GENERALIZED HYERS-ULAM TYPE STABILITY OF THE $2k$-VARIABLE ADDITIVE β-FUNCTIONAL INEQUALITIES AND EQUATIONS IN COMPLEX BANACH SPACES

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ABSTRACT. *In this paper we study to solve two additive $\beta-$functional inequality the $2k-$variables and their $Hyers-Ulam$ stability. First are investigated in complex Banach spaces and last are investigated the $Hyers-Ulam$ stability of additive $\beta-$functional equation associated with the additive $\beta-$functional inequalities in complex Banach spaces. Then $I'll$ show that the solutions of first and second inqualities are additive mappings. Then $Hyers-Ulam$ stability of these inequalities are given and proven. These are the main results of this paper.*

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1. INTRODUCTION

Let $\mathbb{X}$ and $\mathbb{Y}$ be a normed spaces on the same field $\mathbb{K}$, and $f:\mathbb{X}\to\mathbb{Y}$. We use the notation $\Big\|\cdot\Big\|$ for all the norm on both $\mathbb{X}$ and $\mathbb{Y}$. In this paper, we investisgate some additive β-functional inequality when $\mathbb{X}$ is a real or complex normed space and $\mathbb{Y}$ is a complex Banach spaces.
In fact, when $\mathbb{X}$ is a real or complex normed space and $\mathbb{Y}$ is a complex Banach spaces we solve and prove the Hyers-Ulam stability of two forllowing additive β-functional inequality.

$$\begin{aligned}&\left\|f\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\Big(x_j\Big)\right\|_{\mathbb{Y}}\\&\qquad\leq\left\|\beta\left(kf\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\Big(x_j\Big)\right)\right\|_{\mathbb{Y}}\end{aligned} \tag{1.1}$$

$$\left\|kf\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\left(x_j\right)\right\|_{\mathbb{Y}}$$
$$\leq\left\|\beta\left(f\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+1}}{k}\right)-\sum_{j=1}^{k}f\left(x_j\right)\right)\right\|_{\mathbb{Y}}, \tag{1.2}$$

where β is a fixed complex number with $\left|\beta\right| < 1$ and k be a fixed integer with $k \geq 2$.

The Hyers-Ulam stability was first investigated for functional equation of Ulam in [6] concerning the stability of group homomorphisms.
The functional equation

$$f\left(x+y\right)=f\left(x\right)+f\left(y\right)$$

is called the Cauchy equation. In particular, every solution of the Cauchy equation is said to be an additive mapping.

The Hyers [7] gave firts affirmative partial answer to the equation of Ulam in Banach spaces. After that, Hyers' Theorem was generalized by Aoki[1] additive mappings and by Rassias [8] for linear mappings considering an unbouned Cauchy diffrence. Ageneralization of the Rassias theorem was obtained by Găvruta [4] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

$$f\left(\frac{x+y}{2}\right)=\frac{1}{2}f\left(x\right)+\frac{1}{2}f\left(y\right)$$

is called the *Jensen equation*. See [2, 3, 9, 10] for more information on functional equations.

The Hyers-Ulam stability for functional inequalities have been investigated such as in [14, 15]. Gilány showed that is if satisfies the functional inequality

$$\left\|2f\left(x\right)+2f\left(y\right)-f\left(xy^{-1}\right)\right\|\leq\left\|f\left(xy\right)\right\|$$

then f satisfies the Jordan-von Newman functional equation

$$2f\left(x\right)+2f\left(y\right)=f\left(xy\right)+f\left(xy^{-1}\right). \tag{1.3}$$

See also [15,16]. Gilányi [13] and Fechner [9] proved the Hyers-Ulam stability of the functional inequality.
Choonkil *Park* [17, 18] proved the Hyers-Ulam stability of additive β-functional inequalities. Recently, in [17, 18, 19, 22] the authors studied the Hyers-Ulam stability for the following functional inequalities

$$\left\|f\left(x+y\right)-f\left(x\right)-f(y)\right\|\leq\left\|\rho\left(2f\left(\frac{x+y}{2}\right)-f\left(x\right)-f\left(y\right)\right)\right\| \tag{1.4}$$

$$\left\|2f\left(\frac{x+y}{2}\right)-f\left(x\right)-f\left(y\right)\right\|\leq\left\|\rho\left(f\left(x+y\right)-f\left(x\right)-f\left(y\right)\right)\right\|. \tag{1.5}$$

Next

$$\left\|f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x+y}{2}\right)-f\left(z\right)\right\|\leq\left\|\gamma\left(2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f\left(z\right)\right)\right\| \tag{1.6}$$

and

$$\left\|2f\left(\frac{x+y}{2^2}+\frac{z}{2}\right)-f\left(\frac{x+y}{2}\right)-f\left(z\right)\right\|\leq\left\|\gamma\left(f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x+y}{2}\right)-f\left(z\right)\right)\right\|. \tag{1.7}$$

Final

$$\left\|f\left(\sum_{i=1}^{n}x_i\right)-\sum_{i=1}^{n}f\left(x_i\right)\right\|\leq\left\|\gamma\left(nf\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)-\sum_{i=1}^{n}f\left(x_i\right)\right)\right\| \tag{1.8}$$

and

$$\left\|nf\left(\frac{1}{n}\sum_{i=1}^{k}x_i\right)-\sum_{i=1}^{n}f\left(x_n\right)\right\|\leq\left\|\gamma\left(f\left(\sum_{i=1}^{n}x_i\right)-\sum_{i=1}^{n}f\left(x_i\right)\right)\right\| \tag{1.9}$$

in complex Banach spaces.

In this paper, we solve and proved the Hyers-Ulam type stability for two β-functional inequalities (1.1)-(1.2), ie the β -functional inequalities with $2k-variables$. Under suitable assumptions on spaces $\mathbb{X}$ and $\mathbb{Y}$, we will prove that the mappings satisfying the β-functional inequatilies (1.1) or (1.2). Thus, the results in this paper are generalization of those in [17, 18, 19, 22] for β-functional inequatilies with $2k-variables$.

The paper is organized as followns: In section preliminarier we remind some basic notations in [18, 19, 20] such as We only redefine the solution definition of the equation of the additive function.

Section 3: is devoted to prove the Hyers-Ulam stability of the addive β- functional inequalities (1.1) when we assume that $\mathbb{G}$ be a $2k-divisible$ abelian group and $\mathbb{X}$ is a real or complex normed space and $\mathbb{Y}$ complex Banach space.

Section 4: is devoted to prove the Hyers-Ulam stability of the addive β- functional inequalities (1.2) when $\mathbb{X}$ is a real or complex normed space and $\mathbb{Y}$ complex Banach space.

2. PRELIMINARIER

The functional equation

$$f\left(x+y\right)=f\left(x\right)+f\left(y\right)$$

is called the cauchuy equation. In particular, every solution of the cauchuy equation is said to be an *additive mapping*.

Now, we first study the solutions of (1.1). Note that for these inequalities, $\mathbb{X}$ is a real or complex normed space and $\mathbb{Y}$ is a complex Banach spaces. Under this setting, we can show that the mapping satisfying (1.1) is additive. These results are give in the following.

Lemma 3.1. *A mapping* $f : \mathbb{G} \to \mathbb{Y}$ *satilies*

$$\begin{aligned} &\left\| f\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} f\big(x_j\big)\right\|_{\mathbb{Y}} \\ &\quad \leq \left\| \beta\left(kf\left(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} + \frac{1}{k}\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} f\big(x_j\big)\right)\right\|_{\mathbb{Y}} \end{aligned} \tag{3.1}$$

for all $x_j, x_{k+j} \in \mathbb{G}$ *for all* $j = 1 \to k$ *if and only if* $f : \mathbb{G} \to \mathbb{Y}$ *is additive.*

Proof. Assume that $f : \mathbb{G} \to \mathbb{Y}$ satisfies (3.1).
Letting $x_j = x_{k+j} = 0, j = 1 \to k$ in (3.1), we get

$$\left(\left|2k-1\right| - \left|k\beta\right|\right)\left\|f\big(0\big)\right\|_{\mathbb{Y}} \leq 0.$$

So $f\big(0\big) = 0$.
Letting $x_{k+j} = 0$ and $x_j = x$ for all $j = 1 \to k$ in (3.1), we get

$$\left\|f\big(kx\big) - kf\big(x\big)\right\|_{\mathbb{Y}} \leq 0$$

and so $\mathrm{f}\big(\mathrm{kx}\big) = \mathrm{kf}\big(\mathrm{x}\big)$ for all $x \in \mathbb{G}$.
Thus

$$f\left(\frac{x}{k}\right) = \frac{1}{k}f(x) \tag{3.2}$$

for all $x \in \mathbb{G}$ It follows from (3.1) and (3.2) that:

$$\begin{aligned} &\left\| f\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} f\big(x_j\big)\right\|_{\mathbb{Y}} \\ &\qquad \leq \left\| \beta\left(kf\left(\frac{1}{k^2}\sum_{j=1}^{k} x_{k+j} + \frac{1}{k}\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f\left(\frac{x_{k+1}}{k}\right) - \sum_{j=1}^{k} f\big(x_j\big)\right)\right\|_{\mathbb{Y}} \\ &\qquad = \left|\beta\right|\left\| f\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+1} + \sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} f(x_j)\right)\right\|_{\mathbb{Y}} \end{aligned} \tag{3.3}$$

and so

$$f\left(\frac{1}{k}\sum_{j=1}^{k} x_{k+j} + \sum_{j=1}^{k} x_j\right) = \sum_{j=1}^{k} f\left(\frac{x_{k+1}}{k}\right) + \sum_{j=1}^{k} f\big(x_j\big)$$

for all $x_j, x_{k+j} \in \mathbb{G}$ for all $j = 1 \to k$. Hence $f : \mathbb{G} \to \mathbb{Y}$ is additive.
The coverse is obviously true. ☐

Lemma 3.2. *A mapping* $f:\mathbb{G}\to\mathbb{Y}$ *satisfy* $f(0)=0$ *and*

$$\left\|kf\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\Big(x_j\Big)\right\|_{\mathbb{Y}} \leq \left\|\beta\left(f\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\Big(x_j\Big)\right)\right\|_{\mathbb{Y}} \tag{3.4}$$

for all $x_j, x_{k+j}\in\mathbb{G}$ for all $j=1\to k$ if and if $f:\mathbb{G}\to\mathbb{Y}$ is additive.

Proof. Assume that $f:\mathbb{G}\to\mathbb{Y}$ (3.4).
Letting $x_1=x$ and $x_{j+1}=x_{k+j}=0$ for all $j=1\to k$ in (3.4), we get

$$\left\|kf\left(\frac{x}{k}\right)-f\Big(x\Big)\right\|_{\mathbb{Y}}\leq 0$$

and so

$$f\left(\frac{x}{k}\right)=\frac{1}{k}f\Big(x\Big) \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned}&\left\|kf\left(\frac{1}{k^2}\sum_{j=1}^{k}x_{k+j}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\Big(x_j\Big)\right\|_{\mathbb{Y}}\\ &=\left\|f\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+1}}{k}\right)-\sum_{j=1}^{k}f\Big(x_j\Big)\right\|_{\mathbb{Y}}\\ &\leq\Big|\beta\Big|\left\|f\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\Big(x_j\Big)\Big)\right\|_{\mathbb{Y}}\end{aligned}$$

and so

$$f\left(\frac{1}{k}\sum_{j=1}^{k}x_{k+j}+\sum_{j=1}^{k}x_j\right)=\sum_{j=1}^{k}f\left(\frac{x_{k+1}}{k}\right)+\sum_{j=1}^{k}f\Big(x_j\Big)$$

for all $x_j, x_{k+j}\in\mathbb{G}$ for all $j=1\to k$. Hence $f:\mathbb{G}\to\mathbb{Y}$ is additive.
The converse is obvoiusly true. □

From Lemma 3.1 and Lemma 3.2 we have Corollarys.

Corollary 3.3. *A mapping* $f:\mathbb{G}\to\mathbb{Y}$*satilies*

$$\begin{aligned}&f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}z_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\Big(x_j\Big)\\ &=\beta\left(kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\Big(x_j\Big)\right)\end{aligned} \tag{3.6}$$

for all $x_j, x_{k+j}\in\mathbb{G}$ *for all* $j=1\to k$ *if and only if* $f:\mathbb{G}\to\mathbb{Y}$ *is additive.*

Corollary 3.4. *A mapping $f : \mathbb{G} \to \mathbb{Y}$ satilies $f(0) = 0$ and*

$$kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\Big(x_j\Big)$$
$$=\beta\left(f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\Big(x_j\Big)\right) \tag{3.7}$$

for all $x_j, x_{k+j} \in \mathbb{G}$ for all $j = 1 \to k$ if and only if $f : \mathbb{G} \to \mathbb{Y}$ is additive.

* Note: The equations (3.6) and (3.7) is called an additive $\beta - functional\ equations$.

Theorem 3.5. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : \mathbb{X} \to \mathbb{Y}$ be mapping such that*

$$\left\|f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\Big(x_j\Big)\right\|_{\mathbb{Y}}$$
$$\leq\left\|\beta\left(kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\Big(x_j\Big)\right)\right\|_{\mathbb{Y}}$$
$$+\theta\left(\sum_{j=1}^{k}\Big\|x_j\Big\|^r+\sum_{j=1}^{k}\Big\|x_{k+j}\Big\|^r\right) \tag{3.8}$$

for all $x_j, x_{k+j} \in \mathbb{X}$, for all $j = 1 \to k$. Then there exists a unique additive mapping $Q : \mathbb{X} \to \mathbb{Y}$ such that

$$\Big\|f\Big(x\Big)-Q\Big(x\Big)\Big\|_{\mathbb{Y}}\leq\frac{k\theta}{k^r-k}\Big\|x\Big\|^r \tag{3.9}$$

for all $x \in \mathbb{X}$.

Proof. Letting $x_j = x_{k+j} = 0$ for all $j = 1 \to k$ in (3.8), we get

$$\left(\Big|2k-1\Big|-\Big|k\beta\Big|\right)\Big\|f\Big(0\Big)\Big\|_{\mathbb{Y}}\leq 0. \tag{3.10}$$

So

$$f\Big(0\Big)=0.$$

Letting $x_{k+j} = 0$, $x_j = x$ for all $j = 1 \to k$ in (3.8), we get

$$\left\|f\Big(kx\Big)-kf\Big(x\Big)\right\|_{\mathbb{Y}}\leq\theta k\Big\|x\Big\|^r \tag{3.11}$$

$$\left\|f\Big(x\Big)-kf\Big(\frac{x}{k}\Big)\right\|_{\mathbb{Y}}\leq\frac{k\theta}{k^r}\Big\|x\Big\|^r$$

$$\left\|k^l f\left(\frac{x}{k^l}\right) - k^m f\left(\frac{x}{k^m}\right)\right\|_{\mathbb{Y}} \leq \sum_{j=l}^{m-1} \left\|k^j f\left(\frac{x}{k^j}\right) - k^{j+1} f\left(\frac{x}{k^{j+1}}\right)\right\|_{\mathbb{Y}} \leq \frac{k\theta}{k^r} \sum_{j=l}^{m-1} \frac{k^j}{k^{rj}} \left\|x\right\|^r \tag{3.12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbb{X}$. It follows from (3.12) that the sequence $\left\{k^n f\left(\frac{x}{k^n}\right)\right\}$ is a cauchy sequence for all $x \in \mathbb{X}$. Since $\mathbb{Y}$ is complete space, the sequence $\left\{k^n f\left(\frac{x}{k^n}\right)\right\}$ coverges.
So one can define the mapping $Q : \mathbb{X} \to \mathbb{Y}$ by

$$Q(x) := \lim_{n\to\infty} k^n f\left(\frac{x}{k^n}\right)$$

for all $x \in \mathbb{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.12), we get (3.9).
Now, It follows from (3.8) that

$$\begin{aligned}
&\left\|Q\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k} + \sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} Q\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} Q\left(x_j\right)\right\|_{\mathbb{Y}} \\
&= \lim_{n\to\infty} k^n \left\|f\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^{n+1}} + \frac{1}{k^n}\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k^{n+1}}\right) - \sum_{j=1}^{k} f\left(\frac{x_j}{k^n}\right)\right\|_{\mathbb{Y}} \\
&\leq \lim_{n\to\infty} k^n \left\|\beta\left(kf\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^{n+2}} + \frac{1}{k^n}\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k^{n+1}}\right) - \sum_{j=1}^{k} f\left(\frac{x_j}{k^n}\right)\right\|_{\mathbb{Y}} \\
&+ \lim_{n\to\infty} \frac{k^n\theta}{k^{nr}} \left(\sum_{j=1}^{k} \left\|x_j\right\|^r + \sum_{j=1}^{k} \left\|x_{k+j}\right\|^r\right) \\
&= \left\|\beta\left(kf\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^2} + \frac{1}{k}\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} f\left(x_j\right)\right)\right\|_{\mathbb{Y}}
\end{aligned} \tag{3.13}$$

for all $x_j, x_{k+j} \in \mathbb{X}$, for all $j = 1 \to k$.
So

$$\begin{aligned}
&\left\|Q\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k} + \sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} Q\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} Q\left(x_j\right)\right\|_{\mathbb{Y}} \\
&\quad \leq \left\|\beta\left(kQ\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^2} + \frac{1}{k}\sum_{j=1}^{k} x_j\right) - \sum_{j=1}^{k} Q\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} Q\left(x_j\right)\right\|_{\mathbb{Y}}
\end{aligned} \tag{3.14}$$

for all $x_j, x_{k+j} \in \mathbb{X}$, for all $j = 1 \to k$. By Lemma (3.1), the mapping $Q : \mathbb{X} \to \mathbb{Y}$ is additive.
Next, suppose that $T : \mathbb{X} \to \mathbb{Y}$ be another additive mapping satisfying (3.9). Then we

have

$$\begin{aligned}\left\|Q\left(x\right)-T\left(x\right)\right\|_{\mathbb{Y}} &= k^{n}\left\|Q\left(\frac{x}{k^{n}}\right)-T\left(\frac{x}{k^{n}}\right)\right\|_{\mathbb{Y}} \\ &\leq k^{n}\left(\left\|Q\left(\frac{x}{k^{n}}\right)-f\left(\frac{x}{k^{n}}\right)\right\|_{\mathbb{Y}}+\left\|T\left(\frac{x}{k^{n}}\right)-f\left(\frac{x}{k^{n}}\right)\right\|_{\mathbb{Y}}\right) \\ &\leq \frac{2k^{n+1}\theta}{\left(k^{r}-k\right)k^{nr}}\left\|x\right\|^{r} \qquad (3.15)\end{aligned}$$

which tends to zero as$n\to\infty$ for all $x\in\mathbb{X}$. So we can conclude that $Q\left(x\right)=T\left(x\right)$ for all $x\in\mathbb{X}$. This proves the uniqueness of Q. Thus the mapping $Q:\mathbb{X}\to\mathbb{Y}$ is a unique additive mapping satisfying (3.9). □

Theorem 3.6. *Let $r<1$ and θ be nonnegative real numbers, and let $f:\mathbb{X}\to\mathbb{Y}$ be mapping such that*

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_{j}\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f(x_{j})\right\|_{\mathbb{Y}} \\ &\leq\left\|\beta\left(kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^{2}}+\frac{1}{k}\sum_{j=1}^{k}x_{j}\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f(x_{j})\right)\right\|_{\mathbb{Y}} \\ &+\theta\left(\sum_{j=1}^{k}\left\|x_{j}\right\|^{r}+\sum_{j=1}^{k}\left\|x_{k+j}\right\|^{r}\right) \qquad (3.16)\end{aligned}$$

for all $x_j, x_{k+j}\in\mathbb{X}$, for all $j=1\to k$. Then there exists a unique additive mapping $Q:\mathbb{X}\to\mathbb{Y}$ such that

$$\left\|f\left(x\right)-Q\left(x\right)\right\|_{\mathbb{Y}}\leq\frac{k\theta}{k-k^{r}}\left\|x\right\|^{r} \qquad (3.17)$$

for all $x\in\mathbb{X}$

Proof. Letting $x_j=x_{k+j}=0$ for all $j=1\to k$ in (3.16), we get

$$\left(\left|2k-1\right|-\left|k\beta\right|\right)\left\|f\left(0\right)\right\|_{\mathbb{Y}}\leq 0. \qquad (3.18)$$

So

$$f\left(0\right)=0.$$

Letting $x_{k+j}=0$, $x_j=x$ for all $j=1\to k$ in (3.14), we get

$$\left\|f\left(kx\right)-kf\left(x\right)\right\|_{\mathbb{Y}}\leq\theta k\left\|x\right\|^{r} \qquad (3.19)$$

$$\left\|f\left(x\right)-\frac{1}{k}f\left(kx\right)\right\|_{\mathbb{Y}}\leq\theta\left\|x\right\|^{r}$$

Hence

$$\left\|\frac{1}{k^l}f\left(k^lx\right)-\frac{1}{k^m}f\left(k^mx\right)\right\|_{\mathbb{Y}}\leq\sum_{j=l}^{m-1}\left\|\frac{1}{k^j}f\left(k^jx\right)-\frac{1}{k^{j+1}}f\left(k^{j+1}x\right)\right\|_{\mathbb{Y}}\leq\theta\sum_{j=l}^{m-1}\frac{k^{jr}}{k^j}\left\|x\right\|^r \tag{3.20}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbb{X}$. It follows from (3.20) that the sequence $\left\{\frac{1}{k^n}f\left(k^nx\right)\right\}$ is a cauchy sequence for all $x \in \mathbb{X}$. Since $\mathbb{Y}$ is complete space, the sequence $\left\{\frac{1}{k^n}f\left(k^nx\right)\right\}$ coverges.
So one can define the mapping $Q : \mathbb{X} \to \mathbb{Y}$ by

$$Q\left(x\right):=\lim_{n\to\infty}\frac{1}{k^n}f\left(k^nx\right)$$

for all $x \in \mathbb{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.20), we get (3.17). The rest of the proof is similar to the proof of Theorem (3.3).
By the triangle, we have

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f(x_j)\right\|_{\mathbb{Y}}\\&-\left\|\beta\left(kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f(x_j)\right)\right\|_{\mathbb{Y}}\\&\leq\left\|f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\left(x_j\right)\right.\\&\left.-\beta\left(kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\left(x_j\right)\right)\right\|_{\mathbb{Y}}.\end{aligned} \tag{3.21}$$

□

From Theoem 3.5 and Theoem 3.6 we have Corollarys.

Corollary 3.7. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : \mathbb{X} \to \mathbb{Y}$ be mapping such that*

$$\begin{aligned}&\left\|f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\left(x_j\right)\right\|_{\mathbb{Y}}\\&\leq\left\|\beta\left(kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\left(x_j\right)\right)\right\|_{\mathbb{Y}}\\&+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|^r+\sum_{j=1}^{k}\left\|x_{k+j}\right\|^r\right)\end{aligned} \tag{3.22}$$

for all $x_j, x_{k+j} \in \mathbb{X}$, *for all* $j = 1 \to k$. *Then there exists a unique additive mapping* $Q : \mathbb{X} \to \mathbb{Y}$ *such that*

$$\left\| f\left(x\right) - Q\left(x\right) \right\|_{\mathbb{Y}} \leq \frac{k\theta}{k^r - k} \left\| x \right\|^r \tag{3.23}$$

for all $x \in \mathbb{X}$.

Corollary 3.8. *Let* $r < 1$ *and* θ *be nonnegative real numbers, and let* $f : \mathbb{X} \to \mathbb{Y}$ *be mapping such that*

$$\begin{aligned} &\left\| f\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k} + \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k} \right) - \sum_{j=1}^{k} f(x_j) \right\|_{\mathbb{Y}} \\ &\leq \left\| \beta \left(kf\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^2} + \frac{1}{k} \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k} \right) - \sum_{j=1}^{k} f(x_j) \right) \right\|_{\mathbb{Y}} \\ &+ \theta \left(\sum_{j=1}^{k} \left\| x_j \right\|^r + \sum_{j=1}^{k} \left\| x_{k+j} \right\|^r \right) \end{aligned} \tag{3.24}$$

for all $x_j, x_{k+j} \in \mathbb{X}$, *for all* $j = 1 \to k$. *Then there exists a unique additive mapping* $Q : \mathbb{X} \to \mathbb{Y}$ *such that*

$$\left\| f\left(x\right) - Q\left(x\right) \right\|_{\mathbb{Y}} \leq \frac{k\theta}{k - k^r} \left\| x \right\|^r \tag{3.25}$$

for all $x \in \mathbb{X}$

4. Additive β-functional inequality in complex Banach space

Now, we study the solutions of (1.2). Note that for these inequalities, $\mathbb{X}$ is a real or complex normed space and $\mathbb{Y}$ is complex Banach spaces. Under this setting, we can show that the mapping satisfying (1.2) is additive. These results are give in the following.

Theorem 4.1. *Let* $r > 1$ *and* θ *be nonnegative real numbers, and let* $f : \mathbb{X} \to \mathbb{Y}$ *be mapping with* $f\left(0\right) = 0$ *such that*

$$\begin{aligned} &\left\| kf\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^2} + \frac{1}{k} \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k} \right) - \sum_{j=1}^{k} f\left(x_j\right) \right\|_{\mathbb{Y}} \\ &\leq \left\|_{\mathbb{Y}} \beta \left(f\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k} + \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k} \right) - \sum_{j=1}^{k} f(x_j) \right) \right\|_{\mathbb{Y}} \\ &+ \theta \left(\sum_{j=1}^{k} \left\| x_j \right\|^r + \sum_{j=1}^{k} \left\| x_{k+j} \right\|^r \right) \end{aligned} \tag{4.1}$$

for all $x_j, x_{k+j} \in X$, *for all* $j = 1 \to k$. *Then there exists a unique additive mapping* $Q : \mathbb{X} \to \mathbb{Y}$ *such that*

$$\left\| f\left(x\right) - Q\left(x\right) \right\|_{\mathbb{Y}} \leq \frac{k^r\theta}{k^r - k} \left\| x \right\|^r \tag{4.2}$$

for all $x \in \mathbb{X}$.

Proof. Letting $x_{j+1} = x_{k+j} = 0$ and $x_1 = x$ for all $j = 1 \to k$ in (4.1), we get

$$\left\| kf\left(\frac{x}{k}\right) - f\left(x\right) \right\|_{\mathbb{Y}} \leq \theta \left\| x \right\|^r \tag{4.3}$$

for all $x \in \mathbb{X}$. Hence

$$\left\| k^l f\left(\frac{1}{k^l}x\right) - k^m f\left(\frac{1}{k^m}x\right) \right\|_{\mathbb{Y}} \leq \sum_{j=l}^{m-1} \left\| k^j f\left(\frac{x}{k^j}\right) - k^{j+1} f\left(\frac{x}{k^{j+1}}\right) \right\|_{\mathbb{Y}} \leq \theta \sum_{j=l}^{m-1} \frac{k^j}{k^{rj}} \left\| x \right\|^r \tag{4.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbb{X}$. It follows from (4.4) that the sequence $\left\{ k^n f\left(\frac{x}{k^n}\right) \right\}$ is a cauchy sequence for all $x \in \mathbb{X}$. Since $\mathbb{Y}$ is complete space, the sequence $\left\{ k^n f\left(\frac{x}{k^n}\right) \right\}$ coverges.
So one can define the mapping $Q : \mathbb{X} \to \mathbb{Y}$ by

$$Q\left(x\right) := \lim_{n\to\infty} k^n f\left(\frac{x}{k^n}\right)$$

for all $x \in \mathbb{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4,4), we get (4.2). The rest of the proof is similar to the proof of theorem (3.5) □

Theorem 4.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : \mathbb{X} \to \mathbb{Y}$ be mapping with $f\left(0\right) = 0$ such that*

$$\begin{aligned} &\left\| kf\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k^2} + \frac{1}{k} \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} f\left(x_j\right) \right\|_{\mathbb{Y}} \\ &\leq \left\| \beta\left(f\left(\sum_{j=1}^{k} \frac{x_{k+j}}{k} + \sum_{j=1}^{k} x_j \right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} f(x_j) \right) \right\|_{\mathbb{Y}} \\ &+ \theta\left(\sum_{j=1}^{k} \left\| x_j \right\|^r + \sum_{j=1}^{k} \left\| x_{k+j} \right\|^r \right) \end{aligned} \tag{4.5}$$

for all $x_j, x_{k+j} \in \mathbb{X}$, for all $j = 1 \to k$. Then there exists a unique additive mapping $Q : \mathbb{X} \to \mathbb{Y}$ such that

$$\left\| f\left(x\right) - Q\left(x\right) \right\|_{\mathbb{Y}} \leq \frac{k^r\theta}{k - k^r} \left\| x \right\|^r \tag{4.6}$$

for all $x \in \mathbb{X}$

Proof. Letting $x_{j+1} = x_{k+j} = 0$, $x_1 = x$ for all $j = 1 \to k$ in (4.5), we get

$$\left\| kf\left(\frac{x}{k}\right) - f\left(x\right) \right\|_{\mathbb{Y}} \leq \theta \left\| x \right\|^r \tag{4.7}$$

for all $x \in \mathbb{X}$.
So

$$\left\| f\left(x\right) - \frac{1}{k} f\left(kx\right) \right\|_{\mathbb{Y}} \leq \theta \frac{k^r}{k} \left\| x \right\|^r \tag{4.8}$$

for all $x \in \mathbb{X}$.
Hence

$$\left\|\frac{1}{k^l}f\left(k^lx\right)-\frac{1}{k^m}f\left(k^mx\right)\right\|_{\mathbb{Y}}\leq\sum_{j=l}^{m-1}\left\|\frac{1}{k^j}f\left(k^jx\right)-\frac{1}{k^{j+1}}f\left(k^{j+1}x\right)\right\|_{\mathbb{Y}}\leq\frac{k^r\theta}{k}\sum_{j=l}^{m-1}\frac{k^{jr}}{k^j}\left\|x\right\|^r \tag{4.9}$$

for all nonnegative integers m and l with $m > l$ and all $x \in \mathbb{X}$. It follows from (4.8) that the sequence $\left\{\frac{1}{k^n}f\left(k^nx\right)\right\}$ is a cauchy sequence for all $x \in \mathbb{X}$. Since $\mathbb{Y}$ is complete space, the sequence $\left\{\frac{1}{k^n}f\left(k^nx\right)\right\}$ coverges.
So one can define the mapping $Q : \mathbb{X} \to \mathbb{Y}$ by

$$Q\left(x\right) := \lim_{n\to\infty}\frac{1}{k^n}f\left(k^nx\right)$$

for all $x \in \mathbb{X}$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (4.8), we get (4.6). The rest of the proof is similar to the proof of theorem (3.6). □

By thetriangle inequality, we have

$$\begin{aligned}&\left\|kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{1}{k}\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\frac{1}{k}\sum_{j=1}^{k}f(x_j)\right\|_{\mathbb{Y}}\\&-\left\|\beta\left(f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f(x_j)\right)\right\|_{\mathbb{Y}}\\&\leq\left\|kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\frac{1}{k}\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\frac{1}{k}\sum_{j=1}^{k}f(x_j)\right.\\&\left.-\beta\left(f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f(x_j)\right)\right\|_{\mathbb{Y}}.\end{aligned} \tag{4.10}$$

From Theoem 4.1 and Theoem 4.2 we have Corollarys.

Corollary 4.3. *Let $r > 1$ and θ be nonnegative real numbers, and let $f : \mathbb{X} \to \mathbb{Y}$ be mapping with $f\left(0\right) = 0$ such that*

$$\begin{aligned}&\left\|kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2}+\frac{1}{k}\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\left(x_j\right)\right\|_{\mathbb{Y}}\\&\leq\left\|_{\mathbb{Y}}\beta\left(f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k}+\sum_{j=1}^{k}x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f(x_j)\right)\right\|_{\mathbb{Y}}\\&+\theta\left(\sum_{j=1}^{k}\left\|x_j\right\|^r+\sum_{j=1}^{k}\left\|x_{k+j}\right\|^r\right)\end{aligned} \tag{4.11}$$

for all $x_j, x_{k+j} \in X$, *for all* $j = 1 \to k$. *Then there exists a unique additive mapping* $Q : \mathbb{X} \to \mathbb{Y}$ *such that*

$$\left\| f\left(x\right) - Q\left(x\right)\right\|_{\mathbb{Y}} \leq \frac{k^r\theta}{k^r - k}\left\|x\right\|^r \tag{4.12}$$

for all $x \in \mathbb{X}$.

Corollary 4.4. *Let* $r < 1$ *and* θ *be nonnegative real numbers, and let* $f : \mathbb{X} \to \mathbb{Y}$ *be mapping with* $f\left(0\right) = 0$ *such that*

$$\begin{aligned}&\left\|kf\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k^2} + \frac{1}{k}\sum_{j=1}^{k}x_j\right) - \sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k}f\left(x_j\right)\right\|_{\mathbb{Y}}\\ &\leq \left\|\beta\left(f\left(\sum_{j=1}^{k}\frac{x_{k+j}}{k} + \sum_{j=1}^{k}x_j\right) - \sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k}f(x_j)\right)\right\|_{\mathbb{Y}}\\ &+ \theta\left(\sum_{j=1}^{k}\left\|x_j\right\|^r + \sum_{j=1}^{k}\left\|x_{k+j}\right\|^r\right)\end{aligned} \tag{4.13}$$

for all $x_j, x_{k+j} \in \mathbb{X}$, *for all* $j = 1 \to k$. *Then there exists a unique additive mapping* $Q : \mathbb{X} \to \mathbb{Y}$ *such that*

$$\left\| f\left(x\right) - Q\left(x\right)\right\|_{\mathbb{Y}} \leq \frac{k^r\theta}{k - k^r}\left\|x\right\|^r \tag{4.14}$$

for all $x \in \mathbb{X}$.

Remak: If β is a real number such that $-1 < \beta < 1$ and is $\mathbb{Y}$ is a real Banach space, then all the assertions in this sections remain valid

5. Conclusion

In this paper, I have shown that the solutions of the first and second $2k$-*variable* β-functional inequalities are additive mappings. The Hyers-Ulam stability for these given from theorems. These are the main results of the paper , which are the generalization of the results [17, 18, 19, 22] .

References

[1] T. Aoki, On the stability of the linear transformation in Banach space, J. Math. Soc. Japan 2(1950), 64-66.

[2] A.Bahyrycz, M. Piszczek, Hyers stability of the Jensen function equation, Acta Math. Hungar.,142 (2014),353-365. .

[3] M.Balcerowski, On the functional equations related to a problem of z Boros and Z. Dr*óczy*, Acta Math. Hungar.,138 (2013), 329-340.

[4] Pascus. Găvruta, A generalization of the Hyers-Ulam -Rassias stability of approximately additive mappings, Journal of mathematical Analysis and Aequations 184 (3) (1994), 431-436 https://doi.org/10.1006/jmaa.1994.1211.

[5] C. Park,Y. Cho, M. Han. *Functional inequalities associated with Jordan-von Newman-type additive functional equations , J. Inequality .Appl. ,2007(2007), Article ID 41820, 13 pages .*

[6] S M. ULam. A collection of Mathematical problems, volume 8, Interscience Publishers. New York,1960.

[7] Donald H. Hyers, On the stability of the functional equation, Proceedings of the National Academy of the United States of America
27 (4) (1941), 222.https://doi.org/10.1073/pnas.27.4.222. .

[8] Themistocles M. Rassias, On the stability of the linear mapping in Banach space, proceedings of the American Mathematical Society, 27 (1978), 297-300. https: //doi.org/10.2307 /s00010-003-2684-8 .

[9] W. Fechner, Stability of a functional inequlities associated with the Jordan-von Neumann functional equation, Aequationes Math. 71 (2006), 149-161

[10] W. P and J. Schwaiger, A system of two inhomogeneous linear functional equations, Acta Math. Hungar 140 (2013), 377-406 .

[11] L. Maligranda. Tosio Aoki (1910-1989). In *International symposium on Banach and function spaces: 14/09/2006-17/09/2006*, pages 1–23. Yokohama Publishers, 2008.

[12] A. Najati and G. Z. Eskandani. Stability of a mixed additive and cubic functional equation in quasi-Banach spaces. *J. Math. Anal. Appl.*, 342(2):1318–1331, 2008.

[13] Attila Gilányi, On a problem by K. Nikodem, Math. Inequal. Appl., 5 (2002), 707-710. . .

[14] Attila Gilányi, Eine zur parallelogrammleichung *äquivalente* ungleichung, Aequations 5 (2002),707-710. https: //doi.org/ 10.7153/mia-05-71 .

[15] *Jürg* Rätz On inequalities assosciated with the jordan-von neumann functional equation, Aequationes matheaticae, 66 (1) (2003), 191-200 https;// doi.org/10-1007/s00010-0032684-8. .

[16] W Fechner, On some functional inequalities related to the logarithmic mean, Acta Math., Hungar., 128 (2010,)31-45, 303-309.

[17] *Choonkil. Park. Additive β-functional inequalities, Journal of Nonlinear Science and Appl. 7(2014), 296-310* .

[18] *Choonkil. Park. Additive ρ-functional inequalities and equations, Journal of Mathmatical inequalities Volum 9, Number 1 (2015), 17- 26 doi: 10.7153/jmi-09-02.* .

[19] *Ly Van An. Hyers-Ulam stability of functional inequalities with three variable in Banach spaces and Non-Archemdean Banach spaces International Journal of Mathematical Analysis Vol.13, 2019, no. 11. 519-53014), 296-310.* .https://doi.org/10.12988/ijma.2019.9954

[20] *Choonkil Park,. functional in equalities in Non-Archimedean normed spaces. Acta Mathematica Sinica, English Series, 31 (3), (2015), 353-366* .https://doi.org/10.1007/s10114-015-4278-5.

[21] *Jung Rye Lee, Choonkil Park, and DongYunShin. Additive and quadratic functional in equalities in Non-Archimedean normed spaces, International Journal of Mathematical Analysis, 8 (2014), 1233-1247.*. .https://doi.org/10.

[22] *Ly Van An. Hyers-Ulam stability additive β-functional inequalities with three variable in Banach spaces and Non-Archemdean Banach spaces International Journal of Mathematical Analysis Vol.14, 2020, no. 5-8. 519-53014), 296-310.* .https://doi.org/10.12988/ijma.2020.91169.