The symptoms can vary from less than one week to more than one year. Almost every case resulted in death once the symptoms appear. It outspread when an infected animal or human bites or scratches another animal or human. Rabies can also be gotten from the saliva of an infected animal if it comes in contact with the mouth, eyes or nose of the susceptible animal. Generally, the common animal involved in the spread of rabies is the dog. More than 99% of rabies cases in countries where dogs commonly have the disease are the direct result of dog bites (Addo 2012).

Vaccination and animal control have reduced the chance of rabies from dog s in 1some part s of the world. People who are at high risk like those who work with dogs, or who spend a prolonged period in areas where rabies is common can be immunized before exposure. Vaccination (Rabies vaccines) and rabies immunoglobulin are potent to forestall the disease if the person undergoes the treatment before the appearance of the symptoms. Washing the site of exposure for 15 minutes with soap and water, iodine, povidone or detergent may somewhat be potent at preventing the spread of the diseases.

In early models of rabies dynamics, the populations were subdivided into specific classes as susceptible (S), exposed (E), infectious (I), and removed (R) individuals (Anderson et al., 1991). A system of ordinary differential equations (ODEs) was used to explain the dynamics of rabies representing either single populations or linked metapopulations from which a sort of predictions can be drawn concerning temporal and spatial patterns. Several critical features of disease emergence and spread were derived using the basic SEIR compartmental framework in the early model of rabies. The critical threshold for epidemic emergence and the basic reproductive number (R_0) for the virus were calculated from the model equation. The infection will spread, when R_0>1, and the disease will persist. It is possible to suggest what level of population culling would be needed in order to bring threshold density below epizootic level using. Although the formulation of their model using the SEIR compartmental framework, they do not consider the R class, after all, there is evidence of the development of natural immunity and vaccination which translates susceptible into the removed class, they failed to investigate.

Smith et al., (2002) constructed a stochastic spatial model to describe the spread of rabies in Connecticut. Predicting the Spatial Dynamics of Rabies Epidemics on Heterogeneous Landscapes. From their result, they suggested that rivers act as a semipermeable barrier to the transmission of rabies resulting in a sevenfold decrease in the rate of transmission. The Connecticut data, the influence of habitat and long-distance translocation events were used to evaluate the effect of long-distance translocation and spatial heterogeneity in the raccoon rabies epidemic in Connecticut. The results of the reanalysis showed that rivers interact to further decrease the spatial spread of raccoon rabies (Smith et al., 2002).

Russell et al., (2005) used the stochastic spatial model constructed by Smith et al., (2002) to analyze data from Ohio. They later authored another paper using an ODE model to show that the transmission of rabies may be curb by distributing vaccines behind barriers such as rivers. This SIR model included the three classes in nine spatial compartments giving a total of 27 ODEs. Their results showed that a higher rate of vaccination is required for a large population and a lower rate at a higher cost.

Asano et al., (2008) applied optimal control to an epidemic model for rabies in raccoons using a SIR Metapopulation model. They included space through subpopulation arrangement connected by movement. The optimal control vector gives account for the cost of administering the vaccine and gives the rate of vaccination in each subpopulation that reduces the infected class overall subpopulations.

Zhang et al., (2011) considered the actual situation of rabies spreading in China. They formulated two mathematical models to study both the transmission dynamics of rabies in dogs and human and control strategies. They compared the efficiency of three strategies for controlling rabies: culling, vaccination, culling and vaccination and found that vaccination is the best choice to control rabies. Hong-tao et al., (2014) established another mathematical model of rabies with similar controlling strategies in China. Their model considered people infected by exposed dogs, infected dogs and seemingly healthy dogs carrying the virus. Their mathematical analysis and simulation showed the culling strategy as most efficient, then vaccination and culling and vaccination as the last effective.

According to Zhang et al., (2011), human rabies is one of the main public health problems in China. They constructed a model to explore effective control and prevention measures. They proposed a deterministic model to study the transmission dynamics of rabies in China. Their model made use of SEIR
subpopulations of both dogs and humans and consider the spread of rabies among dogs and from infectious dogs to humans. The model simulations agree with the human rabies data reported by the Chinese Ministry of Health. From their analysis, they estimated that the R_0 = 2 for the rabies transmission in China and forecast that the number of human rabies is decreasing but may reach another peak around 2030. They also carried out some sensitivity analysis of in terms of the model parameters and compared the effects of culling and immunization of dogs.

Ega et al., (2015) in their paper; Modeling the Dynamics of Rabies Transmission with Vaccination and Stability Analysis, formulated a deterministic mathematical model for the transmission dynamics of rabies in humans and animals within and around Addis Ababa, Ethiopia. Their model considered vaccination program for dog population. The basic reproduction number and effective reproduction numbers they computed. Their results are completely depending on the parameters of the dog population, which shows that the dog population is responsible for human and livestock infection. For some sets of values of parameters, they estimated from the data they get from the Ethiopian Public Health Institute of Addis Ababa, the basic reproduction number and the effective reproduction number they found to be 2 and 1.6 respectively, which shown that the disease will be endemic. Their numerical simulation of reproduction ratio shows that when vaccination, culling of stray dogs and controlling annual crop of newborn puppies are combined, yield the best method to control rabies spread within and around Addis Ababa.

According to Salahot (2016), rabies is among the top ten infectious diseases that kill the majority of rural people in Tanzania. She modified the rabies model by Zhang et al. (2011) by incorporating controls: vaccination of susceptible dogs and exposed humans and culling of infected dogs. When basic reproduction number R_0>1, the analysis of the simple model shows that the rabies-free equilibrium points are both locally and globally asymptotically stable when. She carried out the sensitivity analysis on the basic reproduction number, which shows that the dog's transmission rate βdd and rabies mortality rate αd are the most sensitive parameters. Therefore, intervention strategies that will reduce dog's transmission rates and increase rabies mortality rates should be applied, in order to reduce the new rabies cases, she claimed. She used estimated parameter values and carried out a simulation to the extended model with vaccination and culling. Her results suggested that the combination of vaccination and culling is the optimal intervention strategy that should be applied by Kilombero, Ulanga and Serengeti districts in Tanzania based on her specified conditions, to limit rabies transmission. Though this also agreed with others' work she failed to considered man to man rabies transmission.

Ruan (2017) worked on modeling the transmission dynamics and control of rabies in China where he first constructed a basic susceptible, exposed, infectious, and recovered (SEIR) type model for the spread of rabies virus among dogs and from dogs to humans. He used the model to simulate the human rabies data in China from 1996 to 2010. Then he later modified the basic model by including both domestic and stray dogs. Then used the model to simulate the human rabies data from Guangdong Province, China. He studied the seasonality of rabies, by proposing an SEIR model with periodic transmission rates. He used the model to simulate the monthly data of human rabies cases reported by the Chinese Ministry of Health from January 2004 to December 2010. He studied the spatial spread of rabies, by introducing diffusion to the dog population in the basic SEIR model and obtained a reaction-diffusion equation model. He determined the minimum wave speed linking the disease-free equilibrium to the endemic equilibrium. He proposed a multi-patch model to study the impact of the movement of dogs on the geographically inter-provincial transmission dynamics of rabies between dogs and humans in

Mainland China. He used a two-patch submodel to investigate the rabies virus clades and used this to simulate human rabies data from Guizhou and Guangxi, Hebei and Fujian, and Sichuan and Shaanxi, respectively. He failed to considered man to man rabies transmission.

Laager and Nord (2018) in their inaugural dissertation; Mathematical modeling of dog rabies transmission in N'Djamena, Chad, developed a deterministic metapopulation model with the introduction of latent dogs to investigate the potential reasons for the resurgence experienced in the area. They claimed that vaccinating dogs is effective and is an equitable way that human exposure can be reduced and suggested that it should be an integral part of rabies control programs in endemic settings. However, in their claims lack dog population management, increased population quickly decrease vaccination coverage and re-introduction from surrounding areas jeopardize the merits of mass dog vaccination campaigns. They concluded that continuous vaccination of dog or mass vaccination campaigns, likewise oral vaccination of reservoirs (though not covered by their work), as a control measure to maintain high vaccination coverage in settings like N'Djamena. Also, they failed to consider man to man rabies transmission.

In this work, we shall be looking at the dog to dog transmission, dog to human transmission and human to human transmission and its contribution to the rabies population. Moreso, we are considering SEIR compartmental framework for the human population and SEIV compartmental framework for dog population where V stands for Vaccinated class.

II. MODEL FORMULATION

In this work, we construct an SEIR model of rabies for human and SEIV model for the dog population. We classified the human population into susceptible, exposed, infected, recovered class and dog population into susceptible, exposed, infected, vaccinated class. The susceptible class have no disease, but they are likely to be infected in case of contact with rabid dogs, Exposed individuals are those who contracted the virus through bites or scratch, but still, they have not shown symptoms, Infected individuals are those who develop clinical symptoms. The Recovered classes are those who recovered through Treatment before they reach to the infectious stage, whereas the rest get infected and die.

The human population is classified into Susceptible, S_h , Exposed, E_h Infectious, I_h , and Recovered, R_h, Individuals are recruited to susceptible class at . A susceptible man bitten by a rabid dog becomes exposed. If post-exposure treatment is not given, the person becomes infectious and ends up with death since there is no recovery at the infectious stage.

The dog population is divided into Susceptible, S_d , Exposed, E_d , Infectious, I_d and Vaccinated V_d. Individuals are recruited into susceptible class at φ_d . For a susceptible dog, Treatment is applied at a rate of . This is because it is the dog population which infects human. An exposed dog moves to the infectious class directly if there's no treatment at the stage of exposure.

A. Assumptions of the SEIR Model

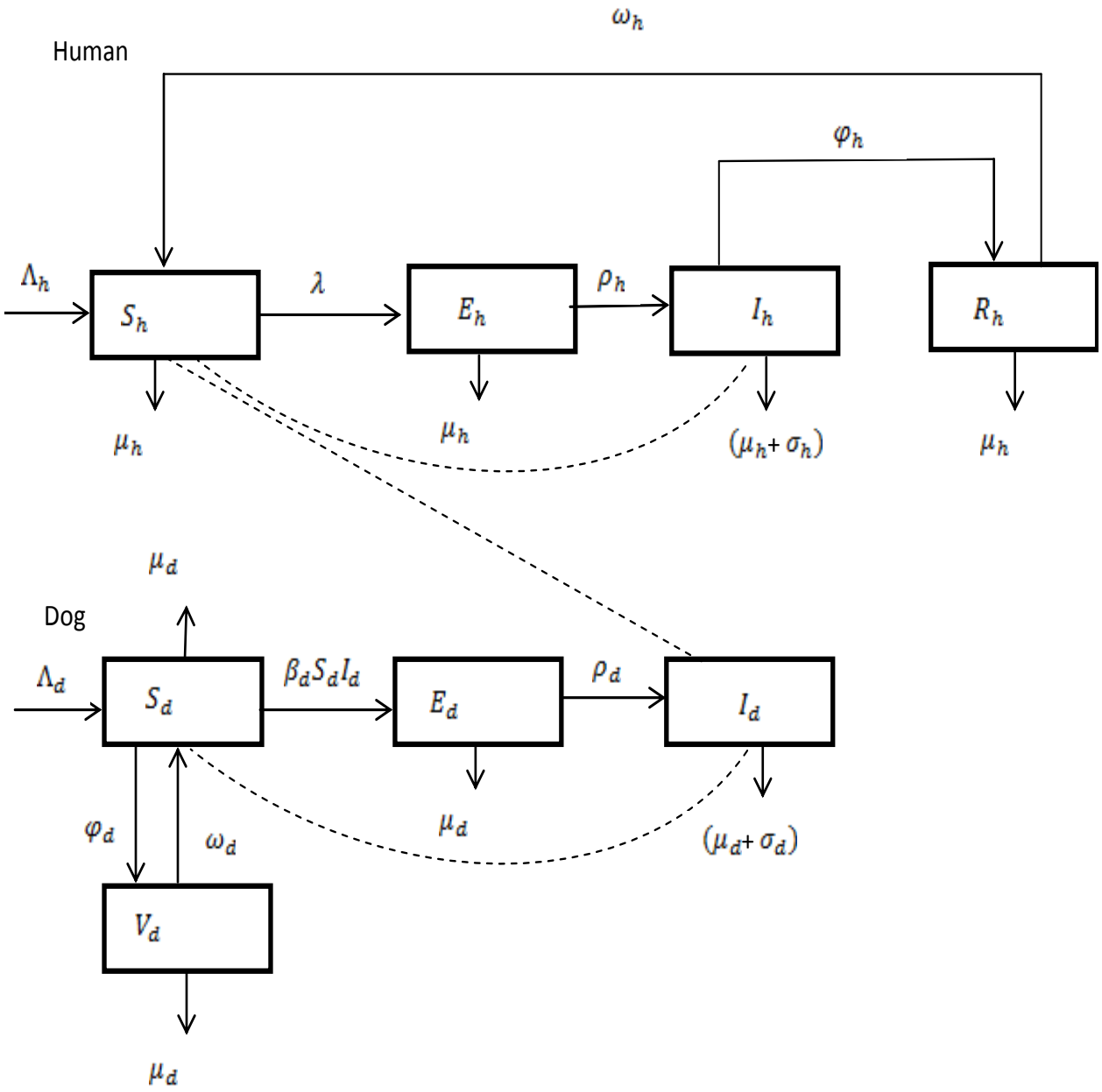
- Susceptible populations of dog and human are recruited at a rate level $\Lambda_h \; and \; \Lambda_d$.
- Rabies transmission among humans to humans was not ignored.
- An individual who is bitten by rabid dog become infected.
- Individuals in each compartment have an equal natural death rate.

B. Model Variables and Parameters

The following variables and parameters shall be used in this model.

S_h(t): the number of Susceptible humans at time t.
E_h(t): the number of Exposed humans at time t.
I_h(t): the number of Infected humans at time t.
R_h(t): the number of Recovered humans at time t.
S_d(t): the number of Susceptible dogs at time t.
E_d(t): the number of Exposed dogs at time t.
I_d(t): the number of Infected dogs at time t.
V_d(t): the number of Vaccinated dogs at time t.
N_h(t): the total human population size at time t.
N_d(t): the total dog population size at time t.
$\Lambda_{h,}\Lambda_d$: the recruitment level of human and dog population, respectively.
σ_h, σ_d: death rate due to rabies for human and dog population, respectively.
μ_h, μ_d: natural death rate of human and dog population, respectively.
ρ_h, ρ_d: the incubation period in human and dog population, respectively.
φ_h: Treatment rate of exposed human population.
φ_d: Vaccination rate of Susceptible dog population.
ω_h, ω_d: the loss rate of vaccination immunity for human and dog population respectively.

III. MODEL DIAGRAM



IV. THE MODEL EQUATION

Base on the assumption and interrelation between the variables and parameters in fig 4.2, Rabies transmission dynamics can be described by using ordinary differential equations.

A. The SEIR Model with Vaccination

$$\frac{dS_h}{dt} = \Lambda_h + \omega_h R_h - (\beta_{h1} S_h I_h + \beta_{h2} S_h I_d) - \mu_h S_h,$$

$$\frac{dE_h}{dt} = (\beta_{h1} S_h I_h + \beta_{h2} S_h I_d) - (\varphi_h + \rho_h + \mu_h) E_h,$$

$$\frac{dI_h}{dt} = \rho_h E_h - (\mu_h + \sigma_h) I_h,$$

$$\frac{dR_h}{dt} = \varphi_h E_h - (\omega_h + \mu_h) R_h,$$

$$\frac{dS_d}{dt} = \Lambda_d - \beta_d S_d I_d + \omega_d V_d - (\varphi_d + \mu_d) S_d, \qquad (4.4.1)$$

$$\frac{dE_d}{dt} = \beta_d S_d I_d - (\rho_d + \mu_d) E_d,$$

$$\frac{dI_d}{dt} = \rho_d E_d - (\mu_d + \sigma_d) I_d,$$

$$\frac{dV_d}{dt} = \varphi_d S_d - (\omega_d + \mu_d) V_d.$$

With initial conditions $S_h(0) = 0$, $E_h(0) = 0$, $I_h(0) = 0$, $R_h(0) = 0$, $S_d(0) = 0$, $E_d(0) = 0$, $I_d(0) = 0$ *and* $V_d(0) = 0$.

B. Invariant Region

Lemma 1
The solution set $\{S_h, E_h, I_h, R_h, S_d, E_d, I_d, V_d\} \in R_+^8$ of model system (4.4.1) is contained in the feasible region Ω.

Proof.
Suppose $\{S_h, E_h, I_h, R_h, S_d, E_d, I_d, V_d\} \in R_+^8$ for all $t > 0$. We want to show that the region Ω is positively invariant, so that it becomes sufficient to look at the dynamics of the model system (4.4.1). Therefore, given that
$N_h(t) = S_h(t) + E_h(t) + I_h(t) + R_h(t)$
and
$N_d(t) = S_d(t) + E_d(t) + I_d(t) + V_d(t)$
Where $N_h(t)$ is the total population of human at time (t) and $N_d(t)$ is the total population of dog at time (t).
Therefore

$\frac{dN_h}{dt} = \frac{dS_h}{dt} + \frac{dE_h}{dt} + \frac{dI_h}{dt} + \frac{dR_h}{dt}$, we obtain

$$\frac{dN_h}{dt} = \Lambda_h - \mu_h N_h - \sigma_h I_h \quad ---------- \quad (4.5.2)$$

$\frac{dN_d}{dt} = \frac{dS_d}{dt} + \frac{dE_d}{dt} + \frac{dI_d}{dt} + \frac{dV_d}{dt}$, we obtain

$$\frac{dN_d}{dt} = \Lambda_d - \mu_d N_d - \sigma_d I_d \quad ---------- \quad (4.5.3)$$

If there is no disease induced death rate, then ($\sigma_h = \sigma_d = 0$), hence equation (4.5.2) and (4.5.3) becomes

$$\frac{dN_h}{dt} = \Lambda_h - \mu_h N_h \quad ---------- \quad (4.5.4)$$

$$\frac{dN_d}{dt} = \Lambda_d - \mu_d N_d \quad ---------- \quad (4.5.5$$

Suppose

$\frac{dN_h}{dt} \leq 0$ and $\frac{dN_d}{dt} \leq 0$,

$N_h \leq \frac{\Lambda_h}{\mu_h}$ and $N_d \leq \frac{\Lambda_d}{\mu_d}$.

Now imposing the theorem proposed by Birkhoff and Rotta (1989) on differential inequality result in $0 \leq N_h \leq \frac{\Lambda_h}{\mu_h}$ and $0 \leq N_d \leq \frac{\Lambda_d}{\mu_d}$. Therefore equation (4.5.4) and (4.5.5) becomes

$$\frac{dN_h}{dt} \leq \Lambda_h - \mu_h N_h \quad ---------- \quad (4.5.6)$$

$$\frac{dN_d}{dt} \leq \Lambda_d - \mu_d N_d \quad ---------- \quad (4.5.7)$$

Solving equation (4.5.6) and (4.5.7) using integrating factor (I.F) method, we have

$\frac{dy}{dx}$ + p(t)y = Q

I.F = $e^{\int p(t)dt}$.

Given the following deductions

$e^{\mu_h t}\frac{dN_h}{dt} + \mu_h N_h e^{\mu_h t} \leq \Lambda_h e^{\mu_h t}$

$\int d\, N_h e^{\mu_h t} \leq \Lambda_h e^{\mu_h t} dt,$

$N_h = \frac{\Lambda_h}{\mu_h} + K_1 e^{-\mu_h t},$

$\frac{\Lambda_h}{\mu_h} \leq N_h - K_1 e^{-\mu_h t}$ where $K_1 e^{-\mu_h t} \leq 0.$

This gives

$\Lambda_h - \mu_h N_h \geq K_1 e^{-\mu_h t}$

Where K_1 is the constant of integration.

Hence the feasible solution of the humans population in the system model (4.4.1) is in the region:

$\Omega_h = \{S_h, E_h, I_h, R_h,\} \in R_+^4.$

Similarly the dogs population follows suit and from (4.5.7) this implies that

$\Lambda_d - \mu_d N_d \geq K_2 e^{-\mu_d t}.$

Where K_2 is the constant of integration. Therefore, the feasible solution of the dog population of the system (4.4.1) is in the region

$\Omega_d = \{S_d, E_d, I_d, V_d,\} \in R_+^4.$

Therefore the feasible solution are contained in Ω. Thus $\Omega = \Omega_h \, x \, \Omega_d$. From the standard comparison theorem used on differential inequality, it is implies that

$N_h(t) \leq N_h(0)\, e^{-\mu_h t} + \frac{\Lambda_h}{\mu_h}(1 - e^{-\mu_h t})$

and

$N_d(t) \leq N_d(0)\, e^{-\mu_d t} + \frac{\Lambda_d}{\mu_d}(1 - e^{-\mu_d t})$

Hence, the vector population size $N_d(t) \rightarrow \frac{\Lambda_d}{\mu_d}$ as $t \rightarrow \infty$.

Similarly for the host population size $N_h(t) \rightarrow \frac{\Lambda_h}{\mu_h}$ as $t \rightarrow \infty$. This means that the infected state $(E_h, I_h E_d, I_d)$ of the two populations tends to zero as time goes to infinity. This means region Ω is pulling(attracting) all the solution in $R^8{}_+$.

Hence, the model system (4.4.1) is mathematically well-posed, biologically and epidemiologically correct.

C. POSITIVITY OF SOLUTION

Let the initial data be $\{S_h(0), E_h(0), I_h(0), R_h(0), S_d(0), E_d(0), I_d(0), V_d(0) \geq 0\} \in \Omega$. Then the solution set $(S_h, E_h, I_h, R_h, S_d, E_d, I_d, V_d)$ of the system is positive for all $t > 0$.

Proof:
From the model equation, we have
$\frac{dS_h}{dt} = \Lambda_h + \omega_h R_h - (\beta_{h1} S_h I_h + \beta_{h2} S_h I_d) - \mu_h S_h \geq -(\beta_{h1} I_h + \beta_{h2} I_d + \mu_h) S_h$

$\frac{dS_h}{dt} \geq -(\beta_{h1} I_h + \beta_{h2} I_d + \mu_h) S_h$

$\frac{dS_h}{S_h} \geq -(\beta_{h1} I_h + \beta_{h2} I_d + \mu_h)\, dt$

$\int \frac{dS_h}{S_h} \geq -\int (\beta_{h1} I_h + \beta_{h2} I_d + \mu_h)\, dt$

$\text{In} S_h \geq -(\beta_{h1} I_h + \beta_{h2} I_d + \mu_h) t + C_1$

$S_h \geq e^{-(\beta_{h1} I_h + \beta_{h2} I_d + \mu_h) t} . e^{C_1}$ where $e^{C_1} = A$

$S_h \geq A e^{-(\beta_{h1} I_h + \beta_{h2} I_d + \mu_h) t}$

At t = 0, $S_h(0) = S_{h_0} \geq A$

$S_h \geq S_{h_0}\, e^{-(\beta_{h1} I_h + \beta_{h2} I_d + \mu_h) t} \geq 0$

Similarly,

$E_h \geq E_{h_0}\, e^{-(\varphi_h + \rho_h + \mu_h) t} \geq 0$, $I_h \geq I_{h_0}\, e^{-(\mu_h + \sigma_h) t} \geq 0$, $R_h \geq R_{h_0}\, e^{-(\omega_h + \mu_h) t} \geq 0$,
$S_d \geq S_{d_0}\, e^{-(\beta_d I_d + \varphi_d + \mu_d) t} \geq 0$, $E_d \geq E_{d_0}\, e^{-(\rho_d + \mu_d) t} \geq 0$, $I_d \geq I_{d_0}\, e^{-(\mu_d + \sigma_d) t} \geq 0$ and
$V_d \geq V_{d_0}\, e^{-(\omega_d + \mu_d) t} \geq 0$.

Hence, from the above solutions, it is clear that the model solution is invariant and has positive results.

D. Basic Reproduction Number (R_O)

The basic reproduction number is an important threshold parameter that determines whether an infectious disease can invade the population.
To calculate R_O we use the next generation matrix approach (Diekmann et al,, 1990).

$$f_i = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} (\beta_{h1} S_h I_h + \beta_{h2} S_h I_d) \\ 0 \\ \beta_d S_d I_d \\ 0 \end{bmatrix}$$

$$v_i = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} (\varphi_h + \rho_h + \mu_h) E_h \\ (\mu_h + \sigma_h) I_h - \rho_h E_h \\ (\mu_d + \rho_d)\, E_d \\ (\mu_d + \sigma_d) I_d - \rho_d E_d \end{bmatrix}$$

$$\text{F} = \begin{pmatrix} 0 & \beta_{h1} S_h & 0 & \beta_{h2} S_h \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_d S_d \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{V} = \begin{pmatrix} (\varphi_h + \rho_h + \mu_h) & 0 & 0 & 0 \\ -\rho_h & (\mu_h + \sigma_h) & 0 & 0 \\ o & 0 & (\mu_d + \rho_d) & 0 \\ 0 & 0 & -\rho_d & (\mu_d + \sigma_d) \end{pmatrix}$$

$$V^{-1} = \begin{pmatrix} \frac{1}{(\varphi_h + \rho_h + \mu_h)} & 0 & 0 & 0 \\ \frac{\rho_h}{(\varphi_h + \rho_h + \mu_h)(\mu_h + \sigma_h)} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{(\mu_d + \rho_d)} & 0 \\ 0 & 0 & \frac{\rho_d}{(\mu_d + \rho_d)(\mu_d + \sigma_d)} & \frac{1}{(\mu_d + \sigma_d)} \end{pmatrix}$$

$$F\,V^{-1} = \begin{pmatrix} \frac{\beta_{h1} S_h \rho_h}{(\varphi_h + \rho_h + \mu_h)(\mu_h + \sigma_h)} & 0 & \frac{\beta_{h2} S_h \rho_d}{(\mu_d + \rho_d)(\mu_d + \sigma_d)} & \frac{\beta_{h2} S_h}{(\mu_d + \sigma_d)} \\ 0 & 0 & o & 0 \\ 0 & 0 & \frac{\beta_d S_d \rho_d}{(\mu_d + \rho_d)(\mu_d + \sigma_d)} & \frac{\beta_d S_d}{(\mu_d + \sigma_d)} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We then solve for the eigen values

Therefore

$$R_0 = \frac{\left(\frac{\beta_{h1} S_h \rho_h}{(\varphi_h + \rho_h + \mu_h)(\mu_h + \sigma_h)} + \frac{\beta_d S_d \rho_d}{(\mu_d + \rho_d)(\mu_d + \sigma_d)}\right) + \sqrt{\left(\frac{\beta_{h1} S_h \rho_h}{(\varphi_h + \rho_h + \mu_h)(\mu_h + \sigma_h)} + \frac{\beta_d S_d \rho_d}{(\mu_d + \rho_d)(\mu_d + \sigma_d)}\right)^2 - 4\left(\frac{\beta_{h1} S_h \rho_h}{(\varphi_h + \rho_h + \mu_h)(\mu_h + \sigma_h)}\right)\left(\frac{\beta_d S_d \rho_d}{(\mu_d + \rho_d)(\mu_d + \sigma_d)}\right)}}{2}$$

E. Disease-free Equilibrium (DFE).

To find the disease-free equilibrium point, we set the right-hand side of equations of system (4.4) equal to zero. In the absence of rabies virus, all other compartments will be zero except the susceptible.

$\Lambda_h + \omega_h R_h - (\beta_{h1} S_h I_h + \beta_{h2} S_h I_d) - \mu_h S_h = 0$
$(\beta_{h1} S_h I_h + \beta_{h2} S_h I_d) - (\varphi_h + \rho_h + \mu_h) E_h = 0$
$\rho_h E_h - (\mu_h + \sigma_h) I_h = 0$
$\varphi_h E_h - (\omega_h + \mu_h) R_h = 0$ (4.7)
$\Lambda_d - \beta_d S_d I_d + \omega_d V_d - (\varphi_d + \mu_d) S_d = 0$
$\beta_d S_d I_d - (\rho_d + \mu_d) E_d = 0$
$\rho_d E_d - (\mu_d + \sigma_d) I_d = 0$
$\varphi_d S_d - (\omega_d + \mu_d) V_d = 0$

$(S_h^0, E_h^0, I_h^0, R_h^+, S_d^0, E_d^0, I_d^0, V_d^0)$ is the steady state of (4.7) which can be obtained by solving (4.7).

The diseased classes for human and dog can be defined as E_h,I_h,E_d, I_d.

In the absence of disease- or disease-free state, it implies that (E_h= I_h=E_d = I_d = 0)

and so (4.7) becomes $\Lambda_h - \mu_h N_h$ where N_h= $S_h + E_h + I_h + R_h$

but $E_h = I_h = 0$

Therefore equation (4.7) reduced to

$\Lambda_h - \mu_h {S^0}_h = 0$
$\Lambda_d - \mu_d {S^0}_d = 0$

Which implies that

$${S^0}_h = \frac{\Lambda_h}{\mu_h}$$
$${S^0}_d = \frac{\Lambda_d}{\mu_d}$$
$${V^0}_d = \frac{\varphi_d \Lambda_d}{\mu_d(\mu_d + \omega_d)}$$

Then the disease-free equilibrium (DFE) ε_o will be

$\varepsilon_o = ({S^0}_h, 0,0,0, {S^0}_d, 0,0, {V^0}_d)$

For the dog population, in the case of disease-free equilibrium points V_d cannot be zero because susceptible dogs which are vaccinated transferred to Vaccinated class. Therefore, the disease free equilibrium points of system (4.2) exist and its given as

$\varepsilon_o = (\frac{\Lambda_h}{\mu_h}, 0,0,0, \frac{\Lambda_d}{\mu_d}, 0,0, \frac{\varphi_d \Lambda_d}{\mu_d(\mu_d + \omega_d)})$.

F. Endemic Equilibrium

The endemic equilibrium point is the point where disease cannot be totally eradicated but remains in the population. Here, it is assumed that there is rabies in the population. And for rabies to persist in the population,

the eight classes most not be zero at equilibrium state. This implies that $S_h(t) \neq 0, E_h(t) \neq 0, I_h(t) \neq 0, R_h(t) \neq 0, S_d(t) \neq 0, E_d(t) \neq 0, I_d(t) \neq 0, V_d(t) \neq 0$. In other words , if $E_+ = (S_h^*, E_h^*, I_h^*, S_d^*, R_d^*, E_d^*, I_d^*, V_d^*) \neq (0,0,0,0,0,0,0,0)$. In order to obtain the endemic equilibrium point, we solve the system of equations (4.4.1) simultaneously.

$$S_d^* = \frac{(\mu_d + \sigma_d)(\mu_d + \rho_d)}{\beta_d \rho_d} \quad ---------- (4.8.1)$$

$$E_d^* = \frac{(\mu_d + \sigma_d) I^* d}{\rho_d} \quad ---------- (4.8.2)$$

$$I_d^* = \frac{\Lambda_d \rho_d}{(\mu_d + \sigma_d)(\mu_d + \rho_d)} + \frac{\omega_d \varphi_d}{\beta_d(\mu_d + \omega_d)} - \frac{(\mu_d + \varphi_d)}{\beta_d} \quad ---------- (4.8.3)$$

$$V_d^* = \frac{\varphi_d(\mu_d + \sigma_d)(\mu_d + \rho_d)}{\beta_d \rho_d(\mu_d + \omega_d)} \quad ---------- (4.8.4)$$

$$S_h^* = \frac{\Lambda_h + \omega_h R_h^*}{\beta_{h1} I_h^* + \beta_{h2} I_d^* + \mu_h} \quad ---------- (4.8.5)$$

$$E_h^* = \frac{(\mu_h + \sigma_h) I^* h}{\rho_h} \quad ---------- (4.8.6)$$

$$I_h^* = \frac{-B_1 \pm \sqrt{B_1^2 + 4A_1 C_1}}{2A_1} \quad ---------- (4.8.7)$$

$$R_h^* = \frac{\varphi_h(\mu_h + \sigma_h) I^* h}{\rho_h(\omega_h + \mu_h)} \quad ---------- (4.8.8)$$

Where

$A_1 = [(\varphi_h + \rho_h + \mu_h)(\mu_h + \sigma_h)\rho_h \beta_{h1}(\omega_h + \mu_h) - \rho_h \beta_{h1} \omega_h \varphi_h(\mu_h + \sigma_h)]$

$B_1 = [\rho_h(\varphi_h + \rho_h + \mu_h)(\mu_h + \sigma_h)(\omega_h + \mu_h)(\beta_{h2} I_d^* + \mu_h) - \beta_{h1} \rho_h^2 \Lambda_h (\omega_h + \mu_h) - \beta_{h2} I_d^* \rho_h \omega_h \varphi_h(\mu_h + \sigma_h)]$

$C_1 = -\beta_{h2} I_d^* \rho_h^2 \Lambda_h (\omega_h + \mu_h)$

G. Stability Analysis of Disease-free Equilibrium

The disease-free equilibrium of the model system (4.4.1) is locally asymptotically stable if $R_O < 1$ and unstable if $R_O > 1$.

Thus, it implies that rabies can be eliminated from the community (when $R_O < 1$). By local stability, we mean the disease may or may not exist depending on the initial population/condition provided, the initial population lies within the neighborhood of the DFE. i.e if and only if $R_O < 1$.

We would verify LAS using Jacobian matrix for the model equation given by

Let

$M_1 = (\beta_{h1} I_h + \beta_{h2} I_d)$, $M_2 = \beta_d I_d$, $K_1 = (\varphi_h + \rho_h + \mu_h)$, $K_2 = (\mu_h + \sigma_h)$, $K_3 = (\omega_h + \mu_h)$, $K_4 = (\mu_h + \varphi_h)$, $K_5 = (\mu_d + \rho_d)$, $K_6 = (\mu_d + \sigma_d)$ and $K_7 = (\mu_d + \omega_d)$

$$J = \begin{bmatrix} -M_1 - \mu_h & 0 & -\beta_{h1} S_h & \omega_h & 0 & 0 & -\beta_{h2} S_h & 0 \\ M_1 & -K_1 & \beta_{h1} S_h & 0 & 0 & 0 & \beta_{h2} S_h & 0 \\ 0 & \rho_h & -K_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \varphi_h & 0 & -K_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -(M_3 + K_4) & 0 & -\beta_d S_d & \omega_d \\ 0 & 0 & 0 & 0 & M_2 & -K_5 & \beta_d S_d & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_d & -K_6 & 0 \\ 0 & 0 & 0 & 0 & \varphi_d & 0 & 0 & -K_7 \end{bmatrix}$$

Evaluating J at DFE (J^0), the characteristics equation is given by

$\Rightarrow -\mu_h - \lambda(-K_3 -)(-K_1 - \lambda)(-K_2 - \lambda)(-K_4 - \lambda)(-K_7 - \lambda)(-K_5 - \lambda)(-K_6 - \lambda)\{[1 - R_O] - \varphi_d \omega_d [1 - R_O]\} = 0$

If $R_O < 1$ then,

$\lambda_1 = -\mu_h$

$\lambda_2 = -K_1$

$\lambda_3 = -K_2$

$\lambda_4 = -K_3$

$\lambda_5 = -K_4$

$\lambda_6 = -K_5$

$\lambda_7 = -K_6$
$\lambda_8 = -K_7$
Since all the eigen values of the Jacobian matrix are negative, its implies that the disease free equilibrium is locally asymptotically stable if $R_O < 1$ and unstable if $R_O > 1$.

H. Stability Analysis of Endemic Equilibrium
Based on the parameters obtained in section (4.6), we have:
Let
$A_1 = (\beta_{h1} I_h^* + \beta_{h2} I_d^*) + \mu_h$
$A_2 = \beta_{h1} S_h^*$
$A_3 = \beta_{h2} S_h^*$
$A_4 = (\beta_{h1} S_h^* I_h^* + \beta_{h2} S_h^* I_d^*)$
$A_5 = (\varphi_h + \rho_h + \mu_h)$
$A_6 = \beta_{h1} S_h^*$
$A_7 = \beta_{h2} S_h^*$
$A_8 = (\mu_h + \sigma_h)$
$A_9 = (\omega_h + \mu_h)$
$A_{10} = (\beta_d I_d^* + \mu_h + \varphi_h)$
$A_{11} = \beta_d I_d^*$
$A_{12} = \beta_d I_d^*$
$A_{13} = (\mu_d + \rho_d)$
$A_{14} = \beta_d S_d^*$
$A_{15} = (\mu_d + \sigma_d)$
$A_{16} = (\mu_d + \omega_d)$
And

$$S_d^* = = \frac{(\mu_d + \sigma_d)(\mu_d + \rho_d)}{\beta_d \rho_d}$$

$$I_d^* = = \frac{\Lambda_d \rho_d}{(\mu_d + \sigma_d)(\mu_d + \rho_d)} + \frac{\omega_d \varphi_d}{\beta_d (\mu_d + \omega_d)} - \frac{(\mu_d + \varphi_d)}{\beta_d}$$

$$S_h^* = \frac{\Lambda_h + \omega_h R_h^*}{\beta_{h1} I_h^* + \beta_{h2} I_d^* + \mu_h}$$

$$I_h^* = \frac{-B_1 \pm \sqrt{B_1^2 + 4 A_1 C_1}}{2 A_1}$$

$$J(E_+) = \begin{pmatrix} -A_1 & 0 & -A_2 & \omega_h & 0 & 0 & -A_3 & 0 \\ A_4 & -A_5 & A_6 & 0 & 0 & 0 & A_7 & 0 \\ 0 & \rho_h & -A_8 & 0 & 0 & 0 & 0 & 0 \\ 0 & \varphi_h & 0 & -A_9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_{10} & 0 & -A_{11} & \omega_d \\ 0 & 0 & 0 & 0 & A_{12} & -A_{13} & A_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_d & -A_{15} & 0 \\ 0 & 0 & 0 & 0 & \varphi_d & 0 & 0 & -A_{16} \end{pmatrix}$$

$$|J(E_+) - \lambda I| = \begin{bmatrix} -A_1 - \lambda & 0 & -A_2 & \omega_h & 0 & 0 & -A_3 & 0 \\ A_4 & -A_5 - \lambda & A_6 & 0 & 0 & 0 & A_7 & 0 \\ 0 & \rho_h & -A_8 - \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \varphi_h & 0 & -A_9 - \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -A_{10} - \lambda & 0 & -A_{11} & \omega_d \\ 0 & 0 & 0 & 0 & A_{12} & -A_{13} - \lambda & A_{14} & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_d & -A_{15} - \lambda & 0 \\ 0 & 0 & 0 & 0 & \varphi_d & 0 & 0 & -A_{16} - \lambda \end{bmatrix}$$

Through computation, we derive the characteristic polynomial to be
$\lambda^8 + D_7\lambda^7 + D_6\lambda^6 + D_5\lambda^5 + D_4\lambda^4 + D_3\lambda^3 + D_2\lambda^2 + D_1\lambda + D_0$ ---------- (4.11)
Where
$D_7 = A_9 + A_5 + A_1 + [(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d) + A_{13} + A_{15} + A_8]$

$\Rightarrow D_7 > 0$

$D_6 = A_9[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d) + A_{13} + A_{15} + A_8] + A_5[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d) + A_{13} + A_{15} + A_8] + [(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)(A_{13} + A_{15}) + A_8) + \mu_d\varphi_d(\beta_d I_d^* + \mu_d)(\mu_d + \omega_d) + A_{13}A_{15} + A_8A_{13} + A_8A_{15}] + A_1[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d) + A_{13} + A_{15} + A_8] + A_5A_9 + A_1A_9 + A_1A_5 - \rho_d A_{11}$

$\Rightarrow D_6 > 0$

$D_5 = A_5A_9[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d) + A_{13} + A_{15} + A_8] + A_9[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)(A_{13} + A_{15}) + A_8) + \mu_d\varphi_d(\beta_d I_d^* + \mu_d)(\mu_d + \omega_d) + A_{13}A_{15} + A_8A_{13} + A_8A_{15}] + A_1A_9[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d) + A_{13} + A_{15} + A_8] + A_5[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)(A_{13} + A_{15}) + A_8) + \mu_d\varphi_d(\beta_d I_d^* + \mu_d)(\mu_d + \omega_d) + A_{13}A_{15} + A_8A_{13} + A_8A_{15}] + A_1A_5[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d) + A_{13} + A_{15} + A_8] + [\mu_d\varphi_d(A_{13} + A_{15} + A_8) + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)(A_{13} + A_{15} + A_8) + (\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)(A_{13}A_{15} + A_8A_{13} + A_8A_{15}) + A_8A_{13}A_{15}] + A_1[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)(A_{13} + A_{15}) + A_8) + \mu_d\varphi_d(\beta_d I_d^* + \mu_d)(\mu_d + \omega_d) + A_{13}A_{15} + A_8A_{13} + A_8A_{15}] + A_1A_5A_9 + A_2A_4\rho_h - \rho_d A_{11}(A_5 + A_8 + A_9 + A_{16} + (\mu_d + \varphi_d) + A_1)$

$\Rightarrow D_5 > 0$

$D_4 = A_5A_9[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)(A_{13} + A_{15}) + A_8) + \mu_d\varphi_d(\beta_d I_d^* + \mu_d)(\mu_d + \omega_d) + A_{13}A_{15} + A_8A_{13} + A_8A_{15}] + A_1A_5A_9[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d) + A_{13} + A_{15} + A_8] + A_9[\mu_d\varphi_d(A_{13} + A_{15} + A_8) + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)(A_{13} + A_{15} + A_8) + (\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)(A_{13}A_{15} + A_8A_{13} + A_8A_{15}) + A_8A_{13}A_{15}] + A_1A_9[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)(A_{13} + A_{15}) + A_8) + \mu_d\varphi_d(\beta_d I_d^* + \mu_d)(\mu_d + \omega_d) + A_{13}A_{15} + A_8A_{13} + A_8A_{15}] + A_5[\mu_d\varphi_d(A_{13} + A_{15} + A_8) + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)(A_{13} + A_{15} + A_8) + (\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)(A_{13}A_{15} + A_8A_{13} + A_8A_{15}) + A_8A_{13}A_{15}] + A_1A_5[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)(A_{13} + A_{15}) + A_8) + \mu_d\varphi_d(\beta_d I_d^* + \mu_d)(\mu_d + \omega_d) + A_{13}A_{15} + A_8A_{13} + A_8A_{15}] + [\mu_d\varphi_d(A_{13}A_{15} + A_8A_{13} + A_8A_{15}) + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)(A_{13} + A_{15} + A_8) + (\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)(A_{13}A_{15} + A_8A_{13} + A_8A_{15}) + A_8A_{13}A_{15}] + [(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d) + A_{15} + A_{13} + A_9] A_2A_4\rho_h - \rho_d A_{11}(A_1A_5 + A_1A_8 + A_1A_9 + A_1A_{16} + A_1b + A_5A_8 + A_5A_9 + A_5A_{16} + A_5b + A_8A_9 + A_8A_{16} + A_8b + A_9A_{16} + A_9b + A_{16}b) + \rho_d\omega_d\varphi_d A_{14}$

$\Rightarrow D_4 > 0$

$D_3 = A_5A_9[\mu_d\varphi_d(A_{13} + A_{15} + A_8) + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)(A_{13} + A_{15} + A_8) + (\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)(A_{13}A_{15} + A_8A_{13} + A_8A_{15}) + A_8A_{13}A_{15}] + A_1A_5A_9[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)(A_{13} + A_{15}) + A_8) + \mu_d\varphi_d(\beta_d I_d^* + \mu_d)(\mu_d + \omega_d) + A_{13}A_{15} + A_8A_{13} + A_8A_{15}] + A_9[\mu_d\varphi_d(A_{13}A_{15} + A_8A_{13} + A_8A_5) + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d) + A_{13}A_{15} + A_8A_{13} + A_8A_{15}) + A_{13}A_{15}(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)] + A_1A_9[\mu_d\varphi_d(A_{13} + A_{15} + A_8) + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)(A_{13} + A_{15} + A_8) + (\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)(A_{13}A_{15} + A_8A_{13} + A_8A_{15}) + A_8A_{13}A_{15}] + A_5[\mu_d\varphi_d(A_{13}A_{15} + A_8A_{13} + A_8A_5) + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d) + A_{13}A_{15} + A_8A_{13} + A_8A_{15}) + A_{13}A_{15}(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)] + A_1A_5[\mu_d\varphi_d(A_{13} + A_{15} + A_8) + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)(A_{13} + A_{15} + A_8) + (\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)(A_{13}A_{15} + A_8A_{13} + A_8A_{15}) + A_8A_{13}A_{15}] + A_8A_{13}A_{15}[\mu_d\varphi_d + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)] + A_1[\mu_d\varphi_d(A_{13}A_{15} + A_8A_{13} + A_8A_5) + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d) + A_{13}A_{15} + A_8A_{13} + A_8A_{15}) + A_{13}A_{15}(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)] + [A_9A_{13} + A_9A_{15} + A_9[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)] + A_{13}A_{15} + A_{13}[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)] + A_{15}[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)] + \mu_d\varphi_d + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)] A_2A_4\rho_h - \rho_d A_{11}[A_5A_8A_9 + A_5A_8A_{16} + A_5A_8(\mu_d + \varphi_d) + A_5A_9A_{16} + A_8A_9A_{16} + A_5A_9(\mu_d + \varphi_d) + A_5A_{16}(\mu_d + \varphi_d) + A_8A_9(\mu_d + \varphi_d) + A_8A_{16}(\mu_d + \varphi_d) + A_9A_{16}(\mu_d + \varphi_d) + A_1A_5A_8 + A_1A_5A_9 + A_1A_5A_{16} + A_1A_5(\mu_d + \varphi_d) + A_1A_8A_9 + A_1A_8A_{16} + A_1A_8(\mu_d + \varphi_d) + A_1A_9A_{16} + A_1A_9(\mu_d + \varphi_d) + A_1A_{16}(\mu_d + \varphi_d)] + \rho_d\omega_d\varphi_d A_{14}(A_1 + A_5 + A_8 + A_9) + A_2A_4A_{14}\rho_d\rho_h$

$\Rightarrow D_3 > 0$

$D_2 = A_5A_9[\mu_d\varphi_d(A_{13}A_{15} + A_8A_{13} + A_8A_5) + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d) + A_{13}A_{15} + A_8A_{13} + A_8A_{15}) + A_{13}A_{15}(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)] + A_1A_5A_9[\mu_d\varphi_d(A_{13} + A_{15} + A_8) + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)(A_{13} + A_{15} + A_8) + (\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)(A_{13}A_{15} + A_8A_{13} + A_8A_{15}) + A_8A_{13}A_{15}] + A_9A_8A_{13}A_{15}[\mu_d\varphi_d + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)] + A_1A_9[\mu_d\varphi_d(A_{13}A_{15} + A_8A_{13} + A_8A_5) + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d) + A_{13}A_{15} + A_8A_{13} + A_8A_{15}) + A_{13}A_{15}(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)] + A_5A_8A_{13}A_{15}[\mu_d\varphi_d + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)] + A_1A_5[\mu_d\varphi_d(A_{13}A_{15} + A_8A_{13} + A_8A_5) + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d) + A_{13}A_{15} + A_8A_{13} + A_8A_{15}) + A_{13}A_{15}(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)] + A_1A_8A_{13}A_{15}[\mu_d\varphi_d + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)] + A_1A_5 + (A_9A_{13}A_{15} + A_9A_{13}[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)] + A_9A_{15}[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)] + A_9[\mu_d\varphi_d + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)] + A_{13}A_{15}[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)] + A_{13}[\mu_d\varphi_d + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)] + A_{15}[\mu_d\varphi_d + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)]A_2A_4\rho_h - \rho_d A_{11}[A_5A_8A_9A_{16} + A_5A_8A_9(\mu_d + \varphi_d) + A_5A_8A_{16}(\mu_d + \varphi_d) + A_5A_9A_{16}(\mu_d + \varphi_d) + A_8A_9A_{16}(\mu_d + \varphi_d) + A_1A_5A_8A_9 + A_1A_5A_8A_{16} + A_1A_5A_8(\mu_d + \varphi_d) + A_1A_5A_9A_{16} + A_1A_5A_9(\mu_d + \varphi_d) + A_1A_5A_{16}(\mu_d + \varphi_d)] + \rho_d\omega_d\varphi_d A_{14}(A_1A_9 + A_1A_5 + A_1A_8 + A_5A_9 + A_8A_9 + A_5A_8) + A_2A_4A_{14}\rho_d\rho_h(A_9 + A_{10} + A_{16})$

$\Rightarrow D_2 > 0$

$D_1 = A_5A_9A_8A_{13}A_{15}[\mu_d\varphi_d + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)] + A_1A_5A_9[\mu_d\varphi_d(A_{13}A_{15} + A_8A_{13} + A_8A_5) + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d) + A_{13}A_{15} + A_8A_{13} + A_8A_{15}) + A_{13}A_{15}(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)] + A_1A_9A_8A_{13}A_{15}[\mu_d\varphi_d + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)] + A_1A_5A_8A_{13}A_{15}[\mu_d\varphi_d + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)] + (A_9A_{13}A_{15}[(\beta_d I_d^* + 2\mu_d + \varphi_d + \omega_d)] + A_9A_{13}[\mu_d\varphi_d + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)] + A_9A_{15}[\mu_d\varphi_d + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)] + A_{13}A_{15}[\mu_d\varphi_d + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)])A_2A_4\rho_h - \rho_d A_{11}[A_5A_8A_9A_{16}(\mu_d + \varphi_d) + A_1A_5A_8A_9A_{16} + A_1A_5A_8A_9(\mu_d + \varphi_d) + A_1A_5A_8A_{16}(\mu_d + \varphi_d) + A_1A_5A_9A_{16}(\mu_d + \varphi_d) + A_1A_8A_9A_{16}(\mu_d + \varphi_d)] + \rho_d\omega_d\varphi_d A_{14}(A_1A_5A_9 + A_1A_8A_9 + A_1A_5A_8 + A_1A_8A_9) + A_2A_4A_{14}\rho_d\rho_h(A_9A_{10} + A_9A_{16} + A_1A_{16}) + A_2A_4A_{14}\rho_d\rho_h\omega_d\varphi_d$

$\Rightarrow D_1 > 0$

$D_0 = A_1A_5A_9A_8A_{13}A_{15}[\mu_d\varphi_d + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)] + A_9A_{13}A_{15}[\mu_d\varphi_d + (\beta_d I_d^* + \mu_d)(\mu_d + \omega_d)] - \rho_d A_1A_5A_8A_9A_{11}A_{16}(\mu_d + \varphi_d) + A_1A_5A_8A_9A_{14}\rho_d\omega_d\varphi_d + A_2A_4A_9A_{10}A_{14}A_{16}\rho_d\rho_h + A_2A_4A_9A_{14}\rho_d\rho_h\omega_d\varphi_d$

$\Rightarrow D_0 > 0$

Hence, we determine the signs of the eigenvalues of (4.11) by using Descartes' rules of signs theorem which state

Let P(x) = $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial with real coefficient.

1. The number of positive zeros of P is either equal to the number of variations in sign of P(x) or less than this by an even number.
2. The number of negative zeros of P is either equal to then number of variations in sign of P($-x$) or less than this by an even number.

Considering the characteristics polynomial of (4.11)

Let $P(x) = \lambda^8 + D_7\lambda^7 + D_6\lambda^6 + D_5\lambda^5 + D_4\lambda^4 + D_3\lambda^3 + D_2\lambda^2 + D_1\lambda + D_0$

$\Rightarrow P(-x) = \lambda^8 - D_7\lambda^7 + D_6\lambda^6 - D_5\lambda^5 + D_4\lambda^4 - D_3\lambda^3 + D_2\lambda^2 - D_1\lambda + D_0$

The signs of the coefficients of P(x) in the descending powers of λ are

+ + + + + + + + +

Obviously, there is no change in sign.

By Descartes's Rules of signs (4.11) will have no positive eigenvalue.

Also, the signs of the coefficients of P($-x$) in the descending powers of λ are

+ − + − + − + − +

Clearly, there are eight (8) change in signs.

Since all the coefficients of the polynomial and all parameters are positive, by Descartes' rule of signs, all the roots are negative.

Hence Endemic Equilibrium Point is locally asymptotically stable.

III. ANALYTICAL SOLUTION OF EQUATION (4.4.1) USING HOMOTOPY PERTURBATION METHOD

The system of equation

$$\frac{dS_h}{dt} = \Lambda_h + \omega_h R_h - (\beta_{h1} S_h I_h + \beta_{h2} S_h I_d) - \mu_h S_h$$

$$\frac{dE_h}{dt} = (\beta_{h1} S_h I_h + \beta_{h2} S_h I_d) - (\varphi_h + \rho_h + \mu_h) E_h$$

$$\frac{dI_h}{dt} = \rho_h E_h - (\mu_h + \sigma_h) I_h$$

$$\frac{dR_h}{dt} = \varphi_h E_h - (\omega_h + \mu_h) R_h$$

$$\frac{dS_d}{dt} = \Lambda_d - \beta_d S_d I_d + \omega_d V_d - (\varphi_d + \mu_d) S_d \qquad (4.11.1)$$

$$\frac{dE_d}{dt} = \beta_d S_d I_d - (\rho_d + \mu_d) E_d$$

$$\frac{dI_d}{dt} = \rho_d E_d - (\mu_d + \sigma_d) I_d$$

$$\frac{dR_d}{dt} = \varphi_d S_d - (\omega_d + \mu_d) V_d$$

Conditions for the differential equation above:

$S_h(0) = C_1, E_h(0) = C_2, I_h(0) = C_3, R_h(0) = C_4,\ S_d(0) = C_5, E_d(0) = C_6, I_d(0) = C_7,\ R_d(0) = C_8.$

We construct an homotopy:

$$(1-p)\frac{dS_h}{dt} + p\left[\frac{dS_h}{dt} - \Lambda_h - \omega_h R_h + (\beta_{h1} S_h I_h + \beta_{h2} S_h I_d) + \mu_h S_h\right] = 0$$

$$(1-p)\frac{dE_h}{dt} + p\left[\frac{dE_h}{dt} - (\beta_{h1} S_h I_h + \beta_{h2} S_h I_d) + (\varphi_h + \rho_h + \mu_h) E_h\right] = 0$$

$$(1-p)\frac{dI_h}{dt} + p\left[\frac{dI_h}{dt} - \rho_h E_h + (\mu_h + \sigma_h) I_h\right] = 0$$

$$(1-p)\frac{dR_h}{dt} + p\left[\frac{dR_h}{dt} - \varphi_h E_h + (\omega_h + \mu_h) R_h\right] = 0 \qquad (4.11.2)$$

$$(1-p)\frac{dS_d}{dt} + p\left[\frac{dS_d}{dt} - \Lambda_d + \beta_d S_d I_d - \omega_d V_d + (\varphi_d + \mu_d) S_d\right] = 0$$

$$(1-p)\frac{dE_d}{dt} + p\left[\frac{dE_d}{dt} - \beta_d S_d I_d + (\rho_d + \mu_d) E_d\right] = 0$$

$$(1-p)\frac{dI_d}{dt} + p\left[\frac{dI_d}{dt} - \rho_d E_d + (\mu_d + \sigma_d) I_d\right] = 0$$

$$(1-p)\frac{dV_d}{dt} + p\left[\frac{dV_d}{dt} - \varphi_d S_d + (\omega_d + \mu_d) V_d\right] = 0$$

The equation (2) can be re-written as:

$$\frac{dS_h}{dt} = p\left[\frac{dS_h}{dt} + \Lambda_h + \omega_h R_h - (\beta_{h1} S_h I_h + \beta_{h2} S_h I_d) - \mu_h S_h - \frac{dS_h}{dt}\right]$$

$$\frac{dE_h}{dt} = p\left[\frac{dE_h}{dt} + (\beta_{h1} S_h I_h + \beta_{h2} S_h I_d) - (\varphi_h + \rho_h + \mu_h) E_h - \frac{dE_h}{dt}\right]$$

$$\frac{dI_h}{dt} = p\left[\frac{dI_h}{dt} + \rho_h E_h - (\mu_h + \sigma_h) I_h - \frac{dI_h}{dt}\right] \qquad (4.11.3)$$

$$\frac{dR_h}{dt} = p\left[\frac{dR_h}{dt} + \varphi_h E_h - (\omega_h + \mu_h) R_h - \frac{dR_h}{dt}\right]$$

$$\frac{dS_d}{dt} = p\left[\frac{dS_d}{dt} + \Lambda_d - \beta_d S_d I_d + \omega_d V_d - (\varphi_d + \mu_d) S_d - \frac{dS_d}{dt}\right]$$

$$\frac{dE_d}{dt} = p\left[\frac{dE_d}{dt} + \beta_d S_d I_d - (\rho_d + \mu_d) E_d - \frac{dE_d}{dt}\right]$$

$$\frac{dI_d}{dt} = p\left[\frac{dI_d}{dt} + \rho_d E_d - (\mu_d + \sigma_d) I_d - \frac{dI_d}{dt}\right]$$

$$\frac{dV_d}{dt} = p\left[\frac{dV_d}{dt} + \varphi_d S_d - (\omega_d + \mu_d) V_d - \frac{dV_d}{dt}\right]$$

The basic assumption is that the solution of equation (1) can be written as a series in powers of p.

$$S_h = S_{ho} + pS_{h1} + p^2 S_{h2} + p^3 S_{h3} + \ldots$$

$$E_h = E_{ho} + pE_{h1} + p^2 E_{h2} + p^3 E_{h3} + \ldots$$

$$I_h = I_{ho} + pI_{h1} + p^2 I_{h2} + p^3 II_{h3} + \ldots$$

$$R_h = R_{ho} + pR_{h1} + p^2 R_{h2} + p^3 R_{h3} + \ldots$$

$$S_d = S_{do} + pS_{d1} + p^2 S_{d2} + p^3 S_{d3} + \ldots \qquad (4.11.4)$$

$$E_d = E_{do} + pE_{d1} + p^2 E_{d2} + p^3 E_{d3} + \ldots$$

$I_d = I_{do} + pI_{d1} + p^2I_{d2} + p^3I_{d3} + \ldots$

$V_d = V_{do} + pV_{d1} + p^2V_{d2} + p^3V_{d3} + \ldots$

If $p \to 1$, we recall the solution of the series

Substituting equation (4) into equation (3), we will have;

$$\frac{d(S_{ho} + pS_{h1} + p^2S_{h2} + \cdots)}{dt} = p[\Lambda_h + \omega_h(R_{ho} + pR_{h1} + p^2R_{h2} + \cdots) - B_{h1}(S_{ho} + pS_{h1} + p^2S_{h2} + \cdots)(I_{ho} + pI_{h1} + p^2I_{h2} + \cdots) - \beta_{h2}(S_{ho} + pS_{h1} + p^2S_{h2} + \cdots)(I_{do} + pI_{d1} + p^2I_{d2} + \cdots) - \mu_h(S_{ho} + pS_{h1} + p^2S_{h2} + \cdots)]$$

$$\frac{d(E_{ho} + pE_{h1} + p^2E_{h2} + \cdots)}{dt} = p[\beta_{h1}(S_{ho} + pS_{h1} + p^2S_{h2} + \cdots)(I_{ho} + pI_{h1} + p^2I_{h2} + \cdots) + \beta_{h2}(S_{ho} + pS_{h1} + p^2S_{h2} + \cdots)(I_{do} + pI_{d1} + p^2I_{d2} + \cdots) - (\varphi_h + \rho_h + \mu_h)(E_{ho} + pE_{h1} + P^2E_{h2} + \cdots)]$$

$$\frac{d(I_{ho} + pI_{h1} + p^2E_{h2} + \cdots)}{dt} = p[\rho_h(E_{ho} + pE_{h1} + p^2E_{h2} + \cdots) - (\mu_h + \sigma_h)(I_{ho} + PI_{h1} + p^2I_{h2} + \cdots)]$$

$$\frac{d(R_{ho} + pR_{h1} + p^2R_{h2} + \cdots)}{dt} = p[\mu_h(E_{ho} + pE_{h1} + p^2E_{h2} + \cdots) - (\omega_h + \mu_h)(R_{ho} + pR_{h1} + p^2R_{h2} + \cdots)]$$

$$\frac{d(S_{do} + pS_{d1} + p^2S_{d2} + \cdots)}{dt} = P[\Lambda_d - \beta_d(S_{do} + pS_{d1} + p^2S_{d2} + \cdots)(I_{do} + pI_{d1} + p^2I_{d2} + \cdots)\,\omega_d(R_{do} + pR_{d1} + p^2R_{d2} + \cdots) - (\mu_d + \mu_d)(S_{do} + pS_{d1} + p^2S_{d2} + \cdots)]$$

$$\frac{d(E_{do} + pE_{d1} + p^2E_{d2} + \cdots)}{dt} = p[\beta_d(S_{do} + pS_{d1} + p^2S_{d2} + \cdots)(I_{do} + pI_{d1} + p^2I_{d2} + \cdots) - (\rho_d + \mu_d)(E_{do} + pE_{d1} + p^2E_d + \cdots)]$$

$$\frac{d(I_{do} + pI_{d1} + p^2I_{d2} + \cdots)}{dt} = p[\rho_d(E_{do} + pE_{d1} + p^2E_{d2} + \cdots) - ((\mu_d + \sigma_d))(I_{do} + pI_{d1} + p^2I_{d2} + \cdots)]$$

$$\frac{d(V_{do} + pV_{d1} + p^2V_{d2} + \cdots)}{dt} = p[\mu_d(S_{do} + pS_{d1} + p^2S_{d2} + \cdots) - (\omega_d + \mu_d)(V_{do} + pV_{d1} + p^2V_{d2} + \cdots)]$$

We now compare the identical powers of p as follow:

$$\frac{ds_{ho}}{dt}+\frac{pds_{h1}}{dt}+\frac{p^2ds_{h2}}{dt}+\cdots$$
$$=p[\wedge_h+\omega_h(R_{ho}+pR_{h1}+p^2R_{h2}+\cdots)$$
$$-\beta_{h1}[S_{ho}\,I_{ho}+pS_{ho}\,I_{h1}+p^2S_{ho}\,I_{h2}+pS_{h1}\,I_{ho}+p^2S_{h1}I_{h1}+p^2S_{h2}\,I_{ho}+\cdots]$$
$$-\beta_{h2}(S_{ho}\,I_{do}+pS_{ho}\,I_{d1}+p^2S_{ho}\,I_{d2}+pS_{h1}\,I_{do}+p^2S_{h1}\,I_{d1}+p^2S_{h2}\,I_{do}+\cdots)$$
$$-\mu_h(S_{ho}+pS_{h1}+p^2S_{h2}+\cdots)]$$

$$\frac{dE_{ho}}{dt}+\frac{pdE_{h1}}{dt}+\frac{p^2dE_{h2}}{dt}+\cdots$$
$$=p[\beta_{h1}(S_{ho}\,I_{ho}+pS_{ho}\,I_{h1}+p^2S_{ho}I_{h2}+pS_{h1}\,I_{ho}+p^2S_{h1}\,I_{h1}+p^2S_{h2}\,I_{ho}+\cdots)$$
$$+\beta_{h2}(S_{ho}\,I_{do}+pS_{ho}\,I_{d1}+p^2S_{ho}\,I_{d2}+pS_{h1}\,I_{do}+p^2S_{h1}\,I_{d1}+p^2S_{h2}\,I_{do}+\cdots)$$
$$-(\varphi_h+\rho_h+\mu_h)(E_{ho}+pE_{h1}+p^2E_{h2}+\cdots)]$$

$$\frac{dI_{ho}}{dt}+\frac{pdI_{h1}}{dt}+\frac{p^2dI_{h2}}{dt}+\cdots=p[\rho_h(E_{ho}\ +pE_{h1}\ +p^2E_{h2}+\cdots)-(\mu_h+\sigma_h)(I_{ho}+pI_{h1}+p^2I_{h2}+\cdots)]$$

$$\frac{dR_{ho}}{dt}+\frac{pdR_{h1}}{dt}+\frac{p^2dR_{h2}}{dt}+\cdots=p[\varphi_h(E_{ho}\ +pE_{h1}\ +p^2E_{h2}+\cdots)-(\omega_h+\mu_h)(R_{ho}+pR_{h1}+p^2R_{h2}+\cdots)]$$

$$\frac{dS_{do}}{dt}+\frac{pdS_{d1}}{dt}+\frac{p^2dS_{d2}}{dt}+\cdots$$
$$=p[\Lambda_d-\beta_d(S_{do}I_{do}\ +pS_{do}I_{d1}\ +p^2S_{do}I_{d2}+pS_{d1}I_{do}+p^2S_{d1}I_{d1}+p^2S_{d2}\,I_{do}+\cdots)$$
$$+\omega_d(V_{do}+pV_{d1}+p^2V_{d2}+\cdots)-(\varphi_d+\mu_d)(S_{do}+pS_{d1}+p^2S_{d2}+\cdots)]$$

$$\frac{dE_{do}}{dt}+\frac{pdE_{d1}}{dt}+\frac{p^2dE_{d2}}{dt}+\cdots$$
$$=p[\beta_d(S_{do}I_{do}\ +pS_{do}I_{d1}\ +p^2S_{do}I_{d2}+pS_{d1}I_{do}+p^2S_{d1}I_{d1}+p^2S_{d2}\,I_{do}+\cdots)$$
$$-(\rho_d+\mu_d)(E_{do}+pE_{d1}+p^2E_{d2}+\cdots)]$$

$$\frac{dI_{do}}{dt}+\frac{pdI_{d1}}{dt}+\frac{p^2dI_{d2}}{dt}+\cdots=p[\rho_d(E_{do}\ +pE_{d1}\ +p^2E_{d2}+\cdots)-(\mu_d+\sigma_d)(I_{do}+pI_{d1}+p^2I_{d2}+\cdots)]$$

$$\frac{dV_{do}}{dt}+\frac{pdV_{d1}}{dt}+\frac{p^2dV_{d2}}{dt}+\cdots=p[\mu_d(S_{do}\ +pS_{d1}\ +p^2S_{d2}+\cdots)-(\omega_d+\mu_d)(V_{do}+pV_{d1}+p^2V_{d2}+\cdots)]$$

The identical powers of p can be selected as follows with the initial conditions

$$P^o:\frac{dS_{ho}}{dt}=0,S_{ho}(o)=C_1$$

$$P^o:\frac{dE_{ho}}{dt}=0,E_{ho}(o)=C_2$$

$$P^o:\frac{dI_{ho}}{dt}=0,I_{ho}(o)=C_3$$

$$P^o: \frac{dR_{ho}}{dt} = 0, R_{ho}(o) = C_4$$

$$P^o: \frac{dS_{do}}{dt} = 0, S_{do}(o) = C_5$$

$$P^o: \frac{dE_{do}}{dt} = 0, E_{do}(o) = C_6$$

$$P^o: \frac{dI_{do}}{dt} = 0, I_{do}(o) = C_7$$

$$P^o: \frac{dR_{do}}{dt} = 0, V_{do}(o) = C_8$$

$$P^1: \frac{dS_{h1}}{dt} = \wedge_h + \omega_h R_{ho} - \beta_{h1} S_{ho} I_{ho} - \beta_{h2} S_{ho} I_{do} - \mu_h S_{ho}$$

$$P^1: \frac{dE_{h1}}{dt} = \beta_{h1} S_{ho} I_{ho} + \beta_{h2} S_{ho} I_{do} - (\varphi_h + \rho_h + \mu_h) E_{ho}$$

$$P^1: \frac{dI_{h1}}{dt} = \rho_h E_{ho} - (\mu_h + \sigma_h) I_{ho}$$

$$P^1: \frac{dR_{h1}}{dt} = \varphi_h E_{ho} - (\omega_h + \mu_h) R_{ho}$$

$$P^1: \frac{dS_{d1}}{dt} = \wedge_d - \beta d S_{do} I_{do} + \omega_d R_{do} - (\varphi_d + \mu_d) S_{do}$$

$$P^1: \frac{dE_{d1}}{dt} = \beta_d S_{do} I_{do} - (\rho_d + \mu_d) E_{do}$$

$$P^1: \frac{dI_{do}}{dt} = \rho_d E_{do} - (\mu_d + \sigma_d) I_{do}$$

$$P^1: \frac{dV_{d1}}{dt} = \varphi_d S_{do} - (\omega_d + \mu_d) V_{do}$$

$$P^2: \frac{dS_{h2}}{dt} = \omega_h R_{h1} - \beta_{h1}(S_{ho} I_{h1} + S_{h1} I_{ho}) - \beta_{h2}(S_{ho} I_{d1} + S_{h1} I_{do}) - \mu_h S_{h1}$$

$$P^2: \frac{dE_{h2}}{dt} = \beta_{h1}(S_{ho} I_{h1} + S_{h1} I_{ho}) - \beta_{h2}(S_{ho} I_{d1} + S_{h1} I_{do}) - (\varphi_h + \rho_h + \mu_h) E_{h1}$$

$$P^2: \frac{dI_{h2}}{dt} = \rho_h E_{h1} - (\mu_h + \sigma_h) I_{h1}$$

$$P^2: \frac{dR_{h2}}{dt} = \varphi_h E_{h1} - (\omega_h + \mu_h) R_{h1}$$

$$P^2: \frac{dSd_2}{dt} = -\beta_d (S_{do} I_{d1} + S_{d1} I_{do}) + \omega_d V_{d1} - (\varphi_d + \mu_d) S_{d1}$$

$$P^2: \frac{dEd_2}{dt} = \beta_d (S_{do} I_{d1} + S_{d1} I_{do}) - (\rho_d + \mu_d) E_{d1}$$

$$P^2: \frac{dId_2}{dt} = \rho_d E_{d1} - (\mu_d + \sigma_d) I_{d1}$$

$$P^2: \frac{dVd_2}{dt} = \varphi_d S_{d1} - (\omega_d + \mu_d)V_{d1}$$

Subsequently, the solution of the series can be generated as follows:

For powers of P^1

$$S_{ho}(t) = C_1, E_{ho}(t) = C_2, I_{ho}(t) = C_3, R_{ho}(t) = C_4, S_{do}(t) = C_5, \quad E_{do}(t) = C_6, I_{do}(t) = C_7, V_{do}(t) = C_8.$$

$$\frac{dS_{h1}}{dt} = \Lambda_h + \omega_h C_4 - \beta_{h1} C_9 - \beta_{h2} C_{10} - \mu_h C_1$$

$$\frac{dE_{h1}}{dt} = \beta_{h1} C_{11} + \beta_{h2} C_{12} - (\varphi_h + \rho_h + \mu_h)C_2$$

$$\frac{dI_{h1}}{dt} = \rho_h C_2 - (\mu_h + \sigma_h)C_3$$

$$\frac{dR_{h1}}{dt} = \varphi_h C_2 - (\omega_h + \mu_h)C_4$$

$$\frac{dS_{d1}}{dt} = \Lambda_d - \beta_d C_{13} + \omega_d C_8 - (\varphi_d + \mu_d)C_5$$

$$\frac{dE_{d1}}{dt} = \beta_d C_{14} - (\rho_d + \mu_d)C_6$$

$$\frac{dI_{d1}}{dt} = \rho_d C_6 - (\mu_d + \sigma_d)C_7$$

$$\frac{dV_{d1}}{dt} = \varphi_d C_5 - (\omega_d + \mu_d)C_8$$

Integrating with respect to t

$$S_{h1} = \Lambda_h t + \omega_h C_4 t - \beta_{h1} C_9 t - \beta_{h2} C_{10} t - \mu_h C_1 t$$

$$E_{h1} = \beta_{h1} C_{11} t + \beta_{h2} C_{12} t - (\varphi_h + \rho_h + \mu_h)C_2 t$$

$$I_{h1} = \rho_h C_2 t - (\mu_h + \sigma_h)C_3 t$$

$$R_{h1} = \varphi_h C_2 t - (\omega_h + \mu_h)C_4 t$$

$$S_{d1} = \Lambda_d t - \beta_d C_{13} t + \omega_d C_8 t - (\varphi_h + \mu_h)C_5 t$$

$$E_{d1} = \beta_d C_{14} t - (\rho_d + \mu_d)C_6 t$$

$$I_{d1} = \rho_d C_6 t - (\mu_d + \sigma_d)C_7 t$$

$$V_{d1} = \varphi_d C_5 t - (\omega_d + \mu_d)C_8 t$$

For powers of p to 2 i.e., P^2

$$\begin{aligned}\frac{dS_{h2}}{dt} = {} & \omega_h \varphi_h C_2 t - \omega_h(\omega_h + \mu_h)C_4 t - \beta_{h1}(\rho_h C_2 t - (\mu_h + \sigma_h)C_4 t) \\ & - \beta_{h1}(\Lambda_h C_3 t + \omega_h C_{17} t - \beta_{h1} C_{18} t - \beta_{h2} C_{19} t - \mu_h C_{20} t) - \beta_{h2}(\rho_d C_{21} t - (\mu_d + \sigma_d)C_{22} t) \\ & - \beta_{h2}(\Lambda_h C_7 t + \omega_h C_{23} t - \beta_{h1} C_{24} t - \beta_{h2} C_{25} t - \mu_h C_{26} t) \\ & - \mu_h(\Lambda_h t + \omega_h C_4 t - \beta_{h1} C_9 t - \beta_{h2} C_{10} t - \mu_h C_1 t)\end{aligned}$$

$$S_{h2} = \frac{\omega_h \varphi_h C_2 t^2}{2} - \frac{\omega_h(\omega_h + \mu_h)C_4 t^2}{2} - \beta_{h1}\left(\frac{\rho_h C_{15} t}{2} - \left(\frac{\mu_h + \sigma_h}{2}\right)C_{14} t\right) - \beta_{h1}\left(\frac{\Lambda_h C_3 t^2}{2} + \frac{\omega_h C_{17} t^2}{2} - \frac{\beta_{h1} C_{18} t^2}{2} - \frac{\beta_{h2} C_{19} t^2}{2} - \frac{\mu_h C_{20} t^2}{2}\right) - \beta_{h2}\left(\frac{\rho_d C_{21} t^2}{2} - \frac{(\mu_d + \sigma_d)C_{22} t^2}{2}\right) - \beta_{h2}\left(\frac{\Lambda_h C_7 t^2}{2} + \frac{\omega_h C_{23} t^2}{2} - \frac{\beta_{h1} C_{24} t^2}{2} - \frac{\beta_{h2} C_{25} t^2}{2} - \frac{\mu_h C_{26} t^2}{2}\right) - \mu_h\left(\frac{\Lambda_h t^2}{2} + \frac{\omega_h C_4 t^2}{2} - \frac{\beta_{h1} C_9 t^2}{2} - \frac{\beta_{h2} C_{10} t^2}{2} - \frac{\mu_h C_1 t^2}{2}\right)$$

$$\frac{dE_{h2}}{dt} = \beta_{h1}(\rho_h C_2 t - (\mu_h + \sigma_h)C_4 t) + \beta_{h1}(\Lambda_h C_3 t + \omega_h C_{17} t - \beta_{h1} C_{18} t - \beta_{h2} C_{19} t - \mu_h C_{20} t) + \beta_{h2}(\rho_d C_{21} t - (\mu_d + \sigma_d)C_{22} t) + \beta_{h2}(\Lambda_h C_7 t + \omega_h C_{23} t - \beta_{h1} C_{24} t - \beta_{h2} C_{25} t - \mu_h C_{26} t) - (\varphi_h + \rho_h + \mu_h)(\beta_{h1} C_{11} t + \beta_{h2} C_{12} t - (\varphi_h + \rho_h + \mu_h)C_2 t)$$

$$E_{h2} = \beta_{h1}\left(\frac{\rho_h C_2 t^2}{2} - \frac{(\mu_h + \sigma_h)C_4 t^2}{2}\right) + \beta_{h1}\left(\frac{\Lambda_h C_3 t^2}{2} + \frac{\omega_h C_{17} t^2}{2} - \frac{\beta_{h1} C_{18} t^2}{2} - \frac{\beta_{h2} C_{19} t^2}{2} - \frac{\mu_h C_{20} t^2}{2}\right) + \beta_{h2}\left(\frac{\rho_d C_{21} t^2}{2} - \frac{(\mu_d + \sigma_d)C_{22} t^2}{2}\right) + \beta_{h2}\left(\frac{\Lambda_h C_7 t^2}{2} + \frac{\omega_h C_{23} t^2}{2} - \frac{\beta_{h1} C_{24} t^2}{2} - \frac{\beta_{h2} C_{25} t^2}{2} - \frac{\mu_h C_{26} t^2}{2}\right) - (\varphi_h + \rho_h + \mu_h)\left(\frac{\beta_{h1} C_{11} t^2}{2} + \frac{\beta_{h2} C_{12} t^2}{2} - \left(\frac{\varphi_h + \rho_h + \mu_h}{2}\right)C_2 t^2\right)$$

$$\frac{dI_{h2}}{dt} = \rho_h(\beta_{h1} C_{11} t + \beta_{h2} C_{12} t - (\varphi_h + \rho_h + \mu_h)C_2 t) - (\mu_h + \sigma_h)(\rho_h C_2 t - (\mu_h + \sigma_h)C_3 t)$$

$$I_{h2} = \rho_h\left(\frac{\beta_{h1} C_{11} t^2}{2} + \frac{\beta_{h2} C_{12} t^2}{2} - \left(\frac{\varphi_h + \rho_h + \mu_h}{2}\right)C_2 t^2\right) - (\mu_h + \sigma_h)\left(\frac{\rho_h C_2 t^2}{2} - \left(\frac{\mu_h + \sigma_h}{2}\right)C_3 t^2\right)$$

$$\frac{dR_{h2}}{dt} = \varphi_h(\beta_{h1} C_{11} t + \beta_{h2} C_{12} t - (\varphi_h + \rho_h + \mu_h)C_2 t) - (\omega_h + \mu_h)(\varphi_h C_2 t - (\omega_h + \mu_h)C_4 t)$$

$$R_{h2} = \varphi_h\left(\frac{\beta_{h1} C_{11} t^2}{2} + \frac{\beta_{h2} C_{12} t^2}{2} - \left(\frac{\varphi_h + \rho_h + \mu_h}{2}\right)C_2 t^2\right) - (\omega_h + \mu_h)\left(\frac{\varphi_h C_2 t^2}{2} - \left(\frac{\omega_h + \mu_h}{2}\right)C_4 t^2\right)$$

$$\frac{dS_{d2}}{dt} = -\beta_d(\rho_d C_{27} t - (\mu_d + \sigma_d)C_{28} t) - \beta_d(\Lambda_d C_7 t - \beta_d C_{29} t + \omega_d C_{30} t - (\varphi_d + \mu_d)C_{31} t) + \omega_d(\varphi_d C_5 t - (\omega_d + \mu_d)C_8 t) - (\varphi_d + \mu_d)(\Lambda_d t - \beta_d C_{13} t + \omega_d C_8 t - (\varphi_d + \mu_d)C_5 t)$$

$$S_{d2} = -\beta_d\left(\frac{\rho_d C_{27}t^2}{2} - \frac{(\mu_d+\sigma_d)}{2}C_{28}t^2\right) - \beta_d\left(\frac{\Lambda_d C_7 t^2}{2} - \frac{\beta_d C_{29}t^2}{2} + \frac{\omega_d C_{30}t^2}{2} - \frac{(\varphi_d+\mu_d)C_{31}t^2}{2}\right) + \omega_d\left(\frac{\varphi_d C_5 t^2}{2} - \frac{(\omega_d+\mu_d)C_8 t^2}{2}\right) - (\varphi_d+\mu_d)\left(\frac{\Lambda_d t^2}{2} - \frac{\beta_d C_{13}t^2}{2} + \frac{\omega_d C_8 t^2}{2} - \frac{(\varphi_d+\mu_d)C_5 t^2}{2}\right)$$

$$\frac{dE_{d2}}{dt} = \beta_d(\rho_d C_{27}t - (\mu_d+\sigma_d)C_{28}t) + \beta_d(\Lambda_d C_7 t - \beta_d C_{29}t + \omega_d C_{30}t - (\varphi_d+\mu_d)C_{31}t) - (\rho_d+\mu_d)[\beta_d C_{14}t - (\rho_d+\mu_d)C_6 t]$$

$$E_{d2} = \beta_d\left(\frac{\rho_d C_{27}t^2}{2} - \frac{(\mu_d+\sigma_d)}{2}C_{28}t^2\right) + \beta_d\left(\frac{\Lambda_d C_7 t^2}{2} - \frac{\beta_d C_{29}t^2}{2} + \frac{\omega_d C_{30}t^2}{2} - \frac{(\varphi_d+\mu_d)C_{31}t^2}{2}\right) - (\rho_d+\mu_d)\left(\frac{\beta_d C_{14}t^2}{2} - \frac{(\rho_d+\mu_d)C_6 t^2}{2}\right)$$

$$\frac{dI_{d2}}{dt} = \rho_d(\beta_d C_{14}t - (\rho_d+\mu_d)C_6 t) - (\mu_d+\sigma_d)(\rho_d C_6 t - (\mu_d+\sigma_d)C_7 t)$$

$$I_{d2} = \rho_d\left(\frac{\beta_d C_{14}t^2}{2} - \frac{(\rho_d+\mu_d)C_6 t^2}{2}\right) - (\mu_d+\sigma_d)\left(\frac{\rho_d C_6 t^2}{2} - \frac{(\mu_d+\sigma_d)C_7 t^2}{2}\right)$$

$$\frac{dV_{d2}}{dt} = \varphi_d(\Lambda_d t - \beta_d C_{13}t + \omega_d C_8 t - (\varphi_d+\mu_d)C_5 t) - (\omega_d+\mu_d)(\varphi_d C_5 t - (\omega_d+\mu_d)C_8 t)$$

$$V_{d2} = \varphi_d\left(\frac{\Lambda_d t^2}{2} - \frac{\beta_d C_{13}t^2}{2} + \frac{\omega_d C_8 t^2}{2} - \frac{(\varphi_d+\mu_d)C_5 t^2}{2}\right) - (\omega_d+\mu_d)\left(\frac{\varphi_d C_5 t^2}{2} - \left(\frac{\omega_d+\mu_d}{2}\right)C_8 t^2\right)$$

IV. NUMERICAL SIMULATION AND RESULTS

In this chapter, the model will be analyzed using the parameter values as well as estimated initial values of the susceptible, exposed, infected, recovered and vaccinated individuals. The results obtained will be discussed. MATLAB was used to get the numerical solution of the model using ode45.

Table (5.1) **parameter and estimated values for initial conditions for the SEIR and SEIV models**

Description	Parameters	Values	Sources
the recruitment level of human population	Λ_h	2000	Assumed
the recruitment level of dog population	Λ_d	500	Assumed
natural death rate of human population	μ_h	0.0022	Hethcote (2016)
natural death rate of dog population	μ_d	0.083	”
death rate due to rabies for human population	σ_h	0.004	”
death rate due to rabies for dog population	σ_d	0.0002	”
the incubation period in human population	ρ_h	0.71	Addo et al. (2012)
the incubation period in dog population	ρ_d	0.31	”
Treatment rate of exposed human population	φ_h	0.29	”
Vaccination rate of Susceptible dog population	φ_d	0.17	”

the loss rate of vaccination immunity for human population	ω_h	0.19	”
the loss rate of vaccination immunity for dog population	ω_d	0.005	”
The initial number of susceptible human	$S_h(0)$	430	Assumed
The initial number of exposed human	$E_h(0)$	40	Assumed
The initial number of infected human	$I_h(0)$	20	Assumed
The initial number of recovered human	$R_h(0)$	0	Assumed
The initial number of susceptible dog	$S_d(0)$	230	Assumed
The initial number of exposed dog	$E_d(0)$	60	Assumed
The initial number of infected dog	$I_d(0)$	10	Assumed
The initial number of vaccinated dog	$V_d(0)$	0	Assumed
Contact rate between S_d and I_d	β_d	0.29	Global alliance
Contact rate between S_h and I_h	β_{h1}	0.304	Global alliance
Contact rate between S_h and I_d	β_{h2}	0.58	Global alliance

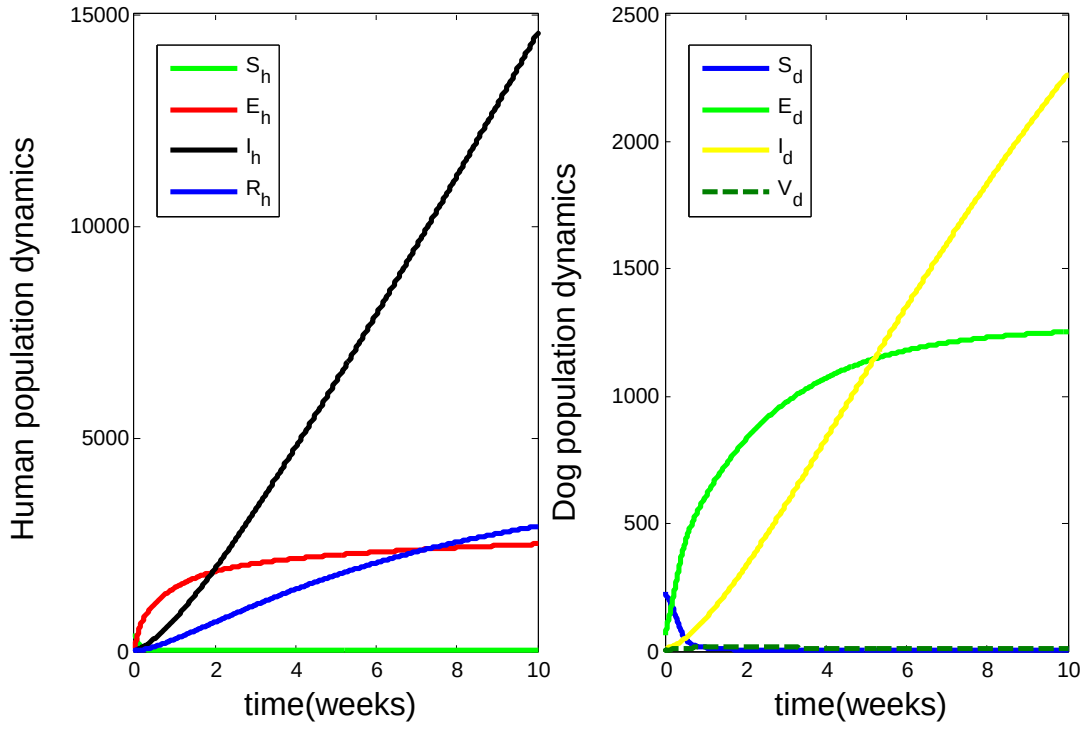


Figure 5.1: General dynamics of Rabies virus in Human and Dog for $R_0 > 1$

From figure 5.1, we tried to show the possible dynamics of Rabies disease in both the human and dog population, and we can see from the graph that at $R_0 > 1$, the infected classes of the two populations are at increase. This means that there is need for reduction of the value of the basic reproduction number to a value less than unity.

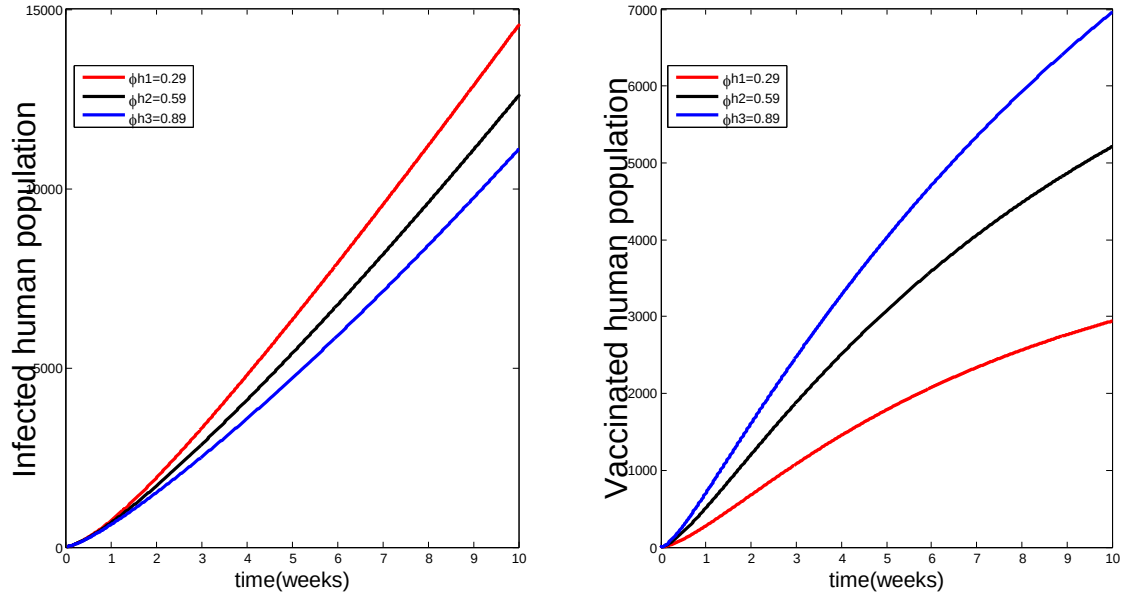


Figure 5.2: Graphical illustration of the effect of the control parameter φ_h which is the rate at which humans are treated, on the infected human population (a) and the recovered human population (b). The values of φ_h are 0.29, 0.59 and 0.89.

From figure 5.2, we can observe that as the rate at which humans are treated (φ_h) increases, the infected population of human's decreases and the recovered human's increases, which shows that φ_h which is the treatment rate of the exposed humans is an effective control measure in reducing Rabies in the human population.

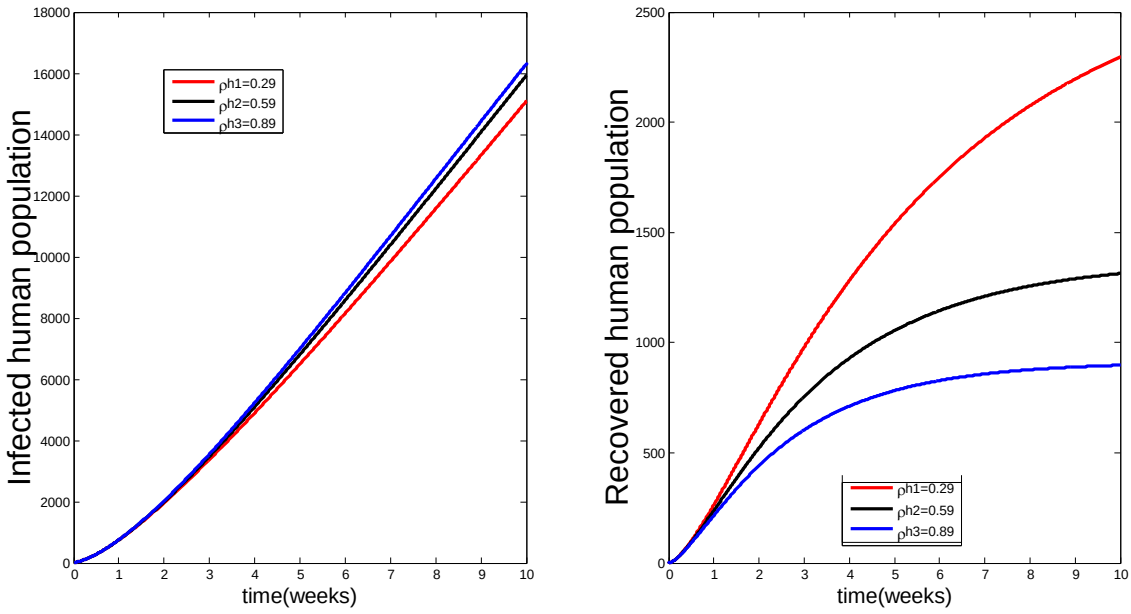


Figure 5.3: Graphical illustration of the effect of rate at which exposed class gets infected, denoted with ρ_h on the infected human population (a) and the recovered human population (b). The values of ρ_h are 0.29, 0.59 and 0.89.

From the figure 5.3 above, we can observe that the rate at which the exposed class gets infected must be reduced for effective eradication of Rabies in the human population, since from the graph it is evident that increase in the

rate of movement to infected class (ρ_h) increases the infected humans and reduces the recovered humans respectively.

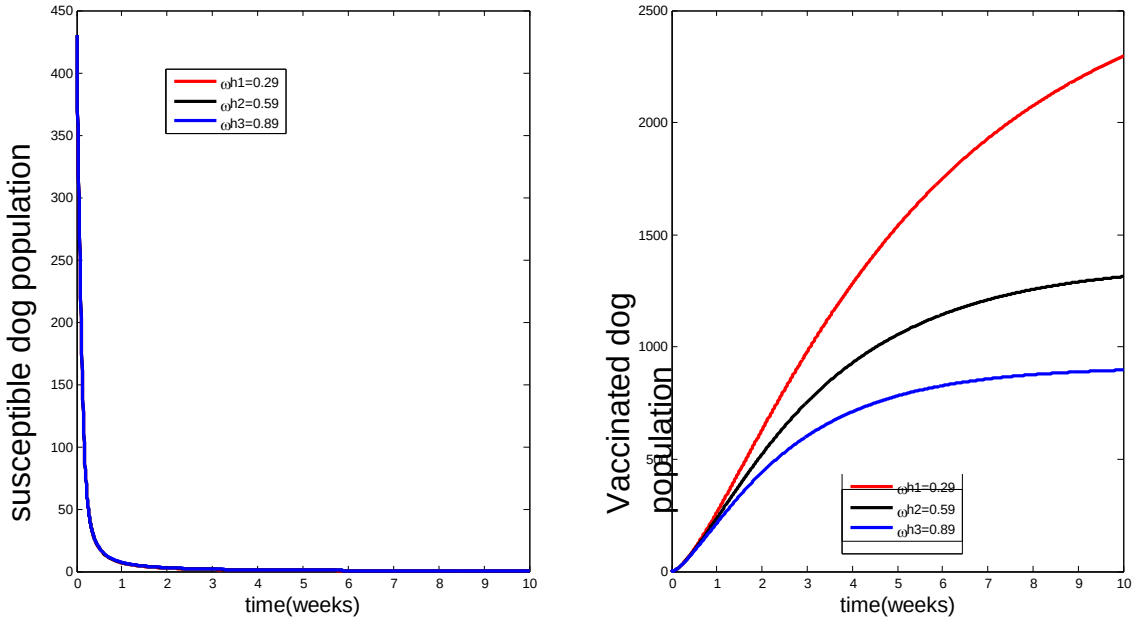


Figure 5.4: Graphical illustration of the effect of the waning rate of vaccine, denoted as ω_h on the susceptible human population (a) and the recovered human population (b). The values of ω_h are 0.29, 0.59 and 0.89.

Also, from figure 5.4, we tried to investigate the impact of rate of waning rate of vaccine (ω_h) on the susceptible and recovered class. From the graph, we can see that when humans lose their immunity over time that the recovered class tends to reduce, so also the susceptible class.

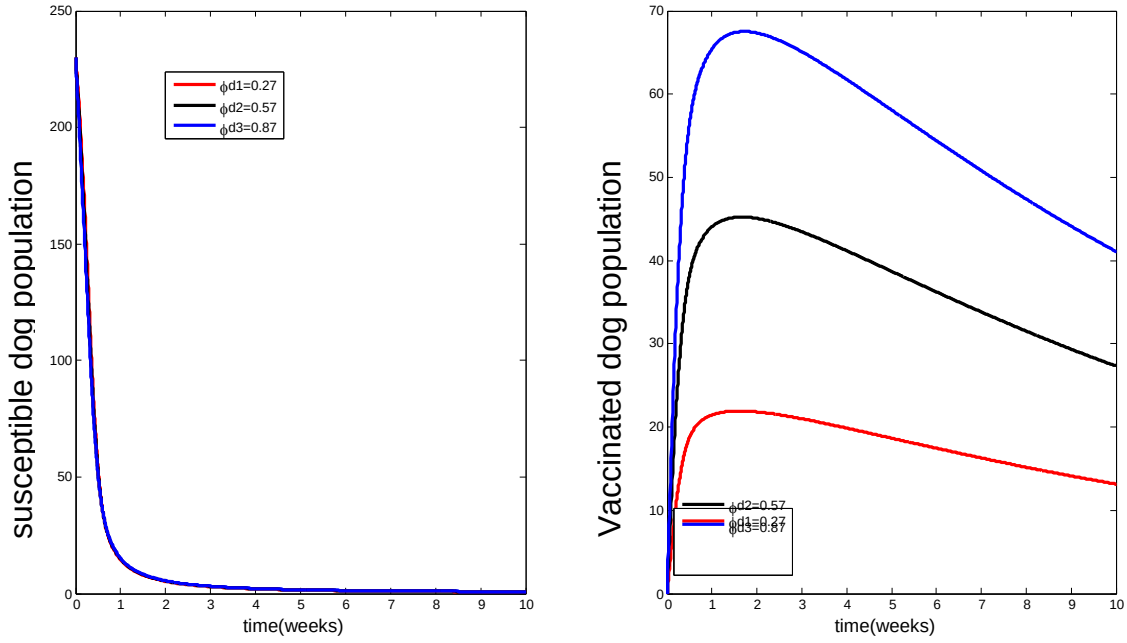


Figure 5.5: Graphical illustration of the effect of the control parameter φ_d which means the rate of vaccine of susceptible dogs, on the susceptible dog population (a) and the vaccinated dog population (b). The values of φ_d are 0.27, 0.57 and 0.87.

From figure 5.5, we can observe that when there is high rate vaccine of dogs, it increases the recovered dogs and reduces susceptible dogs respectively, which means that vaccine of dogs is an effective way to eradicating Rabies in the dog population.

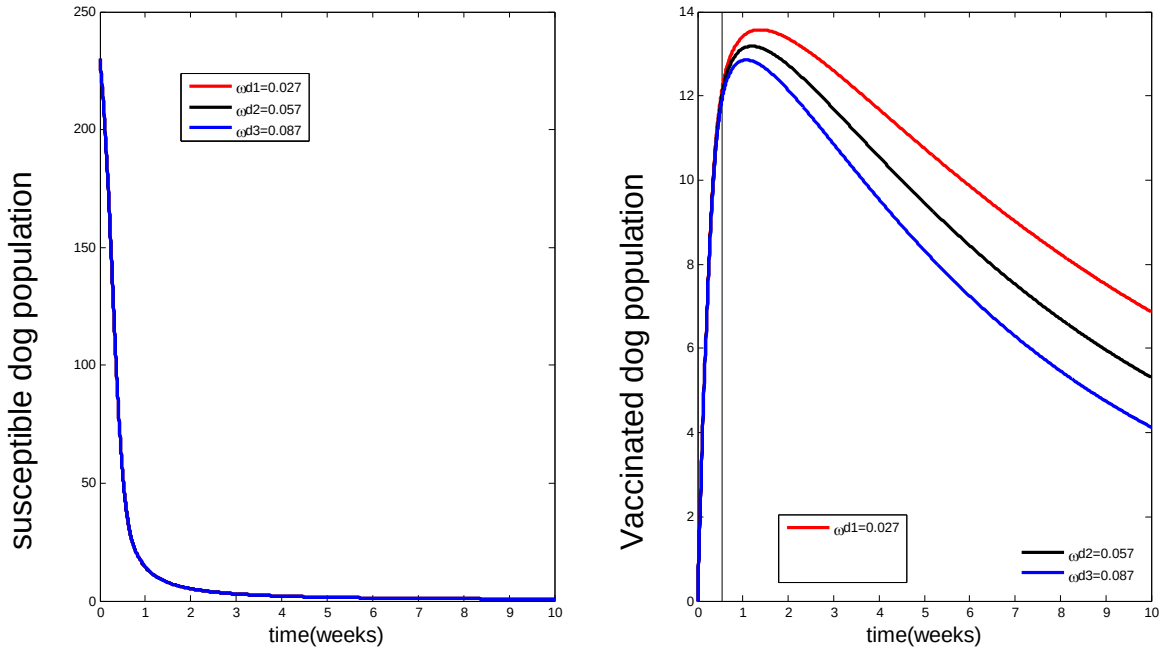


Figure 5.6: Graphical illustration of the effect of the control parameter ω_d describing the waning rate of vaccine in dogs, on the susceptible dog population (a) and the vaccinated dog population (b). The values of ω_d are 0.027, 0.057 and 0.087.

Lastly from figure 5.6, it can be seen that when there is high rate of loss of vaccination immunity in dogs, that this leads to reduction in the vaccinated dogs and subsequent increases in susceptible dogs, which further become infected and that increases the virus. So reduction of loss of immunity will be the best way to check rabies in dogs.

B. Discussion of Results

The dynamics and control intervention measures of Rabies virus disease in human and dogs have been investigated in this work. We carried out detailed study of the dynamics of Rabies virus disease and investigated its transmission pathways from human to human and from dogs to human.

We formulated a mathematical method that discussed this disease transmission in detail, we proceeded in analyzing our formulated model and from our analysis it has shown that significant information concerning the dynamics of Rabies virus disease can be obtained by analyzing an appropriate mathematical epidemiological model.

In analyzing the formulated model, we showed that the model equations were biologically and mathematically well posed, meaning that all the state variables were positive, which agrees with the invariant set theorem. We derived the basic reproduction number of the model and also the equilibrium points, and we showed that in the absence of any control measure that is possible to eradicate Rabies from the entire population, if $R_0 < 1$. We also proceeded to carryout stability analysis of the equilibrium points of the model. We also showed that the disease-free equilibrium is locally asymptotically stable whenever $R_0 < 1$ and unstable otherwise, and this agrees with the intuition that if we can reduce the rate at which secondary infections occur, that the disease will die out from the population. Furthermore, we also showed that the endemic equilibrium point of the model is locally asymptotically stable.

In our simulation result, we carried out an overall dynamics of Rabies virus disease in both human and dogs, and we observed that when $R_0 > 1$, that the infected class increases with time and therefore, we need to make $R_0 < 1$ at all times, if we intend to eradicate Rabies from the entire population.

We further investigated the impact of the different control measures on the susceptible, infected and recovered class of both human and dog population. From our analyses we could see that the control measures initiated had great influence in reducing the infected population of human and dogs respectively, which agrees with our analytical results that reduction of secondary infections reduces the disease and leads to possible eradication of the disease in the entire population.

C. SUMMARY

Rabies is the most dreadful diseases that could be transmitted from animals to man. In order to curb its spread a mathematical model was proposed in this work so as to understand the transmission dynamics of the disease and to proffer solution by introducing vaccination to facilitate the prevention of the spread of the disease.

The population under consideration was a non-constant population. The SEIR and SEIVmodels were considered and the disease free and endemic equilibrium points were obtained. Their basic reproduction number were obtained using the next generation matrix approach. The stability of the disease free and endemic equilibrium points were analyzed. With estimated values for parameters and initial numbers of the population classes, MATLAB was used to get the numerical solution of the model using ode45.

D. CONCLUSION

In Conclusion, we studied a model of rabies transmission dynamics in dogs and humans and the best way of reducing death rate of rabies in humans. The basic reproduction number have been computed using next generation method. We have computed the disease free and endemic equilibrium points. If the basic reproduction number $R_0 < 1$ the disease free equilibrium is locally asymptotically stable and the disease dies out. The endemic equilibrium points have been driven whenever they exist, Using the Descartes' rule of signs, the endemic equilibrium points are locally asymptotically stable. We also obtained a control solution for the model which predicts that the best way of eliminating deaths from rabies as projected by the global alliance for rabies control is using more of pre-exposure prophylaxis in both dogs and humans and public education; however, the results show that pre-exposure prophylaxis and post-exposure prophylaxis in humans with use of vaccination in the dog population is beneficial if total elimination of the disease is desirable. Any combination strategy which involves vaccination in the dogs' population gives a better result of eliminating the disease.

ACKNOWLEDGMENT

I wish to thank Dr. Onah Sunday for his kind and marvelous contribution towards the success of this work. On the other hand, I extend my greeting to my supervisor in the person of Prof. G.C.E. Mbah who acted more like a father. My gratitude goes to my other co-authors for kind suggestions in making this work a success. I thank you all.

REFERENCES

[1] K. M. Addo, An SEIR Mathematical Model for Dog Rabies. Case Study: Bongo District,Ghana, MSc. Dissertation Kwame Nkrumah University of Science and Technology, 2012.
[2] Q. Hou, Z. Jin and S. Ruan, Dynamics of rabies epidemics and the impact of control efforts in Guangdong Province, China, Journal of theoretical biology, 300 (2012)39 - 47.
[3] R. Anderson, H. Jackson, R. May, and A. Smith. Population dynamics of fox rabies in Europe. Nature, 289:765-771, 1981.
[4] L. Andral, A. M. and J. Blancou. Radio-tracking of rabid foxes. Comparative Immunology, Microbiology and Infectious Diseases, 5:285-291, 1982
[5] E. Asano, L. Gross, S. Lenhart, and L. A. Real. Optimal control of vaccine distribution in a rabies metapopulation model. Math. Biosci. Eng, 5:219, 2008
[6] D. Belcher, F. Wurapa, and D. Atuora. Endemic rabies in ghana. Epidemiology and control measures, 1976.
[7] T. Claytona, S. Duke-Sylvesterb, L. J. Grossc, S. Lenhartd, and L. A. Real. Journal of biological dynamics, 2010.
[8] D.Smith, B. Lucey, Waller, C. J., and L. Real. Predicting the spatial dynamics of rabies epidermics on heterogenous landscapes. Proc. Natl. Acad. Sci, USA, 99:3668-3672, 2002.
[9] M. De la Sen, A. Ibeas, and S. Alonso-Quesada. Vaccination rules for a true-mass action SEIR epidemic model based on an observer synthesis.
[10] H. W. Hethcote. The basic epidemiology models: Models expressios for r0, parameter estimation and applicataions.(2016)
[11] T. J. John and R. Samuel. Herd immunity and herd efect: new insights and definitions, 2000.
[12] S. Johnson. Rabies. Ghana Veterinary Medical Association Newsletter, 6:18-19, 2007. 88
[13] M. Yousaf, M. Qasim, S. Zia, R. Khan, U. Ashfaq and S. Khan, Rabies molecular virology, diagnosis, prevention and treatment, Virology Journal, 9 (2012),

[14] D. MacDonald. Rabies and Wildlife: A Biologist's Perspective. Oxford University Press, Oxford, 1980
[15] D. MacDonald, R. Bruce, and P. Bacon. Fox population, habitat characterization and rabies control. J. Biogeogr, 8:145-151, 1981.
[16] J. Murray, E. Stanley, and D. Brown. On the spatial spread of rabies among foxes. Proc. R. Soc, NLond. B Biol. Sci, 229:111-150, 1986.
[17] M. Yousaf, M. Qasim, S. Zia, R. Khan, U. Ashfaq and S. Khan, Rabies molecular virology, diagnosis, prevention and treatment, Virology Journal, 9 (2012), 1 - 5.
[18] J. Zhang, Z. Jinm, G.Q. Sun, T. Zhou, and S. Ruan. Analysis of rabies in China: Transmission dynamics and control, July 2011.
[19] G .Vijay Panjeti and leslie A. Real. The advances in virus research. Mathematical Models for Rabies, 79: 377-395, 2011.
[20] D. Voigt, R. Tinline, and L. Broekhoven. A spatial simulation model for rabies control. Popul.Dyn. Rabies Wildl.,311-349, 1985.
[21] CA. Russell, DL. Smith, Childs JE, Real LA (2005) Predictive Spatial Dynamics and Strategic Planning for Raccoon Rabies Emergence in Ohio. PLoS Biol 3(3): e88. https://doi.org/10.1371/journal.pbio.0030088
[22] Tesfaye Tadesse Ega, S. Livingstone. Luboobi, Dmitry Kuznetsov.(2015) Modeling the Dynamics of Rabies Transmission with Vaccination and Stability Analysis. Applied and Computational Mathematics. Vol. 4, No. 6, pp. 409-419. doi: 10.11648/j.acm.20150406.13
[23] H. Robert; Williams, J .David (1996). Veterinary Medicine: An Illustrated History. Dunlop, Mosby. ISBN 978-0-8016-3209-9.