Spectral Properties of k-Quasi-Parahyponormal Operators

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Abstract: *The spectrum, non zero points of its approximate point spectrum and joint approximate point spectrum of class of k-quasi-parahyponormal operators are characterized in this paper.*

Keywords: *k-Quasi Parahyponormal Operators, Approximate Point Spectrum, Joint Approximate Point Spectrum.*

I. INTRODUCTION

Let H be a separable complex Hilbert space and $B(H)$ denote the C^*- algebra of all bounded linear operators acting on H. Recall that, $T \in B(H)$ is called p - hyponormal for $p > 0$, if $(T^*T)^p \geq (TT^*)^p$ [1], when $p = 1$, T is called hyponormal operator. An operator T is paranormal operator if $\|Tx\|^2 \leq \|T^2x\|\|x\|\ \forall x \in H$ [7,8]. T is called normaloid if $\|T^n\| = \|T\|^n, \forall n \in N$ (equivalently $\|T\| = r(T)$, the spectral radius of T). Mahmoud M Kutkut introduced parahyponormal operator. An operator T is parahyponormal operator if $\|Tx\|^2 \leq \|TT^*x\|\,\|x\|, \forall x \in H$ [14]. Spectral properties of p-hyponormal operators, quasi hyponormal operators and paranormal operators have been studied by many authors and they have also proved many interesting properties similar to those of hyponormal operators[6, 10, 17]. The relations between paranormal and p-hyponormal and loghyponormal operators, Furuta, et al. introduced a very interesting class of bounded linear Hilbert space operators: class A and they showed that class A is a subclass of paranormal and contains p-hyponormal and log-hyponormal operators.

In order to extend the class of parahyponormal operators, we introduce a new class of operators defined as follows:

Definition: For every positive integer k, an operator T is said to be k-quasi-parahyponormal operator, if $\|T^{k+1}x\|^2 \leq \|TT^*T^kx\|\|T^kx\|$ for all $x \in H$ and when $k = 1$, it is quasi-parahyponormal operator. Generally the following implications hold.

Normal $\subset$ hyponormal $\subset$ paranormal $\subset$ parahyponormal $\subset$ quasi parahyponormal $\subset$ k-quasi-parahyponormal

In this paper first we prove some basic structural properties of k-quasi-parahyponormal operator and then the spectrum is continuous, the non-zero points of its approximate point spectrum and approximate joint point spectrum are identical.

II. BASIC PROPERTIES OF K-QUASI-PARAHYPONORMAL OPERATORS

Before describing the properties of k- Quasi parahyponormal operators, we recall a well-known result that, "For any operators A, B and C, $A^*A - 2\lambda B^*B + \lambda^2C^*C \geq 0$ for all $\lambda > 0$ if and only if $\|Bx\|^2 \leq \|Ax\|\|Cx\|$ for all $x \in H$".

Theorem 2.1. An operator $T \in B(H)$ is k-quasi-parahyponormal if and only if

$$T^{*k}(TT^*)^2T^k - 2\lambda T^{*k+1}T^{k+1} + \lambda^2 T^{*k}T^k \geq 0 \text{ for all } \lambda > 0.$$

Now we derive some basic properties of k -quasi-parahyponormal operators as follows.

Theorem 2.2. Let $T \in B(H)$ be a k-quasi-parahyponormal operator for a positive integer k and M be a closed T-invariant sub-space of H. Then the restriction $T|_M$ is also k-quasi-parahyponormal operator.

Proof:

Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = M \oplus M^{\perp}$.

Since T is k-quasi-parahyponormal operators, we have

$$T^{*k}(TT^*)^2T^k - 2\lambda T^{*k+1}T^{k+1} + \lambda^2 T^{*k}T^k \geq 0 \text{ for all } \lambda > 0.$$

$$\Rightarrow \begin{pmatrix} T_1^* & 0 \\ T_2^* & T_3^* \end{pmatrix}^k \left\{ \left(\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \begin{pmatrix} T_1^* & 0 \\ T_2^* & T_3^* \end{pmatrix} \right)^2 - 2\lambda \begin{pmatrix} T_1^* & 0 \\ T_2^* & T_3^* \end{pmatrix} \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} + \lambda^2 \right\} \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}^k \geq 0$$

$\Rightarrow T_1^{*k}((T_1T_1^* + T_2T_2^*)^2 + T_2T_3^*T_3T_2^* - 2\lambda T_1^*T_1 + \lambda^2)T_1^k \geq 0.$
$\Rightarrow T_1^{*k}((T_1T_1^*)^2 - 2\lambda T_1^*T_1 + \lambda^2)T_1^k + T_1^{*k}((T_2T_2^*)^2 + T_2T_3^*T_3T_2^* + 2T_1T_1^*T_2T_2^*)T_1^k \geq 0$
$\Longrightarrow T_1^{*k}((T_1T_1^*)^2 - 2\lambda T_1^*T_1 + \lambda^2)T_1^k \geq 0$ and $T_2T_2^* = 0$.
Therefore $T_1 = T|_M$ is k-quasi-parahyponormal operator.

Theorem 2.3. Let $T \in B(H)$ be k-quasi-parahyponormal operator for any positive integer $k > 0$ and let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T^k)} \oplus \ker(T^{*k})$ be 2×2 matrix expression. Assume that $ran(T^k)$ is not dense if and only if $(T_1T_1^*)^2 - 2\lambda T_1^*T_1 + \lambda^2 \geq 0$ on $\overline{ran(T^k)}$ and $T_3^k = 0$. Furthermore $\sigma(T) = \sigma(T_1) \cup \{0\}$.
Proof:
Let P be the projection of H onto $\overline{ran(T^k)}$. Then $T_1 = TP = PTP$.
Since T is k-quasi-parahyponormal operator, we have

$$P((TT^*)^2 - 2\lambda T^*T + \lambda^2)P \geq 0$$
$$P((TT^*)^2)P - 2\lambda P(T^*T)P + P\big((\lambda^2)\big)P \geq 0$$
$$((PTT^*P)^2 - 2\lambda PT^*TP + \lambda^2) \geq 0$$
$$(T_1T_1^*)^2 - 2\lambda T_1^*T_1 + \lambda^2 \geq 0$$

For any $x = (x_1, x_2) \in H$

$$\langle T_3^k x_2, x_2\rangle = \langle T^k(I-P)x, (I-P)x\rangle = \langle (I-P)x, T^{*k}(I-P)x\rangle = 0$$

This implies $T_3^k = 0$.
Since $\sigma(T) \cup M = \sigma(T_1) \cup \sigma(T_3)$, where M is the union of the holes in $\sigma(T)$, which happens to be a subset of $\sigma(T_1) \cup \sigma(T_3)$ by corollary 7 [11]. $\sigma(T_3) = 0$ and $\sigma(T_1) \cap \sigma(T_3)$ has no interior points we have $\sigma(T) = \sigma(T_1) \cup \{0\}$.
Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T^k)} \oplus \ker(T^{*k})$ where $(T_1T_1^*)^2 - 2\lambda T_1^*T_1 + \lambda^2 \geq 0$, for every $\lambda > 0$ and $T_3^k = 0$.

$$T^k = \begin{pmatrix} T_1^k & \sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \\ 0 & 0 \end{pmatrix}$$

$$(TT^*)^2 = \begin{pmatrix} (T_1T_1^*)^2 & T_1T_1^*T_2T_3^* + T_2T_3^*T_3T_3^* \\ T_3T_2^*T_1T_1^* + T_3T_3^*T_3T_2^* & T_3T_2^*T_2T_3^* + T_3T_3^*T_3T_3^* \end{pmatrix}$$

$$T^kT^{*k} = \begin{pmatrix} T_1^kT_1^{*k} + \sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j}\left(\sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j}\right)^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

Where $A = A^* = T_1^kT_1^{*k} + \sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \left(\sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j}\right)^* \geq 0$ for every $\lambda > 0$.
Therefore $T^kT^{*k}((TT^*)^2 - 2\lambda T^*T + \lambda^2)T^kT^{*k} = \begin{pmatrix} A((T_1T_1^*)^2 - 2\lambda T_1^*T_1 + \lambda^2)A & 0 \\ 0 & 0 \end{pmatrix} \geq 0.$
It follows that $T^{*k}((TT^*)^2 - 2\lambda T^*T + \lambda^2)T^k \geq 0$ for $\lambda > 0$ on $H = \overline{ran(T^k)} \oplus \ker(T^{*k})$. Thus T is k-quasi-parahyponormal operator.

Corollary 2.4. Let T be k-quasi-parahyponormal operator with $ran(T)$ is not dense and $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T^k)} \oplus \ker(T^{*k})$, then T_1 is a parahyponormal operator, $T_3^k = 0$. Further more $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Corollary 2.5. Let T be k-quasi-parahyponormal operator with $ran(T)$ is not dense and $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on H, then T_1 is a parahyponormal operator on $\overline{ran(T)}$.

Corollary 2.6. Let T be k-quasi-parahyponormal operator and $0 \neq \mu \in \sigma_p(T)$. If T is of the form $T = \begin{pmatrix} \mu & B \\ 0 & C \end{pmatrix}$ on $H = N(T - \mu I) \oplus N(T - \mu I)^{\perp}$, then $B = 0$.

III. THE SPECTRAL CONTINUITY OF K-QUASI-PARAHYPONORMAL OPERATORS

For every $T \in B(H)$, $\sigma(T)$ is compact subset of C. The function viewed as a function from $B(H)$ into the set of all compact subset of C, equipped with the Housdorff metric, is well known to be upper semi continuous, but fails to be continuous in general. Conway and Morrel [3] have carried out a detailed study of spectral continuity in $B(H)$. Recently, the continuity of spectrum was considered when restricted to some subsets of the entire manifold of Toeplitz operators in [13]. It has been proved that is continuous in the set of normal operators and hyponormal operators in [9]. And this result has been extended to quasi hyponormal operators by Djordjevic in [8] to p-hyponormal operators, (p,k)-quasi hyponormal operators, *-paranormal and paranormal operators by many authors. In this section we extend this result to k-quasi- *parahyponormal operators.

A complex number λ is said to be in the point spectrum $\sigma_p(T)$ of T if there is a non zero $x \in H$ such that $(T-\lambda I)x = 0$. If in addition $(T^* - \lambda I)x = 0$ then λ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of T. If T is hyponormal then $\sigma_{jp}(T) = \sigma_p(T)$. The approximate point spectrum of an operator T is $\sigma_{ap}(T) = \{\lambda \in C : \exists\ a\ sequence\ of\ unit\ vectors\ x_n\ such\ that\ \|Tx_n - \lambda I x_n\| \to 0\ as\ n \to 0\}$.

Theorem 3.1. Let T be a k-quasi-parahyponormal operator. Then the following assertions hold.
(i) If T is quasi nilpotent, then $T^{k+1} = 0$.
(ii) For every nonzero $\lambda \in \sigma_p(T)$, the matrix representation of T with respect to the decomposition $H = N(T-\mu I) \oplus N(T-\mu I)^{\perp}$ is $T = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$ for some operator satisfying $\lambda \notin \sigma_p(B)$ and $\sigma(T) = \lambda \cup \sigma(B)$.

Proof:

(i) Suppose T is k-quasi- parahyponormal operator. If the range of T^k is dense, then T is parahyponormal operator, which leads to that T is normaloid. Hence $T = 0$.

If the range of T^k is not dense, then $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T^k)} \oplus \ker(T^{*k})$ where T_1 is a parahyponormal operator, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$ (by Theorem 2.3.

Since $\sigma(T_1) = 0$, we have $T_1 = 0$. Thus $T^{k+1} = \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix}^{k+1} = \begin{pmatrix} 0 & T_2 T_3^k \\ 0 & T_3^{k+1} \end{pmatrix} = 0$.

(ii) If $\lambda \neq 0$ and $\lambda \in \sigma_p(T)$, we have that $N(T-\lambda I)$ reduces T by Corollary 2.6. So we have that $\begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$ for some operator B satisfying $\lambda \notin \sigma_p(B)$ and $\sigma(T) = \lambda \cup \sigma(B)$.

Theorem 3.2. [2] Let H be a complex Hilbert space. Then there exists a Hilbert space K such that $H \subset K$and a map $\phi : B(H) \to B(K)$ such that

(i) ϕ is a faithful * representation of the algebra $B(H)$ on K.
(ii) $\phi(A) \geq 0$ for every $A \geq 0$ in $B(H)$.
(iii) $\sigma_a(T) = \sigma_a(\phi(T)) = \sigma_p(\phi(T))$ for any $T \in B(H)$.

Theorem 3.3. [2] Let $\phi : B(H) \to B(K)$ be Berberian's faithful * representation, then $\sigma_{ja}(T) = \sigma_{jp}(\phi(T))$.
Theorem 3.4. The spectrum σ is continuous on the set of k-quasi-parahyponormal operator.
Proof. Suppose T is k -quasi-parahyponormal operator. Let $\phi : B(H) \to B(K)$ be Berberian's faithful * representation of Theorem 3.2. Now we will show that $\phi(T)$is also k-quasi-parahyponormal operator.

Since $$T^{*k}((TT^*)^2 - 2\lambda T^*T + \lambda^2)T^k \geq 0 \text{ for every } \lambda > 0$$
$$(\phi(T))^{*k}\left(\left((\phi(T))(\phi(T))^*\right)^2 - 2\lambda(\phi(T))^*(\phi(T)) + \lambda^2\right)(\phi(T))^k \geq 0 \text{ for every } \lambda > 0$$
$$\phi(T^{*k}((TT^*)^2 - 2\lambda T^*T + \lambda^2)T^k) \geq 0 \text{ for every } \lambda > 0$$

Therefore $\phi(T)$ is also k - quasi-parahyponormal operator, we have T belongs to the set C(i) [5].
Therefore, we have that the spectrum σ is continuous on the set of k - quasi-parahyponormal operators (by corrollary7, [5]).

Theorem 3.5. Let T be a k-quasi-parahyponormal operator and $\lambda \neq 0$ then $Tx = \lambda x$ implies $T^*x = \bar{\lambda}x$.

Proof. We may assume that $x \neq 0$. Let M_0 is a span of $\{0\}$ then is an invariant subspace of T and $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = M_0 \oplus M_0^{\perp}$. Let P be the projection of H onto M_0. It sufficious to show that $T_2 = 0$ in the above equation. Since T is a k -quasi-parahyponormal operator, we have
$$P((TT^*)^2 - 2\lambda T^*T + \lambda^2)P \geq 0$$
By expanding this and by simple calculations we have $\sum T_2 T_3^* = 0$.
Since T is k -quasi-parahyponormal operator, $T^{*k}((TT^*)^2 - 2\lambda T^*T + \lambda^2)T^k \geq 0$
Recall that $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geq 0$ if and only if $X, Z \geq 0$ and $Y = X^{\frac{1}{2}} W Y^{\frac{1}{2}}$ for some contraction W. Therefore $T_2 T_3^k = 0$.
Since $\lambda \neq 0$ and $T_2 = 0$, we have $Tx = \lambda x$ and $T^*x = \bar{\lambda}x$. Hence $(T-\lambda I)x = 0$ and $(T^* - \bar{\lambda} I)x = 0$.

Theorem 3.6. Let T be a k -quasi-parahyponormal operator then $\sigma_{jp}|\{0\} = \sigma_p|\{0\}$ and if $(T-\lambda I)x = 0$, $(T-\mu I)y = 0$, and $\lambda \neq \mu$, then $\langle x, y \rangle = 0$.
Proof. Suppose T is k -quasi-parahyponormal operator. Then $T^{*k}((TT^*)^2 - 2\lambda T^*T + \lambda^2)T^k \geq 0$. Also $(T-\lambda I)x = 0$ and $(T^* - \bar{\lambda} I)x = 0$ for $x \neq 0 \in H$(by Theorem 3.5). By the definition of joint point spectrum,

point spectrum and by the above equation, we have $\sigma_{jp}|\{0\} = \sigma_p|\{0\}$. Without the loss of generality, we may assume that $\mu \neq 0$. Then we have $(T - \mu I)^* y = 0$ (By Theorem 3.5). So $\mu\langle x, y\rangle = \langle x, T^* y\rangle = \langle Tx, y\rangle = \langle x, y\rangle$. Since $\lambda \neq \mu$, we conclude that $\langle x, y\rangle = 0$.

Theorem 3.7. Let T be a k -quasi- parahyponormal operator for a positive integer. Then $\sigma_{ja}(T)|\{0\} = \sigma_a(T)|\{0\}$.

Proof. By continuity of the spectrum and by Theorem 3.5, Theorem 3.6 the result is true. That is the non zero points of its approximate point spectrum and joint approximate point spectrum are identical.

IV. CONCLUSION

We prove more succinct properties of k-quasi-parahyponormal operators and discussed about the spectrum of class of k-quasi-parahyponormal operator is continuous and the non zero points of its approximate point spectrum and joint approximate point spectrum are identical.

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