# On distance spectra of power graphs of finite groups 

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#### Abstract

For a finite group $G$, the power graph $\mathcal{P}(G)$ is a simple connected graph having vertex set as the set of elements of finite group $G$, where two distinct vertices are adjacent if and only if one is a power of the other. In this paper, we obtain the distance spectrum power graphs of finite groups such as cyclic groups, dihedral groups, dicyclic groups, abelian groups, elementary abelian $p$ groups and other non abelian groups.


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## 1 Introduction

In this paper, we consider only connected, undirected, simple and finite graphs. A graph is denoted by $\Gamma(V(\Gamma), E(\Gamma))$, where $V(\Gamma)=\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ is its vertex set and $E(\Gamma)$ is its edge set. The order of $\Gamma$ is the number $n=|V(\Gamma)|$ and its size is the number $m=|E(\Gamma)|$. The set of vertices adjacent to $v \in V(\Gamma)$, denoted by $N(v)$, refers to the neighborhood of $v$. The degree of $v$, denoted by $d_{\Gamma}(v)$ (we simply write $d_{v}$ if it is clear from the context) means the cardinality of $N(v)$. The adjacency matrix $A=\left[a_{i j}\right]$ of $\Gamma$ is a ( 0,1 )-square matrix of order $n$ whose $(i, j)$-entry is equal to 1 , if $v_{i}$ is adjacent to $v_{j}$ and equal to 0 , otherwise.

In $\Gamma$, the distance between two vertices $u, v \in V(\Gamma)$, denoted by $d(u, v)$, is defined as the length of a shortest path between $u$ and $v$. The diameter of $\Gamma$ is the maximum distance between any two vertices of $\Gamma$. The distance matrix of $\Gamma$ is denoted by $D(\Gamma)$ and is defined as $D(\Gamma)=\left[d_{u v}\right]$, where $d_{u v}=d(u, v)$ if $u \neq v \in V(\Gamma)$ and zero otherwise. For more about $D(\Gamma)$, we refer reader to its survey [2]. Kelarev and Quinn [14] defined the directed power graph of a semigroup $S$ as a directed graph with vertex set $S$ in which two vertices $x, y \in S$ are connected by an arc from $x$ to $y$ if and only if $x \neq y$ and $y^{i}=x$ for some positive integer $i$. Motivated by this, Chakrabarty et al. [9] defined the undirected power graph $\mathcal{P}(G)$ of a group $G$ as an undirected graph with vertex set as $G$ and two vertices $x, y \in G$ are adjacent if and only if $x^{i}=y$ or
$y^{j}=x$, for $2 \leq i, j \leq n$. Many researchers have studied various properties of power graphs and their applications in characterising finite groups. Cameron and Gosh [6] proved that two finite abelian groups with isomorphic power graphs are isomorphic. Cameron [5] proved that if two finite groups have isomorphic power graphs, then they have equal number of elements of each possible order. In [9], it was shown that for any finite group $G$ the power graph of a subgroup of $G$ is an induced subgraph of $\mathcal{P}(G)$ and $\mathcal{P}(G)$ is complete if and only if $G$ is a cyclic group of order $p^{n}$, form some prime $p$ and $n$ being positive integer. Various other characterizations of power groups of finite groups are given in $[10,15]$ and their survey was done in [1]. The spectrum of power graphs has attracted many researchers, as Laplacian spectrum of power graphs of finite cyclic and dihedral groups was studied in [7], spectrum and spectral properties was studied in [16], Laplacian spectral properties in [19], signless Laplacian spectra in [3], and other spectral properties in $[12,18]$ and references therein. We denote identity of group $G$ by $e$, by $\mathcal{P}\left(G^{*}\right)=\mathcal{P}(G \backslash\{e\})$, we mean proper power graph of $\mathcal{P}(G)$ by removing the vertex $e$ and by $U(n)$ we denote the set $\left\{a \in \mathbb{Z}_{n} \mid 1 \leq a<n, \operatorname{gcd}(a, n)=1\right\}$. Our other notations are standard, $K_{n}, K_{1, n-1}, P_{n}$ denotes complete graph, star and path, for other undefined notations and terminology from spectral graph theory, and group theory, the readers are referred to $[11,13,17]$.

The rest of the paper is organized as follows. In Section 2, we discuss the distance spectrum of the power graph $\mathcal{P}(G)$ for certain finite groups like cyclic groups, dihedral groups, dicyclic groups, elementary abelian $p$ groups and other non abelian groups.

## 2 Distance spectra of power graphs of finite groups

In this section, we find distance spectrum of power graphs of various group and state some known results.

Consider an $m \times m$ matrix

$$
A=\left(\begin{array}{ccccc}
A_{1,1} & A_{1,2} & \cdots & A_{1, m} & \\
A_{2,1} & A_{2,2} & \cdots & A_{2, m} & \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
A_{m, 1} & A_{m, 2} & \cdots & A_{m, m}
\end{array}\right)
$$

whose rows and columns are partitioned according to a partition $P=\left\{P_{1}, P_{2}, \ldots, P_{m}\right\}$ of $X=$ $\{1,2, \ldots, n\}$. The quotient matrix $M$ [4] is the $m \times m$ matrix whose entries are the average row sums of the blocks $A_{i, j}$ of $A$. The partition $P$ is called equitable if each block $A_{i, j}$ of $A$ has constant row (and column) sum and in such case matrix $M$ is known as equitable quotient matrix. A vertex partition $\left\{V_{1}, V_{2}, \ldots, V_{m}\right\}$ of the vertex set $V(G)$ of the graph $G$ is equitable if for each $i$ and for all $u, v \in V_{i},\left|N(u) \cap V_{j}\right|=\left|N(v) \cap V_{j}\right|$, for all $j$. In general eigenvalues of $M$ interlace the eigenvalues of $A$, while if partition is equitable, then following lemma is helpful and can be found in [4].

Lemma 2.1 If the partition $P$ of $X$ of matrix $A$ is equitable. Then each eigenvalue of $M$ is an eigenvalue of $A$.

The following graph operation appears in the literature with different names, called $H$-join graph operation in [11], and also joined union in [20]. Herein, we use the latter name for it, and define it as follows:
Let $\Gamma=(V, E)$ be a graph of order $n$ and $\Gamma_{i}=\left(V_{i}, E_{i}\right)$ be a graph of order $m_{i}$, where $i=1, \ldots, n$. Then, the joined union $\Gamma\left[\Gamma_{1}, \ldots, \Gamma_{n}\right]$ is the graph $H=(W, F)$ with:

$$
W=\bigcup_{i=1}^{n} V_{i} \quad \text { and } \quad F=\bigcup_{i=1}^{n} E_{i} \cup \bigcup_{\left\{v_{i}, v_{j}\right\} \in E} V_{i} \times V_{j}
$$

In other words, the joined union is obtained from the union of graphs $\Gamma_{1}, \ldots, \Gamma_{n}$ by joining an edge between each pair of vertices from $\Gamma_{i}$ and $\Gamma_{j}$ whenever $v_{i}$ and $v_{j}$ are adjacent in $\Gamma$. Thus, the usual join of two graphs $\Gamma_{1}$ and $\Gamma_{2}$ is a special case of the joined union: $\Gamma_{1} \nabla \Gamma_{2}=K_{2}\left[\Gamma_{1}, \Gamma_{2}\right]$, where $K_{2}$ is the complete graph of order 2.

The following theorem in [20] gives the distance spectrum of the joined union of graphs $\Gamma_{1}, \Gamma_{2}, \ldots, \Gamma_{n}$, in terms of adjacency spectrum of the graphs $\Gamma_{1}, \Gamma_{2}, \ldots, \Gamma_{n}$ and eigenvalues of equitable quotient matrix.

Theorem 2.2 Let $\Gamma=\left(V_{i}, E_{i}\right)$ be a simple graph with $n$ vertices $V(\Gamma)=\left\{v_{1}, \ldots, v_{n}\right\}$ and for $i=1,2, \ldots, n$, let $\Gamma_{i}=\left(V_{i}, E_{i}\right)$ be an $r_{i}$-regular graph of order $m_{i}$ and eigenvalues of the adjacency matrix $\lambda_{i, 1}=r_{i} \geq \lambda_{i, 2} \geq \ldots \geq \lambda_{i, m_{i}}$. Then the distance spectrum of the joined union $\Gamma\left[\Gamma_{1}, \ldots, \Gamma_{n}\right]$ consists of the eigenvalues $-\lambda_{i, j}-2$ for $i=1, \ldots, n$ and $j=2,3, \ldots, m_{i}$ and the eigenvalues of the matrix

$$
M=\left[\begin{array}{ccccc}
2 m_{1}-r_{1}-2 & m_{2} d_{\Gamma}\left(v_{1}, v_{2}\right) & m_{3} d_{\Gamma}\left(v_{1}, v_{3}\right) & \ldots & m_{n} d_{\Gamma}\left(v_{1}, v_{n}\right) \\
m_{1} d_{\Gamma}\left(v_{2}, v_{1}\right) & 2 m_{2}-r_{2}-2 & m_{3} d_{\Gamma}\left(v_{2}, v_{3}\right) & \ldots & m_{n} d_{\Gamma}\left(v_{2}, v_{n}\right) \\
\vdots & \vdots & \ddots & \vdots & \\
m_{1} d_{\Gamma}\left(v_{n}, v_{1}\right) & m_{2} d_{\Gamma}\left(v_{n}, v_{2}\right) & m_{3} d_{\Gamma}\left(v_{n}, v_{3}\right) & \ldots & 2 m_{n}-r_{n}-2
\end{array}\right] .
$$

Corollary 2.3 Let $\Gamma_{i}$ be $r_{i}-$ regular graphs of order $n_{i}$ and let $r_{i}=\lambda_{i, 1} \geq \lambda_{i, 2} \geq \cdots \geq \lambda_{i, n_{i}}$ be the adjacency eigenvalues of $\Gamma_{i}$. Then the distance spectrum of $\Gamma_{1} \vee \Gamma_{2}$ consists of the eigenvalues $-\lambda_{i, j}-2, i=1,2$ and $2 \leq j \leq m_{i}$, and two more eigenvalues

$$
m_{1}+m_{2}-2-\frac{r_{1}+r_{2}}{2} \pm \sqrt{\left(n_{1}-n_{2}-\frac{r_{1}-r_{2}}{2}\right)^{2}+m_{1} m_{2}} .
$$

An integer $d$ is called proper divisor of $n$ if $d \mid n$, for $1<d<n$. Let $d_{1}, d_{2}, \ldots, d_{t}$ be the distinct proper divisors of $n$. Let $\Delta_{n}$ be the simple graph with vertex set $\left\{d_{1}, d_{2}, \ldots, d_{t}\right\}$, in which two distinct vertices are connected by an edge if and only if $d_{i} \mid d_{j}$, for $1 \leq i<j \leq t$. If the prime power factorization of $n=p_{1}^{n_{1}} p_{2}^{n_{2}} \ldots p_{r}^{n_{r}}$, where $r, n_{1}, n_{2}, \ldots, n_{r}$ are positive integers
and $p_{1}, p_{2}, \ldots, p_{r}$ are distinct prime numbers. Then it is easy to see that the size of the $\Delta_{n}$ is given by

$$
\left|V\left(\Upsilon_{n}\right)\right|=\prod_{i=1}^{r}\left(n_{i}+1\right)-2
$$

If $n=2^{2}$ or $n=p q$, where $p<q$ are primes, then it is easy to see that $\Delta_{n}$ is disconnected graph otherwise $\Delta_{n}$.

The next result can be found in [15], stating that power of cyclic group $\mathbb{Z}_{n}$ can be written as the joined union of complete graphs.

Theorem 2.4 Let $\mathbb{Z}_{n}$ be a finite cyclic group. Then power graph

$$
\mathcal{P}\left(\mathbb{Z}_{n}\right)=K_{\phi(n)+1} \nabla \Delta_{n}\left[K_{\phi\left(d_{1}\right)}, K_{\phi\left(d_{2}\right)}, \ldots, K_{\phi\left(d_{t}\right)}\right] .
$$

Now, we compute the distance spectrum of the power graph $\mathcal{P}(G)$ of the finite group with the help of Theorem 2.2 and Theorem 2.4, in terms of adjacency spectrum of $K_{\omega}$ and zeros of the characteristic polynomial of the auxiliary matrix. We recall from [9] that $\mathcal{P}(G)$ is a complete graph if and only if $G$ is cyclic group of order $n=p^{z}, z \in \mathbb{N}$, where $p$ is prime and noting that the adjacency spectrum of $K_{\omega}$ is $\left\{\omega,(-1)^{[\omega-1]}\right\}$. So by Theorem 2.2 , out of $n$ distance eigenvalues of $\mathcal{P}(G), n-t$ of them are known to be non negative integers. The remaining $t$ distance eigenvalues of $\mathcal{P}(G)$ are the non-zero zeros of the characteristic polynomial of the equitable quotient matrix $M$.

We consider a new graph $H=K_{1} \nabla \Delta_{n}$, it is clearly that $H$ is a connected graph of diameter at most two. This graph is very important for finding the distance spectrum of power graph of finite cyclic groups. The following result computes the distance eigenvalues of $\mathbb{Z}_{n}$ with the help of Theorem 2.2 and Theorem 2.4.

Theorem 2.5 The distance spectrum of $\mathcal{P}\left(\mathbb{Z}_{n}\right)$ consists of the eigenvalue -1 with multiplicity $n-t-1$ and the zeros of the characteristic polynomial of the following equitable quotient matrix

$$
M=\left[\begin{array}{cccc}
\phi(n) & \phi\left(d_{1}\right) & \ldots & \phi\left(d_{n}\right) \\
(\phi(n)+1) d\left(v_{2}, v_{1}\right) & \phi\left(d_{2}\right)-1 & \ldots & \phi\left(d_{t}\right) d\left(v_{2}, v_{t+1}\right) \\
\vdots & \vdots & \ldots & \vdots \\
(\phi(n)+1) d\left(v_{t+1}, v_{1}\right) & \left(\phi\left(d_{1}\right)\right) d\left(v_{t+1}, v_{2}\right) & \ldots & \phi\left(d_{t}\right)-1
\end{array}\right] .
$$

Proof. Let $G \cong \mathbb{Z}_{n}$ be a finite cyclic group of order $n$. Then by the definition of power graph, we observe that identity and the generators of $\mathbb{Z}_{n}$, which are $\phi(n)$ in number are adjacent to every other vertex of $\mathcal{P}\left(\mathbb{Z}_{n}\right)$. So by Theorem 2.4 , we have

$$
\mathcal{P}\left(\mathbb{Z}_{n}\right)=K_{\phi(n)+1} \nabla \Delta_{n}\left[K_{\phi\left(d_{1}\right)}, K_{\phi\left(d_{2}\right)}, \ldots, K_{\phi\left(d_{t}\right)}\right]=H\left[K_{\phi(n)+1}, K_{\phi\left(d_{1}\right)}, K_{\phi\left(d_{2}\right)}, \ldots, K_{\phi\left(d_{t}\right)}\right],
$$

where $H=K_{1} \nabla \Delta_{n}$ is the new graph with vertices $\left\{v_{1}, \ldots, v_{t+1}\right\}$. Since $m_{1}=\phi(n)+1$ and $m_{i}=\phi\left(d_{i-1}\right)$, for $i=2, \ldots, t+1$. By Theorem 2.2, we have $\lambda_{1, j}-2=-(-1)-2=-1$
with multiplicity $\phi(n)$. Similarly, we can show that -1 is the distance eigenvalues of $\mathcal{P}\left(\mathbb{Z}_{n}\right)$ with multiplicity $\phi\left(d_{1}\right)-1$. Proceeding in this way, we see that eigenvalue -1 occurs with multiplicity $\phi(n)+\phi\left(d_{1}\right)-1+\ldots \phi\left(d_{t}\right)-1=\phi(n)-\sum_{1, n \neq d \mid n} \phi(d)-t=\phi(n)+n-1-\phi(n)-t=n-t-1$. The remaining distance eigenvalues are the zeros of the characteristic polynomial of the equitable quotient matrix $M$.

From the Theorem 2.5, we note that the distance $d\left(v_{i}, v_{j}\right), i \neq j$ and $1 \leq i, j \leq t+1$ remains most of times unknown. Since $H=K_{1} \nabla \Delta_{n}$ is a graph of diameter at most two, so $d\left(v_{i}, v_{j}\right)=1$ or 2 depending upon whether $v_{i}$ and $v_{j}$ are adjacent or not.

Following are the consequences of the Theorem 2.5, which gives distance Laplacian spectrum of $\mathcal{P}\left(\mathbb{Z}_{n}\right)$ for various values of $n$.

Corollary 2.6 If $n=p^{z}$, where $p$ is prime and $z$ is a non negative integer, then distance Laplacian spectrum of $\mathcal{P}\left(\mathbb{Z}_{n}\right)$ is $\left\{n-1,(-1)^{[n-1]}\right\}$.

Proof. If $n=p^{z}$, where $p$ is prime and $n \in \mathbb{N}$, then as shown in [9], $\mathcal{P}\left(\mathbb{Z}_{n}\right)$ is isomorphic to complete graph $K_{n}$ and result hence follows.

Corollary 2.7 If $n=p q$, where $p<q$ are primes, then distance spectrum of $\mathcal{P}\left(\mathbb{Z}_{n}\right)$ consists of the eigenvalue -1 with multiplicity $n-3$ and the remaining three eigenvalues are given by the matrix in (2.1).

Proof. Let $n=p q$, where $p<q$ are distinct primes. Since there are $\phi(p q)$ generators and identity element which are connected to every other vertex of $\mathcal{P}\left(\mathbb{Z}_{n}\right)$. Then by [10], we have

$$
\mathcal{P}\left(\mathbb{Z}_{n}\right)=K_{p-1} \cup K_{q-1} \nabla K_{\phi(p q)+1}=P_{3}\left[K_{p-1}, K_{\phi(p q)+1}, K_{q-1}\right] .
$$

Now, by using Theorem 2.4, we get eigenvalue $\lambda_{1, j}-2=-1$ with multiplicity $m_{1}-1$. Similarly, we see that -1 is the distance eigenvalue with multiplicity $m_{2}+m_{3}-2$, and the remaining eigenvalues of $\mathcal{P}\left(\mathbb{Z}_{n}\right)$ are given by the following matrix

$$
\left[\begin{array}{ccc}
p-2 & p q-p-q+2 & q-1  \tag{2.1}\\
p-1 & p q-p-q+1 & q-1 \\
p-1 & p q-p-q+2 & q-2
\end{array}\right]
$$

Next result is the generalization of above corollary and gives distance spectrum of $\mathcal{P}\left(\mathbb{Z}_{n}\right)$ when $n$ is product of three distinct primes.

Corollary 2.8 Let $n=p q r$, where $p<q<r$ are primes. Then the distance spectrum of $\mathcal{P}\left(\mathbb{Z}_{n}\right)$ consists of eigenvalue -1 with multiplicity $n-7$ and the remaining five distance eigenvalues of $\mathcal{P}\left(\mathbb{Z}_{n}\right)$ are the zeros of the characteristic polynomial of the matrix in (2.2).

Proof. Let $n=p q r$, where $p<q<r$ are primes. Then by the definition of $\Delta_{n}$, we have edge set of $\Delta_{n}$ as $\{(p, p q),(p, p r),(q, p q),(q, q r),(r, p r),(r, q r)\}$ and $H=K_{1} \nabla \Delta_{n}$. This implies that $\mathcal{P}\left(\mathbb{Z}_{n}\right)=H\left[K_{\phi(n)+1}, K_{\phi\left(d_{1}\right)}, \ldots, K_{\phi\left(d_{6}\right)}\right]$. After labelling the vertices of $\mathcal{P}\left(\mathbb{Z}_{n}\right)$ in a proper way and using Theorem $2.4,-1$ is the distance eigenvalue with multiplicity $n-7$. The remaining eigenvalues are given by the following quotient matrix

$$
\left[\begin{array}{ccccccc}
\phi(n) & \phi(p) & \phi(q) & \phi(r) & \phi(p q) & \phi(p r) & \phi(q r)  \tag{2.2}\\
\phi(n)+1 & \phi(p)-1 & 2 \phi(q) & 2 \phi(r) & \phi(p q) & \phi(p r) & 2 \phi(q r) \\
\phi(n)+1 & 2 \phi(p) & \phi(q)-1 & 2 \phi(r) & \phi(p q) & 2 \phi(p r) & \phi(q r) \\
\phi(n)+1 & 2 \phi(p) & 2 \phi(q) & \phi(r)-1 & 2 \phi(p q) & \phi(p r) & \phi(q r) \\
\phi(n)+1 & \phi(p) & \phi(q) & 2 \phi(r) & \phi(p q)-1 & 2 \phi(p r) & 2 \phi(q r) \\
\phi(n)+1 & \phi(p) & 2 \phi(q) & \phi(r) & 2 \phi(p q) & \phi(p r)-1 & 2 \phi(q r) \\
\phi(n)+1 & 2 \phi(p) & \phi(q) & \phi(r) & 2 \phi(p q) & 2 \phi(p r) & \phi(q r)-1
\end{array}\right] .
$$

Corollary 2.9 The distance spectrum of $\mathcal{P}\left(\mathbb{Z}_{p q^{2}}\right)$ is -1 with multiplicity $n-5$ are the zeros of the characteristic polynomial of the following quotient matrix

$$
\left[\begin{array}{ccccc}
\phi(n) & \phi(p) & \phi(p q) & \phi(q) & \phi\left(q^{2}\right)  \tag{2.3}\\
\phi(n)+1 & \phi(p)-1 & \phi(p q) & 2 \phi(q) & 2 \phi\left(q^{2}\right) \\
\phi(n)+1 & \phi(p) & \phi(p q)-1 & \phi(q) & 2 \phi\left(q^{2}\right) \\
\phi(n)+1 & 2 \phi(p) & \phi(p q) & \phi(q)-1 & \phi\left(q^{2}\right) \\
\phi(n)+1 & 2 \phi(p) & 2 \phi(p q) & \phi(q) & \phi\left(q^{2}\right)-1
\end{array}\right] .
$$

Proof. Let $n=p q^{2}$, where $p$ and $q$ are distinct primes. Since proper divisors of $n$ are $p, q, p q, q^{2}$, so $\Delta_{p^{2} q}$ is the path $P_{4}: p \sim p q \sim q \sim q^{2}$. By Theorem 2.4, we have

$$
\mathcal{P}\left(\mathbb{Z}_{p q^{2}}\right)=K_{\phi\left(p q^{2}\right)+1} \nabla P_{4}\left[K_{\phi(p)}, K_{\phi(p q)}, K_{\phi(q)}, K_{\phi\left(q^{2}\right)}\right]=H\left[K_{\phi\left(p q^{2}\right)+1}, K_{\phi(p)}, K_{\phi(p q)}, K_{\phi(q)}, K_{\phi\left(q^{2}\right)}\right],
$$

where $H=K_{1} \nabla P_{4}$. By Theorem 2.5, the distance spectrum of $\mathcal{P}\left(\mathbb{Z}_{p q^{2}}\right)$ consists of eigenvalues -1 with multiplicities $n-5$. The remaining eigenvalues are given by the equitable quotient matrix in (2.3).

Corollary 2.10 The distance spectrum of $\mathcal{P}\left(\mathbb{Z}_{n}\right)$ for $n=(p q)^{2}$ is -1 wit multiplicity $n-8$ and the zeros of the characteristic polynomial of the matrix in (2.4).

Proof. Let $n=(p q)^{2}$, where $p<q$ are distinct primes. Since proper divisors of $n$ are $p, p^{2}, q, q^{2}, p q, p^{2} q$ and $p q^{2}$, so $\Delta_{n}$ is the graph $G_{7}$ with vertex set as proper divisors and edge set $\left\{\left(p, p^{2}\right),(p, p q),\left(p, p^{2} q\right),\left(p, p q^{2}\right),\left(q, q^{2}\right),(q, p q),\left(q, p^{2} q\right),\left(q, p q^{2}\right),\left(p^{2}, p^{2} q\right),\left(q^{2}, p q^{2}\right)\left(p q, p^{2} q\right),\left(p q, p q^{2}\right)\right\}$. Let $H=K_{1} \nabla G_{7}$ and using Theorem 2.4, we have

$$
\begin{aligned}
\mathcal{P}\left(\mathbb{Z}_{n}\right) & =K_{\phi(n)+1} \nabla G_{7}\left[K_{\phi(p)}, K_{\phi(q)}, K_{\phi\left(p^{2}\right)}, K_{\phi\left(q^{2}\right)}, K_{\phi(p q)}, K_{\phi\left(p^{2} q\right)}, K_{\phi\left(p q^{2}\right)}\right] \\
& =H\left[K_{\phi(n)+1}, K_{\phi(p)}, K_{\phi(q)}, K_{\phi\left(p^{2}\right)}, K_{\phi\left(q^{2}\right)}, K_{\phi(p q)}, K_{\phi\left(p^{2} q\right)}, K_{\phi\left(p q^{2}\right)}\right] .
\end{aligned}
$$

By Theorem 2.5 the distance eigenvalue of $\mathcal{P}\left(\mathbb{Z}_{n}\right)$ is -1 with multiplicity $n-8$ and the remaining eight eigenvalues are given by the following matrix

$$
\left[\begin{array}{cccccccc}
\phi(n) & \phi(p) & \phi(q) & \phi\left(p^{2}\right) & \phi\left(q^{2}\right) & \phi(p q) & \phi\left(p^{2} q\right) & \phi\left(p q^{2}\right)  \tag{2.4}\\
\phi(n)+1 & \phi(p)-1 & 2 \phi(q) & 2 \phi\left(p^{2}\right) & \phi\left(q^{2}\right) & \phi(p q) & \phi\left(p^{2} q\right) & \phi\left(p q^{2}\right) \\
\phi(n)+1 & 2 \phi(p) & \phi(q)-1 & 2 \phi\left(p^{2}\right) & \phi\left(q^{2}\right) & \phi(p q) & \phi\left(p^{2} q\right) & \phi\left(p q^{2}\right) \\
\phi(n)+1 & 2 \phi(p) & 2 \phi(q) & \phi\left(p^{2}\right)-1 & 2 \phi\left(q^{2}\right) & 2 \phi(p q) & \phi\left(p^{2} q\right) & \phi\left(p q^{2}\right) \\
\phi(n)+1 & \phi(p) & \phi(q) & 2 \phi\left(p^{2}\right) & \phi\left(q^{2}\right)-1 & 2 \phi(p q) & 2 \phi\left(p^{2} q\right) & 2 \phi\left(p q^{2}\right) \\
\phi(n)+1 & \phi(p) & \phi(q) & 2 \phi\left(p^{2}\right) & 2 \phi\left(q^{2}\right) & \phi(p q)-1 & 2 \phi\left(p^{2} q\right) & 2 \phi\left(p q^{2}\right) \\
\phi(n)+1 & \phi(p) & \phi(q) & \phi\left(p^{2}\right) & 2 \phi\left(q^{2}\right) & 2 \phi(p q) & \phi\left(p^{2} q\right)-1 & 2 \phi\left(p q^{2}\right) \\
\phi(n)+1 & \phi(p) & \phi(q) & \phi\left(p^{2}\right) & 2 \phi\left(q^{2}\right) & 2 \phi(p q) & 2 \phi\left(p^{2} q\right) & \phi\left(p q^{2}\right)-1
\end{array}\right] .
$$

Next we find the distance spectrum of the dihedral group and dicyclic group for some particular values of $n$. The dihedral group of order $2 n$ and dicyclic groups of order $4 n$ are denoted and presented as follows

$$
\begin{aligned}
D_{2 n} & =<a, b \mid a^{n}=b^{2}=e, b a b=a^{-1}> \\
Q_{n} & =<a, b \mid a^{2 n}=e, b^{2}=a^{n}, a b=b a^{-1}>.
\end{aligned}
$$

If $n$ is a power of 2 , then $Q_{n}$ is called the generalized quaternion group of order $4 n$.
Proposition 2.11 If $n$ is a prime power, then the distance spectrum of $\mathcal{P}\left(D_{2 n}\right)$ is

$$
\left\{(-1)^{[n-2]},(-2)^{[n-1]}\right\}
$$

and the three eigenvalues of matrix in (2.5).
Proof. Since $<a>$ generates a cyclic subgroup of order $n$ and is therefore isomorphic to $\mathbb{Z}_{n}$. The remaining $n$ elements of $D_{2 n}$ form an independent set of $\mathcal{P}\left(D_{2 n}\right)$, sharing the identity element $e$. Therefore, the structure of the power group of the dihedral group $D_{2 n}$ can be obtained from the power graph $\mathcal{P}\left(\mathbb{Z}_{n}\right)$ by adding the $n$ pendent vertices at the identity vertex $e$. If $n=p^{z}$, where $z$ is positive integer, then it is easy to see that

$$
\mathcal{P}\left(D_{2 n}\right)=P_{3}\left[K_{n-1}, K_{1}, \bar{K}_{n}\right] .
$$

By using Theorem 2.2, the distance spectrum of $\mathcal{P}\left(D_{2 n}\right)$ consists of the eigenvalue $\lambda_{1, j}-2=$ $1-2=-1$ with multiplicity $n-2$, the eigenvalue -2 with multiplicity $n-1$ and the remaining three distance eigenvalues are given by the following matrix

$$
\left[\begin{array}{ccc}
n-2 & 1 & 2 n  \tag{2.5}\\
n-1 & 0 & n \\
2 n-2 & 1 & 2 n-2
\end{array}\right]
$$

Proposition 2.12 If $n$ is a power of 2 , then the distance spectrum of $\mathcal{P}\left(Q_{n}\right)$ is -1 with multiplicity $3 n-2$ and the remaining $n+2$ eigenvalues are given by the matrix (2.6).

Proof. As by the definition of power graph, the identity element is always connected to every other vertex of $\mathcal{P}\left(Q_{n}\right)$ and in particular, if $n$ is a power of 2 , then in [19], it was shown that $a^{n}$ is also adjacent to all other vertices of $\mathcal{P}\left(Q_{n}\right)$. Therefore by using these observation and some simple investigation, the power group of $\mathcal{P}\left(Q_{n}\right)$ can be written as

$$
\mathcal{P}\left(Q_{n}\right)=W[K_{2}, K_{2 n-2}, \underbrace{K_{2}, K_{2}, \ldots, K_{2}}_{n}],
$$

where $W=K_{1, n+1}$. By using Theorem 2.2, we see that the distance spectrum of $\mathcal{P}\left(Q_{n}\right)$ consists of the eigenvalue -1 with multiplicity $1+2 n-3+\underbrace{1+\cdots+1}_{n}$ and the remaining $n+2$ eigenvalues of $\mathcal{P}\left(Q_{n}\right)$ are given by the following equitable quotient matrix

$$
M\left(Q_{n}\right)=\left[\begin{array}{cccccc}
1 & 2 n-2 & 2 & \ldots & 2 & 2  \tag{2.6}\\
2 & 2 n-3 & 2 & \ldots & 2 & 2 \\
2 & 2(2 n-2) & 1 & \ldots & 2 & 2 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
2 & 2(2 n-2) & 2 & \ldots & 1 & 2 \\
2 & 2(2 n-2) & 2 & \ldots & 2 & 1
\end{array}\right]
$$

Problem (2). Discuss the distance spectrum of $\mathcal{P}\left(D_{2 n}\right)$ for other values of $n \in\left\{p q, p q r, p^{2} q,(p q)^{2}\right\}$ and generalize for any $n$ ?
Problem (3). Discuss the distance spectrum of $\mathcal{P}\left(Q_{n}\right)$ for other values of $n \in\left\{p q, p q r, p^{2} q,(p q)^{2}\right\}$ and generalize for any $n$ ?

In the next result, we find distance spectrum of finite elementary abelian groups (that is, group whose each non trivial element is of prime order) of prime power order, for that we need following lemma from [10].

Lemma 2.13 Let $G$ be an elementary abelian group of order $p^{n}$ for some prime number $p$ and positive integer $n$. Then $\mathcal{P}(G) \cong K_{1} \nabla\left(\cup_{i=1}^{l} K_{p-1}\right)$, where $l=\frac{p^{n}-1}{p-1}$.

Theorem 2.14 Let $G$ be an elementary abelian group $p^{n}, p$ is prime. Then the distance spectrum of $\mathcal{P}(G)$ is -1 with multiplicity $l(p-2)$ and the zeros of the matrix in (2.7).

Proof. Let $G$ be an elementary abelian group of order $p^{n}$ for some prime $p$ and positive integer $n$. Then by Lemma $2.13, \mathcal{P}(G) \cong K_{1} \nabla\left(\cup_{i=1}^{l} K_{p-1}\right)$ and it follows that

$$
\mathcal{P}(G) \cong S\left[K_{1}, K_{p-1}, \ldots, K_{p-1}\right]
$$

where $K_{p-1}$ occurs $l=\frac{p^{n}-1}{p-1}$ times and $S=K_{1} \nabla \bar{K}_{l}=K_{1, l}$. Then by Theorem 2.2, we see that -1 is the distance eigenvalue of $\mathcal{P}(G)$ with multiplicity $l(p-2)$ and the remaining distance eigenvalues of $\mathcal{P}(G)$ are given by the following equitable quotient matrix

$$
\left[\begin{array}{cccccc}
0 & p-1 & p-1 & \ldots & p-1 & p-1  \tag{2.7}\\
1 & p-2 & 2(p-1) & \ldots & 2(p-1) & 2(p-1) \\
1 & 2(p-1) & p-2 & \ldots & 2(p-1) & 2(p-1) \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
1 & 2(p-1) & 2(p-1) & \ldots & p-2 & 2(p-1) \\
1 & 2(p-1) & 2(p-1) & \ldots & 2(p-1) & p-2
\end{array}\right] .
$$

In the Proposition 2.11, distance spectrum of non abelian groups of order $2 p$, where $p$ is a prime were determined. In the following result we obtain generalization of Proposition 2.11, when order of non abelian group is product of two distinct primes and determine its distance spectrum completely. Before proceeding further we need basic result from [10], which is stated below.

Lemma 2.15 Let $G$ be a finite group of order $p q$, where $p$ and $q$ are primes. Then $G$ is non abelian if and only if $\mathcal{P}(G) \cong K_{1} \nabla\left(q K_{p-1} \cup K_{q-1}\right)$.

Proposition 2.16 If $G$ be a non abelian group of order $n=p q$, where $p<q$ are primes, then the distance spectrum of $\mathcal{P}(G)$ is -1 with multiplicity $q(p-1)-2$ and the eigenvalues of the matrix in (2.8).

Proof. If $G$ be a non abelian group of order $p q$, then by the Lemma 2.15, power group of $\mathcal{P}(G)$ can be written as

$$
\mathcal{P}(G)=K_{1, q+1}[K_{1}, \underbrace{K_{p-1}, K_{p-1}, \ldots, K_{p-1}}_{q}, K_{q-1}] .
$$

By using Theorem 2.2, the distance spectrum of $\mathcal{P}(G)$ consists of the eigenvalue -1 with multiplicity $p q-2 q+q-2=p q-q-2$ and the remaining $q+2$ eigenvalues of $\mathcal{P}(G)$ are given by the following equitable quotient matrix

$$
\left[\begin{array}{cccccc}
0 & q-1 & p-1 & \ldots & p-1 & p-1  \tag{2.8}\\
1 & q-2 & 2 p-2 & \ldots & 2 p-2 & 2 p-2 \\
1 & 2 q-2 & p-2 & \ldots & 2 p-2 & 2 p-2 \\
\vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\
1 & 2 q-2 & 2 p-2 & \ldots & p-2 & 2 p-2 \\
1 & 2 q-2 & 2 p-2 & \ldots & 2 p-2 & p-2
\end{array}\right]
$$

Lemma 2.17 [10] Let $G$ be a finite group of order $n$. Then $\mathcal{P}(G) \cong K_{1, n-1}$ if and only each non identity element is self invertible.

Groups satisfying hypothesis of Lemma 2.17 are abelian of order $n$, following lemma gives the distance spectrum of such type of abelian groups completely.

Lemma 2.18 Let $G$ be a finite group of order $n$ in which each non identity element is self invertible. Then the distance spectrum of $\mathcal{P}(G)$ is $\left\{(-2)^{n-2}, n-2 \pm \sqrt{n^{2}-3 n+1}\right\}$.

Proof Since $G$ is a group of order $n$, such that each non identity element is of order two. Then by Lemma 2.17, $\mathcal{P}(G) \cong K_{1} \nabla \bar{K}_{n-1}$ and using Corollary 2.3, result follows.

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