⊙-Subalgebra and ⊙-Ideal in BN_1-algebra

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Abstract — *This study is the development of the Companion BN_1-algebra concept that researched by A. Mursalima, et al. [1]. The companion concept is developed by adding the subalgebra and ideal concepts to obtain ⊙--Subalgebra and ⊙-Ideal in BN_1-algebra. The final result of this research is in the form of constructing the definitions and properties of ⊙-Subalgebra and ⊙--Ideal which are stated in several theorems, including the nature of the relationship.*

Keywords — *BN_1-algebra, Companion, Subalgebra, Ideal.*

I. INTRODUCTION

In 2002, H. S. Kim and J. Neggers [15] have been defined a new algebra, namely B-algebra. B-algebra is a nonempty set $(X,*,0)$ that statisfies (B1) $x*x=0$, (B2) $x*0=x$ and (B) $(x*y)*z=x*(z*(0*y))$ for all $x,y,z\in X$ [15]. Four years later, in 2006, C.B. Kim and H. S. Kim [5] introduced a new sub class of B-algebra that is BM-algebra. BM-algebra is algebra that satisfies (B2) and (BM) $(z*y)*(z*y)=y*x$. Furthermore in 2007, A. Walendziak [2] introduced a generalization of B-algebra which called BF-algebra.

In 2013, H. S. Kim and C. B. Kim founded a new structure algebra and a sub class of BF-algebra, namely BN-algebra [6]. BN-algebra is a nonempty set $(X,*,0)$ that satisfies the axiom (B1), (B2) and (BN) $(x*y)*z=(0*z)*(y*x)$ for all $x,y,z\in X$. BN-algebra is also one of the wide classes of BM-algebra. In {6}, H. S. Kim and C. B. Kim [6] was also introduced subalgebra of BN-algebra, BN_1-algebra. BN_1-algebra is a BN-algebra that satisfies $x=(x*y)*y$.

Various algebra concepts have been applied in B-algebra. L. D. Naingue and J. P. Vilela [18] developed a companion in B-algebra. Based on companion B-algebra [18], A. Mursalima, et al. developed on companion BN_1-algebra [1]. In this article, by adding subalgebra and ideal concept, we deveoped the definition of ⊙-subalgebra and ⊙-ideal. Let S and I be a companion BN_1-algebra, then S is called a ⊙-subalgebra when $x\odot y\in S$ for all $x,y\in S$. If While I is said to be ⊙-ideal, if $0\odot y\in I$, $x\odot y\in I$ and $y\in I$, then $x\in I$. We also investigate the properties of ⊙-subalgebra and ⊙-ideal BN_1-algebra, the relationship between ⊙-subgebra and ⊙-ideal by using some axioms

II. B-ALGEBRA, BN-ALGEBRA, BN_1-ALGEBRA AND COMPANION BN_1-ALGEBRA

In this section, the definition and some properties of B-Algebra, BN-algebra, BN_1-algebra and companion BN_1-algebra are given. J. Neggers and H. S. Kim [15] gave the definition of B-algebra by:

Definition 1.1 (Definition of B-algebra) B-algebra is a nonempty set $(X,*,0)$ with constanta 0 and binary operation $*$ that satisfies:

(B1) $x*x=0$,

(B2) $x*0=x$,

(B) $(x*y)*z=x*(z*(0*y))$,

for all $x,y,z\in X$.

Companion B-algebra was defined by [18]:

Definition 1.2 (Companion B-algebra) Let $(X,*,0)$ be with companion operation $\odot$. A operation $\odot$ is said to be subcompanion operation it it satisfies $((x \odot y)*x)*y = 0$ for all $x, y \in X$. A operation $\odot$ is a companion operation of X if for $(z*x)*y = 0$, then $z*(x \odot y) = 0$.

BN-algebra was introduced by C. B. Kim and H. S. Kim [6] in 2013 and was defined by :

Definition 1.3 (Definition of BN-Algebra) BN-algebra is a nonempty set $(X,*,0)$ with binary operation $*$ that satisfies the following axioms:

(B1) $x*x = 0$,
(B2) $x*0 = x$,
(BN) $(x*y)*z = (0*z)*(y*x)$,
for any $x, y, z \in X$.

Example 1.1 Let $X := \{0,1,2,3\}$ be a set with Cayley table as follows:

Table 1: Tabel Cayley of BN-algebra

$*$	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	1	0	1
3	3	1	1	0

Then based on definition 1.3, it is easy to show that X is BN-algebra.

BN-algebra is also satisfies some of addition axioms, such as [6]:

Theorem 1.1 If a nonempty set $(X,*,0)$ is a BN-algebra, then for any $x, y, z \in X$

(i) $0*(0*x) = 0$,
(ii) $y*x = (0*x)*(0*y)$,
(iii) $(0*x)*y = (0*y)*x$,
(iv) $x*y = 0 \Rightarrow y*x = 0$,
(v) $0*x = 0*y \Rightarrow x = y$,
(vi) $(x*z)*(y*z) = (z*y)*(z*x)$.

Proof: we can see in [6].

A subalgebra of BN-algebra is defined by [6]:

Definition 1.4 (**Definition of Sub BN-Algebra)** Let $(X,*,0)$ be a BN-algebra and $\emptyset \neq S \subseteq X$. S id called to be a subalgebra of X if $x*y \in S$ for any $x, y \in S$.

One of the subalgebra of BN-algebra is BN1-algebra that is defined C. B. Kim and H. S. Kim [6] define BN1-algebra as follows:

Definition 1.5 (BN_1-algebra) BN_1-algebra is a nonempty set $(X,*,0)$ of BN-algebra satisfies i (BN1) $x = (x*y)*y$ for all $x, y, z \in X$.

Example 1.2 Let $X := \{0,1,2,3\}$ be a set with Cayley table as follows:

Table 2: Tabel Cayley of BN_1-algebra

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then based on definition 1.3 and definition 1.5, it is easy to show that X is BN_1-algebra.

Some additional axioms is satisfied on BN_1-algebra, such as [6]:

Theorem 1.2 If $(X,*,0)$ is a BN_1-algebra, then it satisfies

(i) $0*x = x$,

(ii) $x*y = y*x$

for any $x, y \in X$.

Proof. (i) Let $x, y, z \in X$. By using (BN1) $x = (x*y)*y$ and (B1) $x*x = 0$, for $y = x$, we get $(x*y)*y = (x*x)*x = 0*x = x$.

(ii) From theorem 1.1, we have $y*x = (0*x)*(0*y) = x*y$.

Theorem 1.3 Let $(X,*,0)$ be a BN_1-algebra. If $x*y = 0$, then $x = y$ for any $x, y \in X$.

Proof. We can see in [6].

Based on definiton of subalgebra, the Subalgebra of BN_1-algebra can be defined by

Definition 1.6 (Subalgebra of BN_1-Algebra) Let $(X,*,0)$ be a BN_1-algebra and $\emptyset \neq S \subseteq X$. S is said to be a subalgebra of X if $x*y \in S$ for $x, y \in S$.

While Ideal of BN_1-algebra can be defined as follows:

Definition 1.7 (BN_1-ideal) Let $(X,*,0)$ be a BN_1-algebra and $\emptyset \neq I \subseteq X$. I is called BN_1-ideal ofi X if $0 \in I$, $y \in I$ dan $x*y \in I$, then $x \in I$ for any $x, y \in I$.

Example 1.3 Let $X := 0, 1, 2, 3$ is a BN_1-algebra in example 1.2. $I_1 = \{0,1\}$, $I_2 = \{0,2\}$ and $I_3 = \{0,3\}$ are ideal of BN_1-algebra, but $I_4 = \{0,1,2\}$ is not a ideal because $1*2 = 2*1 = 3$ and $3 \notin I_4$.

Companion concept has been applied in BN_1-algebra. A. Mursalima, et al.[1] defined companion BN_1-algebra as follows:

Definition 1.8 (Companion BN_1-algebra) Let $(X,*,0)$ be a BN_1-algebra with operation $\odot$. Operation $\odot$ is called subcompanion, if it satisfies $((x \odot y)*x)*y = 0$ for any $x, y \in X$ and subcompanion operation $\odot$ is a companion of X if satisfies (C) if $(z*x)*y = 0$, then $z*(x \odot y) = 0$ for all $x, y, z \in X$.

Exampe 1. Let $X := \{0, 1, 2, 3\}$ be a BN_1-Algebra with Cayley table as follows:

Table 3: Companion BN_1-Algebra

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\odot$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Based on definition 1.8, it is easy to show that $(X, *, \odot, 0)$ is a companion BN_1-algebra.

III. ⊙-SUBALGEBRA AND ⊙-IDEAL OF BN_1-ALGEBRA

In this section, we construct the definition ⊙-Subalgebra and ⊙-Ideal BN_1-algebra. We also contruct the propeties of ⊙-subalgebra and ⊙-Ideal BN_1-algebra on some theorems.. ⊙-Sub of BN_1-algebra is defined by

Definition 1.9 (⊙-Subalgebra) Let $(X, *, \odot, 0)$ be a companion BN_1-algebra and S is a nonempty subset of X. S is said to be a ⊙-subalgebra of X if $x \odot y \in I$ for $x, y \in S$.

Example 1.5 Let $X := \{0, 1, 2, 3\}$ is a BN_1-algebra in example 1.4. A set $I_1 = \{0,1\}$ is a ⊙-subalgebra, while $I_2 = \{0,2,3\}$ is not because $2 \odot 3 = 3 \odot 1 = 1 \notin I_2$.

A ⊙-subalgebra of BN_1-algebra has a relation with BN_1-ideal that we can see in this theorem

Theorem 1.4 Let $(X, *, \odot, 0)$ be a companion BN_1-algebra. If I is a ideal of X, then I is a ⊙-subalgebra of X.

Proof. Let $x, y \in I$ and $(x * y) * z = (x * y) * z$. By using (SC), we have $((x \odot y) * x) * y = 0 \in I$. Since I is a BN_1-ideal, $x \in I$ and $y \in I$, we get $(x \odot y) * x \in I$ and $x \odot y \in I$. Hence I is a ⊙-sub BN_1-algebra.

Another relation A ⊙-subalgebra of BN_1-algebra with BN_1-ideal as follows:

Theorem 1.5 Let $(X, *, \odot, 0)$ be a companion BN_1-algebra that satisfies associative law $(x * y) * z = (x * y) * z$. If I is a ⊙-subalgebra of X and $0 \in I$, then I is a BN_1-ideal.

Proof. Let I is a ⊙-subalgebra of X and $0 \in I$, and $b \in I$. Since X satisfies $(x * y) * z = (x * y) \odot z$, thus we obtain

$$a = (a \odot b) * b$$

$$a = (a \odot b) \odot b$$

I is a ⊙-subalgebra, so we have $a \odot b \in I$ and $b \in I$ and then $a \in I$. Based on definition i 1.7, I is a BN_1-ideal.

⊙-ideal of BN_1-algebra is defined by

Definition 1.10 (⊙-ideal) Let $(X, *, \odot, 0)$ be a companion BN_1-algebra and I is a subset of i X. I is said to be ⊙-ideal if satisfies

(i) $0 \in I$
(ii) for $x \odot y \in I$ and $y \in I$, then $x \in I$

for $x, y \in I$.

Example 1.6 Let $X := \{0, 1, 2, 3\}$ is a BN_1-algebra in example 1.4. Set $I_1 = \{0,1\}$ and $I_1 = \{0,2\}$ is a ⊙-ideal of X.

Theorem 1.6 Let $(X,*,\odot,0)$ be a companion BN_1-algebra that satisfies $(x*y)*z=(x*y)\odot z$. If I is a $\odot$-ideal of X, then I is a $\odot$-subalgebra of X.

Proof. X satisfies $(x*y)*z=(x*y)\odot z$, we obtain

$$(x\odot y)*y=x$$
$$(x\odot y)\odot y=x$$

Now $x\in I$, so $(x\odot y)\odot y\in I$. Since $(x\odot y)\odot y\in I$, $y\in I$ and I $\odot$-ideal of X, then $x\odot y\in I$. By using definition 1.6, thus I is a $\odot$-subalgebra of X.

Theorem 1.7 Let $(X,*,\odot,0)$ is a companion BN_1-algebra satisfies $(x*y)*z=(x*y)\odot z$. If I is a $\odot$-subalgebra of X and $0\in I$, then I is a $\odot$-ideal of X.

Proof.. Since X satisfies $(x*y)*z=(x*y)\odot z$, we have

$$(x\odot y)*y=x$$
$$(x\odot y)\odot y=x$$

Now $x\in I$, so we have $(x\odot y)\odot y\in I$. Since $(x\odot y)\odot y\in I$, $y\in I$ and I $\odot$-ideal dari X, we obtain $x\odot y\in I$. By using definition 1.6, I is a $\odot$-subalgebra of X.

Theorem 1.8 Let $(X,*,\odot,0)$ be a companion BN_1-algebra that satisfies $(x*y)*z=(x*y)*z$. If I is a $\odot$-subalgebra of X and $0\in I$, then I is a $\odot$-ideal of X.

Proof. I is a $\odot$-subalgebra of X and $0\in I$. Let $a*b\in I$ and $b\in I$ and X satisfies $(x*y)*z=(x*y)\odot z$, we get

$$a=(a\odot b)*b$$
$$a=(a\odot b)\odot b$$

Since I is a$\odot$-subalgebra, so $a\odot b\in I$ and $b\in I$ and then $a\in I$. By using definition 1.10, I is a$\odot$-ideal of X.

IV. CONCLUSION

In this article, the the definition of $\odot$-subalgebra and $\odot$-ideal are developed from the concepts of Companion B-algebra [15] and companio BN1-algebra [1]. We also obtained the relationship between $\odot$-subalgebra and $\odot$-ideal BN1-algebra which is expressed in several theorems, such as the $\odot$-ideal is definitely a $\odot$-sub BN1-algebra. However, a sub BN_1-algebra is not necessarily be a $\odot$-ideal and can be declared be a $\odot$-idea if it satisfies certain axioms.

V. REFERENCES

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