The Study of Structure of Roots of Transcendental Equation Using Argument Principle

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Abstract. *In acoustics and vibration theory the basic equations are usually linearized. The condition of existence of a nontrivial solution of the linear equations leads to the characteristic transcendental equation with respect to H. Here H is the complex number imaginary part of which is natural frequency of oscillations of the system. Real part is describing the rate of damping of oscillations or instability (in case if real part is positive). The structure of roots of such equations is studied by using argument principle for some specific applied problem from fluid mechanics. It is shown that dispersion equation has two complex-conjugate roots and infinite number of real roots. All roots lie in the left complex half-plane, providing damping of oscillations.*

Keywords - *argument principle, structure of roots, transcendental equation*

I. INTRODUCTION

We consider gas bubbles performing free radial oscillations in a liquid. Such problem was studied in [1-3] in the presence of heat exchange between gas and liquid. The system of basic equations describing the process under assumption of the spherical symmetry of the process was derived in [1-3]. In case of small oscillations analytical solution of heat equation inside the bubble can be obtained. The condition of the existence of a nontrivial solution of the system of linear equations leads to the characteristic transcendental equation relative to H. Here H is a complex number imaginary part of which is natural frequency of oscillations of the system. Real part is describing the rate of damping of oscillations or instability (in case if real part is positive).

II. BASIC EQUATIONS AND RESULTS

For considered problem this equation has form [1-3]:

$$f(H) = f_0(H) + f_1(H)\,\mathrm{Kh}(H) = 0; \qquad (1)$$

$$\mathrm{Kh}(H) = H^{1/2}\coth\left(H^{1/2}\right) - 1,$$

Where $f_0(H)$ and $f_1(H)$ are polynomials of H, the degree of the second polynomial being lower than the degree of the first polynomial.

In common simple case [1], we have:

$$f_0(H) = H^2 + A^2, f_1(H) = \varepsilon H\,, \varepsilon > 0\,.$$

We note that $\mathrm{Kh}(H)$ is a meromorphic function in the entire complex plane ζ(no branching at the points $\zeta = 0$ and$\zeta = \infty$); therefore, the function $\mathrm{Kh}(H)$ does not depend on the choice of the sign ahead of the square root. Therefore, we can set $\sqrt{H} = x + iy$ for $x > 0$ or $x = 0$ and $y \geq 0$. We prove the following lemma.

Lemma 1. The function $\coth(x+iy)$ and its modulus can be represented as

$$\coth(x+iy) = \frac{\sinh(2x) - i\sin(2y)}{\cosh(2x) - \cos(2y)},$$

$$|\coth(x+iy)| = \sqrt{\frac{\cosh(2x) + \cos(2y)}{\cosh(2x) - \cos(2y)}} = \sqrt{\frac{\sinh^2 x + \cos^2 y}{\sinh^2 x + \sin^2 y}}\,.$$

Proof. The function $\coth(x+iy)$ can be written as

$$\coth(x+iy)=\frac{\exp(x+iy)+\exp(-x-iy)}{\exp(x+iy)-\exp(-x-iy)}=\frac{\cosh x\cos y+i\sinh x\sin y}{\sinh x\cos y+i\cosh x\sin y}.$$

Multiplying the numerator and denominator by a number which is complex conjugate to the denominator, we obtain

$$\coth(x+iy)=\frac{\cosh x\sinh x(\cos^2 y+\sin^2 y)+i\sin y\cos y(\sinh^2 x-\cosh^2 x)}{\sinh^2 x\cos^2 y+\cosh^2 x\sin^2 y}.$$

Transforming to binary arguments (by multiplying the numerator and denominator by 2),we obtain

$$\coth(x+iy)=\frac{\sinh(2x)-i\sin(2y)}{(\cosh(2x)-1)\cos^2 y+(\cosh(2x)+1)\sin^2 y}=\frac{\sinh(2x)-i\sin(2y)}{\cosh(2x)-\cos(2y)}.$$

Calculation of the modulus of this expression yields

$$|\coth(x+iy)|=\sqrt{\frac{\sinh^2(2x)+\sin^2(2y)}{(\cosh(2x)-\cos(2y))^2}}.$$

Taking into account that

$$\sinh^2(2x)+\sin^2(2y)=\cosh^2(2x)-1+\sin^2(2y)=\cosh^2(2x)-\cos^2(2y),$$

and decomposing the terms of this expression into multipliers, we obtain:

$$|\coth\,(x+iy)|=\sqrt{\frac{\cosh(2x)+\cos(2y)}{\cosh(2x)-\cos(2y)}}.$$

Transformation from the binary arguments to ordinary arguments leads to the required representation for $|\coth(x+iy)|$. The representation

$$\coth\,(x+iy)=\frac{\sinh(2x)-i\sin(2y)}{\cosh(2x)-\cos(2y)}$$

implies that the poles of the function $\mathrm{Kh}(H)$ are the points

$$x=0,\quad y=\pi n,\qquad H=(iy)^2=-(\pi n)^2,\qquad n=1,2,\ldots,$$

and the zeros are the solutions of the equation $\tan y=y$ and $H=-y^2$.

The poles of the function $\mathrm{Kh}(H)$ are also poles for the characteristic function $f(H)$, and zeros of the form $H_n=-y_n^2$, where $n\pi<y_n<(n+1)\pi$, are also zeros of the function $f(H)$ because the function $f(-y^2)$ takes real values, and in each of the specified intervals, it varies from $-\infty$ to $+\infty$. These zeros [calculated as the real roots of the equation $f_0(-y^2)+f_1(-y^2)(y\cot(y)-1)=0$] define rapidly decreasing solutions without oscillations (H is a negative real number). The last equation can be written as

$$\tan y=\frac{yf_1(-y^2)}{f_0(-y^2)-f_1(-y^2)},$$

from which it follows that, in the interval $n\pi<y_n<(n+1)\pi$, the characteristic equation has an infinite number of real roots.

The question arises of the existence of other zeros (zeros can also appear in the negative part of the straight line). Of special interest are zeros with a positive real part (instability). We consider the region bounded by the straight line $y \le \pi(n+1/4)$ for $\sqrt{H} = x + iy$. In the plane H, this region is bounded by the parabola

$$\operatorname{Re} H \ge \left(\frac{\operatorname{Im} H}{2\pi(n+1/4)}\right)^2 - \pi^2(n+1/4)^2.$$

Because, according to Lemma 1, on the boundary of the region and at infinity, $|\coth(x+iy)| = 1$, it follows that for large H, the inequality $|f_0(H)| > |f_1(H)|$ holds. Consequently, along the boundary of the region considered, the argument of the function $f(H)$ varies in the same manner as the argument of the function $f_0(H)$. From the argument principle [4], it follows that for large n, the difference in number between zeros and poles in this region is equal to $N - P = \deg(f_0)$. In particular, the above characteristic function has not only real zeros but also a pair of complex-conjugate zeros. This property holds for any regular characteristic function (all complex zeros are included in the pairs of complex-conjugate numbers).

In the example considered, the function $f_1(H)$ has a small multiplier. In this case, in the absence of multiple roots of the function $f_0(H) = 0$, the search of nontrivial zeros of the characteristic function reduces to the following iterative procedure. Let H_0 be some zero of $f_0(H)$; then, $H_{k+1} = f_0^{-1}(-f_1(H_k)\mathrm{Kh}(H_k))$[$f_0^{-1}$ is a local inverse function near the corresponding zero of the function $f_0(H)$].

We calculate the roots of the characteristic function

$$H^2 + A^2 + \varepsilon H\left(H^{1/2}\coth\left(H^{1/2}\right) - 1\right) = 0$$

for the first approximation for ε :

$$H \approx iA\sqrt{1 + (\varepsilon i/A)\mathrm{Kh}(iA)} \approx iA - (\varepsilon/2)\mathrm{Kh}(iA). \tag{2}$$

Introducing $a = \sqrt{2A}$, taking into account that $H^{1/2} = a(1+i)/2$, and using Lemma 1, we obtain the expression

$$\mathrm{Kh}(iA) = \frac{a(1+i)}{2}\frac{\sinh a - i\sin a}{\cosh a - \cos a} - 1 = \frac{a(\sinh a + \sin a)}{2(\cosh a - \cos a)} - 1 + i\frac{a(\sinh a - \sin a)}{2(\cosh a - \cos a)}.$$

Substitution of this expression into formula (2) yields approximate values for two conjugate roots:

$$H \approx \pm iA\left(1 - \frac{\varepsilon}{2}\frac{\sinh a - \sin a}{\cosh a - \cos a}\right) - \frac{\varepsilon}{2}\left[\frac{(a/2)(\sinh a + \sin a) - \cosh a + \cos a}{\cosh a - \cos a}\right]. \tag{3}$$

It should be noted that the expression in square brackets is always positive; therefore the real part of the roots is negative. In this case, all the real roots are larger in absolute values than the real parts of the complex roots.

III. CONCLUSION

Thus, it is proved that all roots of the characteristic equation, except for two complex-conjugate roots with a negative real part, are negative real numbers and are larger in absolute value than the real parts of the complex roots, i.e., they decay rapidly and do not make a significant contribution to the general solution. It should be noted that roots with a positive real part that cause instability are absent. This is evidence for the correctness of the result of papers [1, 3], in which characteristic transcendental equations were solved numerically and only complex-conjugate root were found and were then used to calculate the damping decrement.

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