Spatially symmetric operators of Strum - Liouville Problems

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Abstract: *This paper we discusses with spatially symmetric operators of Strum-Liouville problem of eigenvalues and eigenfunctions, of the differential operators* $-\left(\frac{d^2}{dx^2}\right)+P(x)$ *.We simplify the proofs of theorems due to Borg, Levinson,Hochstadt and Lieverman. In the present article Strum-Liouville operators of spatially symmetric type*

Keywords:- *spatially symmetric operators in Strum-Liouville problem of eigenvalues and eigenfunctions,*

Introduction

The spatially symmetric operators of Strum-Liouville problem within the notations for $P \in C^{'}[0,1]$, $h \in R$ and $H \in R$, $A_{P.h.H}$ denotes the realization in $L^2(0,1)$ of the differential operator, For $P \in C^{'}_{s}[0,1]$, and $h \in R$ $A_{P.h.h}$ denotes the symmetric operator, for $P \in C^{'}[0,1]$, $h \in R$ and $\lambda \in R$, $\phi = \phi(x:p,h,\lambda)$ and $\phi^{*} = \phi^{*}(x:p,h,\lambda)$, for $P \in C^{'}[0,1]$, A^{*}_{p} denotes the realization $L^2[0,1]$ of the differential operator, For $n_1 \geq 1$, $P \in C^{'}[0,1]$, and $h \in R$, $W = W(x;p,h,n_1)$. The method of separation of variables utilized within the solutions of boundary value problems of mathematical physics frequently gives rise so called Strum-Liouville eigenvalue problems of spatially symmetric operators ,our attention to small but significant fragment of the theory of Strum-Liouville problems and their solutions.

1.1 Spatially symmetric operators

For $P \in C^{'}[0,1]$, $h \in R$ and $H \in R$, $A_{P.h.H}$ denotes the realization in $L^2(0,1)$ of the differential operator

$$-\left(\frac{d^2}{dx^2}\right)+P(x)$$

With the boundary condition $\left[-\left(\frac{d}{dx}\right)+h\right]\Big|_{x=0}$

$$=\left[-\left(\frac{d}{dx}\right)+h\right]\Big|_{x=0}$$

$= 0$

Let $\sigma(A_{P.h.H})=\{\lambda_n\}_{n=0}^{\infty}$ be the eigenvalues $A_{P.h.H}$ each $\not\subset \lambda_n$ is simple $-\infty<\lambda_0<\lambda_1<\ldots\to\infty$

Put $$C_s^{'}[0,1]=\{P\in C^{'}[0,1]\,|\,P(1-x)=P(x), 0\le x\le 1\} \qquad (1.1)$$

Then $A_{P.h.H}$ is a spatially symmetric operator if and only if $P\in C_s^{'}[0,1]$ and a h = H.

Let a symmetric operator $A_{P.h.h}$ be given, and let $\{\lambda_n\}_{n=0}^{\infty}$

$-\infty<\lambda_0<\lambda_1<\ldots\to\infty$ be $\sigma(A_{P.h.h})$, the eigenvalues of $A_{P.h.h}$

Furthermore let $\{\lambda_m\}_{m=0}^{\infty}$

$(-\infty<\mu_0<\mu_1<\ldots\to\infty)$ be the eigenvalues of another symmetric operator $A_{q.j.j}$.

1.2 Notation (Spatially symmetric operators)

For $P\in C_s^{'}[0,1]$, and $h\in R$ $A_{P.h.h}$ denotes the symmetric operator

$$-\left(\frac{d^2}{dx^2}\right)+P(x) \text{ in } L^2(0,1)$$

With the boundary condition

$$\left[-\left(\frac{d}{dx}\right)+h\right]\Big|_{x=0} \qquad =\left[-\left(\frac{d}{dx}\right)+h\right]\Big|_{x=1} \quad =0$$

Let $$\sigma(A_{P.h.h})=\{\lambda_n\}_{n=0}^{\infty} \qquad (-\infty<\lambda_0<\lambda_1<\ldots\to\infty)$$

the eigenvalues of $A_{P.h.h}$ are denoted by

$$\lambda_n=\lambda_n(P,h) \qquad (n=0,1,2\ldots)$$

Let a symmetric operator

$$A_{P.h.h}\,(P\in C_s^{'}[0,1], h\in R$$

and a set $$\Sigma\subset N\equiv\{0,1,2\ldots\}$$

be given, and put

$$Q_{P.h.\Sigma}=\{(q,j)\in C_s^{'}[0,1]\; \mathrm{X}\, R \;\,|\lambda_n(q,j)$$
$$=\lambda_n(p,h)(n\in N\setminus\Sigma)\} \qquad (1.2)$$

$Q_{P.h.\Sigma}$, denotes the totality of symmetric operator $A_{q.j.j}$

Whose eigenvalues $\lambda_n(q.j)$ coincide with those of $A_{p.h.h}$, expect for $n \in \Sigma$.

1.3 Notation (Spatially symmetric operators)

For $P \in C^{'}[0,1]$, $h \in R$ and $\lambda \in R$,

$\phi = \phi(x : p, h, \lambda)$ and $\phi^* = \phi^*(x : p, h, \lambda)$ denote the solution of

$$\left[-\left(\frac{d^2}{dx^2}\right) + P(x)\right]\phi = \lambda\phi\,(0 \le x \le 1), \phi(0) = 1, \phi^{'}(0) = h \qquad (1.3)$$

And $$\left[-\left(\frac{d^2}{dx^2}\right) + P(x)\right]\phi^* = \lambda\phi^*\,(0 \le x \le 1),$$

$$\phi^*(0) = 1,\ \phi^{*'}(0) = 1 \qquad (1.4)$$

respectively.

Hence forth, means $\frac{d}{dx}$.

For $\lambda = \lambda_n(p, h) \quad n \in N,$

$\phi(.; p, h, \lambda)$ become an eigenfunction of $A_{p.h.h}$

On the other hand $< \phi, \phi^* >$ gives a fundamental system of the solution of\

$$\left[-\left(\frac{d^2}{dx^2}\right) + P(x)\right]\psi = \lambda\psi$$

1.4 Notation (Spatially symmetric operators)

For $P \in C^{'}[0,1]$, A^*_p denotes the realization $L^2[0,1]$ of the differential operator

$-\left(\frac{d^2}{dx^2}\right) + P(x)$ with the (Dirichlet) boundary condition $.|_{x=0} = .|_{x=1} = 0$

The eigenvalues of A^*_p, $\sigma(A^*_p) = \{\lambda^*_n\}_{n=1}^{\infty}$ $\quad \{-\infty < \lambda^*_1 < \lambda^*_2 < \ldots \to \infty\}$

are denoted by $\lambda^*_n = \lambda^*_n(p) \quad for \quad n \in N^* \equiv \{1, 2, \ldots\}$

Note that in our notation, $\sigma(A_{p.h.h})$ are numbered from 0.

while $\sigma(A_p^*)$ are numbered from 1.

For $$\lambda = \lambda_n^*(p)\ \ (n \geq 1),\ \phi^*(.;p,\lambda)$$

Becomes an eigefunction of A^*_p

1.5 Notation (Spatially symmetric operators)

For $n_1 \geq 1,\ P \in C^{'}[0,1]$, and $h \in R$

$$w = w(x; p, h, n_1)$$

$$= \phi^{*'}(x; p, \lambda_{n_1}^*(p))\ \phi(x; p, h, \lambda_{n_1}(p,h))$$

$$-\phi^*(x; p, \lambda_{n_1}^*(p))\ \phi^{'}(x; p, h, \lambda_{n_1}(p,h)) \tag{1.5}$$

As will be shown in

$$P \in C^{'}_s[0,1],\ \text{implies}\ \left(\frac{d^2}{dx^2}\right)\log(w) \in C^{'}_s[0,1].$$

In the case of

$$\lambda_{n_1}^*(p) = \lambda_{n_1}(p,h)$$

W becomes the wronskian for

$$-\psi^{''} + p\psi = \lambda\psi\ \ (\lambda = \lambda_{n_1}^*(p) = \lambda_{n_1}(p,h)).$$

Hence w' = 0 and so

$$\left(\frac{d^2}{dx^2}\right)\log(w) = (w'/\,w') = 0$$

We have conversely, that $(w'/\,w') = 0$

implies $\lambda_{n_1}^* = \lambda_{n_1}(p,h)$

1.6 proposition

For each $f \in c^2(\overline{AC})$ and $g \in c^2(\overline{BC})$ with $f_{|C} = g_{|C}$, there exists a unique $k = k(x,y) \in c^2(\overline{\Omega})$

and $k_{|AC} = f, k_{|BC} = g$ (1.6)

1.7 proposition

For each $f \in c^2(\overline{AB})$ and $g \in c^{'}(\overline{\mathrm{AB}})$ there exists a unique $k = k(x,y) \in c^2(\overline{\Omega})$)

and $k_{|AB} = f, \frac{\partial}{\partial v} k_{|AB} = g$ (1.7)

1.8 proposition

For each $f \in c^2(\overline{AC}),\ g \in c^2(\overline{BC})$ Wight $f_{|A} = g_{|A}$, there exists a unique $k = k(x,y) \in c^2(\overline{\Omega})$

Such that $k_{|AC} = f, k_{|AB} = g$

1.9 proposition

For each $f \in c^2(\overline{AC})$ and $g \in c^{'}(\overline{AB})$ and $h \in R$, there exists a unique $k = k(x,y) \in c^2(\overline{\Omega})$

Such that $k_{|AC} = f, \quad \frac{\partial}{\partial v} k + hk_{|AB} = g$ (1.8)

By the d'Alembert formula,

We can given the solution

$k_0 = k_0(x,y) \in c^2(\overline{\Omega})$ of

$$k_{xx} - k_{yy} = 0 \ on(\overline{\Omega}) \quad (1.9)$$

Satisfying the boundary condition given in these propositions.

On the other hand, for each

$F = F(x,y) \in c^0(\overline{\Omega}),$

The solution $k = k(x,y) \in c^2(\overline{\Omega})$ of

$$k_{xx} - k_{yy} = F(x,y) \ on(\overline{\Omega}) \quad (1.10)$$

Satisfying the bo undary condition for f = g = 0, is also given in a similar way

1.10 Therom

Given $p, q \in C'[0,1]$ and $h, j \in R$, there exists a unique $k = k(x,y) = k(x,y;q,j;p,h) \in c^2(\overline{D})$

such that $k_{xx} - k_{yy} + p(y)k = q(x)k(\overline{D})$ (1.11)

$$k(x,x)=(j-h)+\frac{1}{2}\int_0^x \left(q(s)-p(s)\right)ds \quad 0\le x\le 1 \qquad (1.12)$$

$$k_y(x,0)=hk(x,0) \quad 0\le x\le 1 \qquad (1.13)$$

Proof

To show the existence of $k=k(;,;q,j;p,h)$,

We extend $q\in C'[0,1]$ to $q\in C'[0,2]$

Set $D(x,y)|0<y<x<(2-y)$

By proposition (1.9)

There exists $k=k(x,y)\in c^2(D)$

Such that $k_{xx}-k_{yy}+p(y)k=q(x)k(\overline{D})$ (1.14)

$$k.(x,x)=(j-h)+\frac{1}{2}\int_0^x \left(q(s)-p(s)\right)ds \quad 0\le x\le 1 \qquad (1.15)$$

$$k_y(x,0)=hk(x,0) \quad 0\le x\le 2 \qquad (1.16)$$

The restriction $k=k|\overline{D}\in c^2(\overline{D})$ satisfies (1.10).

To verify the uniqueness, we divide D into

$$\Omega_1=\{(x,y)|0<y<x<1-y\}\, and\, \Omega_2=D\overline{\Omega_1}$$

We prove that (1.11)

$$k(x,x)=0 \quad (0\le x\le 1),$$

(1.12) and $k\in c^2(\overline{D})$ Imply k = 0

In fact, we first have k = 0 on $\overline{\Omega_1}$ by proposition (1.9)

so that we have k = 0 on $\overline{\Omega_2}$ by proposition (1.8)

1.11 Theorem

First deformation formula

The function $\phi=\phi(.;,p,h,\lambda)$ defined in notation (1.3).Then for $k=k(;.;q,j;p,h)$ is theorem (1.10), the identity

$$\phi(x;q,j,\lambda)=\phi(x;p,h,\lambda)+\int_0^x k(x,y;q,j;p,h)\ \phi(y;p,y)dy \qquad (1.17)$$

holds for q, $p\in C'[0,1]\ j,h\in R\ and\ \lambda\in R$

Note

We finally note the following facts on the eigenfunctions

$\{\phi(.;p,h,\lambda_n(p,h)\}_{n=0}^{\infty}$ of a symmetric operator $A_{p.h.h}$

Put, for the moment $\phi_n(x)=\phi(x;p,h,\lambda_n(p,h)) \quad (n=0,1,....)$

Let $p\in C_s^{'}[0,1]\ and\ h\in R$

1.12 Theorem

We have $\phi_n(1-x)=(-1)^n\phi_n(x) \qquad (\mathrm{n}\in\mathrm{N},\mathrm{x}\in[0,1]) \qquad (1.18)$

Proof

Since $A_{p.h.h}$ is symmetric,

$\phi_n(1-x)=c_n\phi_n(x) \qquad (0\le \mathrm{x}\le 1)$

Holds of the for some $c_n\in R$ because of the uniqueness of the Cauchy problem,

From $c_n\neq 1$ and $c_n\neq -1$ follow

$$\phi_n\left(\frac{1}{2}\right)=0\quad and\quad \phi_n^{'}\left(\frac{1}{2}\right)=0$$

respectively, while

$$\phi_n\left(\frac{1}{2}\right)=\ \phi_n^{'}\left(\frac{1}{2}\right)=0$$

cannot occur simultaneously

hence $c_n=\pm 1$ hence

By sturm-Liouville theorem

ϕ_n has non-zeros an (0,1) hence (1.18) holds

Conclusion

In this article we've investigated Strum-Liouvillie problems in spatially symmetric operators of eigenvalues and eigenfunctions , eigenvalue and special function are discussed. Eigenvalues and eigan function expantion of spatially symmetric operators of strum – Liouville are lustrated.

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