Construction and Applications for Two-Parameter Composite Perturbation of Copulas

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***Abstract* -** *On the basis of the quadratic construction of copulas, the two-parameter composite perturbation of copulas is introduced, the range of its perturbation parameters is given, and the one-parameter perturbation of copulas is obtained under its special form. The new two-parameter composite perturbation of copulas contains the characteristics of the original copula and the one-parameter perturbation of copulas, which further widens the range of their measures of association and enhances the adaptability and accuracy of copula in describing the characteristics of different random vectors. Taking the correlation analysis of the maximum temperature in summer between Tianjin and Shanghai as an example, the six appropriate copulas to real data are obtained. By the method of the squared Euclidean distance, it can easily be found that the two-parameter composite perturbed copula is optimal for* $\alpha=0.51$, $\beta=1$.

Keywords - *copula,quadratic construction, two-parameter perturbation, one-parameter perturbation, correlation analysis, optimal*

I. INTRODUCTION

Copula plays an important role in describing the relevance of random variables(e.g.[1]).Constructing a new copula is to construct a new joint distribution function that can fully characterize the probability properties of random variables. There are several construction methods which are convex combinations of copulas, ordinal sums, or other patchwork techniques(e,g.[2,3,4,5]).

Fitting an appropriate copula to given real data requires a large number of copulas, There are some methods for fitting copulas to experimental data(e.g.[13,14,15]).therefore, any new construction method of copulas extends the possibility of their applications. Once we know approximately a copula C appropriate to the model of the observed data, we look for a minor perturbation of C which fits them better than C itself. This method has been used in papers(e.g.[7,9]). However, the classes of perturbed copulas introduced in(e.g.[6,7,8]) and extended in(e.g.[9]).

Therefore, we have concluded that best models were often obtained in form of perturbations with nonzero values of perturbation parameters. This idea inspires us to further study dependence parameters of perturbation of copulas and their superiority.

II. COPULAS

Copula represents a multivariate distribution that captures the dependence structure among random variables. It is a great tool for building flexible multivariate stochastic model. copula offers the choice of an appropriate model for the dependence between random variables independently from the selection of marginal distributions.

Definition 2.1. A function $C:[0,1]^2 \to [0,1]$ is called a bivariate copula which satisfies the properties.

(i) for all $\mathrm{u},\mathrm{v} \in [0,1]$, $C(\mathrm{u},0)=0$, $C(0,\mathrm{v})=0$,

(ii) for all $\mathrm{u},\mathrm{v} \in [0,1]$, $C(\mathrm{u},1)=\mathrm{u}$, $C(1,\mathrm{v})=\mathrm{v}$,

(iii) for all $u_1,u_2,v_1,v_2 \in [0,1]$, $u_1 \le u_2, v_1 \le v_2$,

$$V_C([u_1,u_2]\times[v_1,v_2]) = C(u_2,v_2) - C(u_1,v_2) - C(u_2,v_1) + C(u_1,v_1) \ge 0 .$$

By property(i),0 is the annihilator of each copula C, and by (ii),1 is its neutral element. The third property is the 2-increasing property of C.

For a 2-dimensional random vector (X,Y) with a joint distribution function F_{XY} and continuous marginal distribution functions F_X, F_Y,a copula C satisfying the relations

$$F_{XY}(x,y) = C(F_X(x), F_Y(y)), \tag{1}$$

is the distribution function of the random vector (U,V),where $U=F_X(x)$, $V=F_Y(y)$ have uniform distribution on $[0,1]$. For more details we can refer to monograph (e.g.[1]).

III. CONSTRUCTION FOR TWO-PARAMETER COMPOSITE PERTURBATION OF COPULAS

A quadratic function P is introduced(e.g.[10]),i.e.,

$$\mathrm{P}(\mathrm{u},\mathrm{v},\mathrm{z}) = \mathrm{a}\mathrm{u}^2 + bv^2 + cz^2 + duv + euz + fvz + gu + hv + iz + j, \tag{2}$$

with real coefficients $\mathrm{a,b,c,d,e,f,g,h,i,j} \in R$ and and $z=C(\mathrm{x},\mathrm{y})$ (with $(x,y)\in[0,1]^2$) is a copula. It is important to find conditions under which the function P defined by (2) is a copula for any copula C .By the boundary conditions (i),we can simplify the formula(2)and write it as:

$$P(u,v,C(u,v)) = cC(u,v)^2 + duv - cuC(u,v) - cvC(u,v) + (1+c-d)C(u,v),\ c,d \in R \tag{3}$$

To distinguish the coefficient c from the density function of copula , let $\alpha=c, \beta=b$ and $C^{\alpha,\beta}(u,v) = P(u,v,C(u,v))$,then formula (3) can be written as

$$C^{\alpha,\beta}(u,v) = \alpha C(u,v)^2 + \beta uv - \alpha uC(u,v) - \alpha vC(u,v) + (1+\alpha-\beta)C(u,v).$$

Consequently,

$$C^{\alpha,\beta}(u,v) = (1+\alpha-\beta)C(u,v) + \beta\Pi - \alpha C(u,v)(u+v-C(u,v)). \tag{4}$$

Formula (4) contains $C(u,v)$, Π and $C(u,v)(u+v-C(u,v))$ .Then the construction (4) can be seen as the composite copula $C^{\alpha,\beta}$ .Where $C(u,v)$ is a fixed copula and $C(u,v)(u+v-C(u,v))$ is a continuous function. Coefficients α and β are called perturbation parameters. In the next theorem, we give sufficient and necessary conditions so that $C^{\alpha,\beta}$ to be a copula for any copula C .

Theorem 3.1 For a copula C:$[0,1]^2 \to [0,1]$,let $C^{\alpha,\beta}$ is a function defined on$[0,1]^2$ by (4).Then conditions (I) and (II) are equivalent.

(I) For any copula C ,$C^{\alpha,\beta}$ is a copula.

(II) $C^{\alpha,\beta}$ is given by formula (4) with coefficients α ,β satisfying conditions

$$0 \le \beta \le 1, -1 \le \alpha \le 1, 0 \le \beta - \alpha \le 1.$$

Proof.(I)$\Rightarrow$ (II)

The restrictions of coefficients α and β can be found by distinguishing copulas $C = M$,W and Π .

Because $C^{\alpha,\beta}$ is a copula. firstly, let us consider that $C = M$ and investigate the 2-increasing property of $C^{\alpha,\beta}$,we have

$$C^{\alpha,\beta} = (\beta - \alpha)uv + (1 + \alpha - \beta)\min\{u, v\},$$

the $C^{\alpha,\beta}$ -volune of any rectangle $R \in [u_1, u_2] \times [v_1, v_2]$ (with $R = \{(u,v) \in [0,1]^2 \mid u \le v\}$) is

$$V_{C^{\alpha,\beta}}(R) = (\beta - \alpha)(u_2 - u_1)(v_2 - v_1),$$

and thus $V_{C^{\alpha,\beta}}(R) \ge 0$,then we can obtain $\beta \ge \alpha$ .Moreover for all $0 \le a \le b \le 1$ we have

$$V_{C^{\alpha,\beta}}([a,b]^2) = (b - a)(1 - (\beta - \alpha)(1 - b + a)),$$

and thus $V_{C^{\alpha,\beta}}[\mathrm{a},\mathrm{b}]^2 \ge 0$ if and only if $\beta - \alpha \le 1$.

In summary, for $C = M$,the necessary conditions for $C^{\alpha,\beta}$ to be a copula are

$$0 \le \beta - \alpha \le 1.$$

Next, let us suppose that $C = W$,then

$$C^{\alpha,\beta} = \beta uv + (1 - \beta)\max\{0, u + v - 1\},$$

necessarily $\beta \ge 0$ and because of the monotonicity of $C^{\alpha,\beta}$ leads to $\beta \le 1$ .Therefore, we have

$$0 \le \beta \le 1.$$

Finally, assume $C = \Pi$,as

$$C^{\alpha,\beta} = uv + \alpha uv(1 - u)(1 - v),$$

we obtain the function $C^{\alpha,\beta}$ is a copula if and only if $\alpha \in [-1,1]$,see e.g.[1].It is known that $C^{\alpha,\beta}$ is Farlie-Gumbel-Morgenstern family.

Consequently, the function $C^{\alpha,\beta}$ given by (4) to be a copula for each copula C ,we require necessary conditions are

$$0 \le \beta \le 1, -1 \le \alpha \le 1, 0 \le \beta - \alpha \le 1.$$

(2)$\Rightarrow$ (1)

For $0 \le \beta \le 1, -1 \le \alpha \le 1, 0 \le \beta - \alpha \le 1$,then the set of all mentioned coefficients can be denoted by Ω ,i.e.,

$$\Omega = \{(\alpha,\beta) \in \mathrm{R}^2 | 0 \le \beta \le 1, -1 \le \alpha \le 1, 0 \le \beta - \alpha \le 1\}.$$

All possible (α, β) of the set Ω form a parallelogram with vertices $(0,1)$,$(1,1)$,$(-1,0)$ and $(0,0)$ .thus, the four vertices corresponds to the functions, i.e.,

$C^{0,1} = uv = \Pi$,

$C^{1,1} = C^2 + uv - uC - vC + C$,

$C^{-1,0} = -C^2 + uC + vC = C(u + v - C)$

$C^{0,0} = C$

Let C be a copula. Then, where $C^{1,1}$ and $C^{-1,0}$ (e.g.[10]) are also copulas. Because each point $(\alpha, \beta) \in \Omega$ is a convex combination of its vertices. Therefore the function $C^{\alpha,\beta}$ is the same convex combination of copulas $C^{0,1}, C^{1,1}, C^{-1,0}$ and $C^{0,0}$ .Thus, for any copula C , we can obtain the two-parameter composite perturbation of copulas, i.e., $C^{\alpha,\beta}$ (with $(\alpha, \beta) \in \Omega$).

Moreover, let us suppose that $\alpha = \beta$,Then (4) is equivalent to

$$C_\beta(u,v) = C(u,v) + \beta D(u,v) \qquad \beta \in [0,1] \qquad (5)$$

where $D(u,v) = (u - C(u,v))(v - C(u,v))$, for any copula C ,the function C_β is a copula, we also call it one-parameter perturbation of copulas. For more details on (5), refer to monographs(e.g.[7,8]).

IV. DEPENDENCE MEASURES FOR TWO-PARAMETER COMPOSITE PERTURBATION OF COPULAS

A Kendall's tau

The sample version of the measure of association known as Kendall's tau is defined in terms of concordance. Let (X_1, Y_1) and (X_2, Y_2) be independent and identically distributed random vectors with a common joint distribution. Then the population version of Kendall's tau is defined as the probability of concordance minus the probability of discordance:

$$\tau_{X,Y} = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0]$$

If X and Y are continuous random variables with copula C , then the population version of Kendall's tau is

$$\tau(C) = 4\iint_{I^2} C(u,v)dC(u,v) - 1 = 1 - 4\iint_{I^2} C_u(u,v)C_v(u,v)\,du\,dv \qquad (8)$$

We also use notations $C_u(u,v)$ and $C_v(u,v)$ for the partial derivatives of C . Moreover, if C is an absolutely continuous copula, then

$$dC(u,v) = \frac{\partial C(u,v)}{\partial u \partial v} = c(u,v)dudv \quad ,$$

for details, refer to (e.g.[1]).

***Theorem I*:**

Let $C : [0,1]^2 \to [0,1]$ be a copula with Kendall's tau by (8), for the two-parameter composite perturbation $C^{\alpha,\beta} = (1 + \alpha - \beta)C + \beta \prod - \alpha H$ (with $(H = C(u + v - C))$),then, we have

$$\tau(C^{\alpha,\beta}) = 1 - \beta^2 -$$

$$-4\iint_{I^2} AC_uC_v + B(C_uH_v + C_vH_u) + D(vC_v + uC_u) + E(vH_v + uH_u) + FH_uH_v\,dudv \quad ,(9)$$

where $A = (1 + \alpha - \beta)^2$, $B = -(1 + \alpha - \beta)\alpha$, $D = (1 + \alpha - \beta)\beta$, $E = -\alpha\beta$, $F = \alpha^2$.

Proof. The formula (9) follows directly from the second part of formula (8),for the partial derivatives

$$C_u^{\alpha,\beta} = (1+\alpha-\beta)C_u + \beta v - \alpha H_u ,$$

and

$$C_v^{\alpha,\beta} = (1+\alpha-\beta)C_v + \beta u - \alpha H_v ,$$

Consequently

$$C_u^{\alpha,\beta} C_v^{\alpha,\beta} = \beta^2 uv + (1+\alpha-\beta)^2 C_u C_v - (1+\alpha-\beta)\alpha C_u H_v - (1+\alpha-\beta)\alpha C_v H_u +$$
$$+ (1+\alpha-\beta)\beta u C_u + (1+\alpha-\beta)\beta v C_v - \alpha\beta v H_v - \alpha\beta u H_u + \alpha^2 H_u H_v . \quad (10)$$

Substitution $A = (1+\alpha-\beta)^2$, $B = -(1+\alpha-\beta)\alpha$, $D = (1+\alpha-\beta)\beta$, $E = -\alpha\beta$ and $F = \alpha^2$,then we can simplify (10) and write it as:

$$C_u^{\alpha,\beta} C_v^{\alpha,\beta} = \beta^2 uv + (1+\alpha-\beta)^2 C_u C_v - (1+\alpha-\beta)\alpha (C_u H_v + C_v H_u)$$
$$+ (1+\alpha-\beta)\beta (uC_u + vC_v) - \alpha\beta\, (vH_v + uH_u) + \alpha^2 H_u H_v \quad (11)$$

Next, combining (8) and (11) we get that (9) holds.

B Spearman's rho

Let (X_1, Y_1), (X_2, Y_2) and (X_3, Y_3) be independent and identically distributed random vectors with a common joint distribution. Spearman's rho is defined to be proportional to the probability of concordance minus the probability of discordance of the pairs (X_1, Y_1) and (X_2, Y_3):

$$\rho_{X,Y} = 3\{P[(X_1 - X_2)(Y_1 - Y_3) > 0] - P[(X_1 - X_2)(Y_1 - Y_3) < 0]\}$$

If X and Y are continuous random variables with copula C ,then the population version of Spearman's rho is

$$\rho(C) = 12 \iint_{I^2} C(u,v)\,dudv - 3 . \quad (12)$$

***Theorem II*:** Let $C : [0,1]^2 \to [0,1]$ be a copula with Spearman's rho by (12), for the two-parameter composite perturbation $C^{\alpha,\beta} = (1+\alpha-\beta)C + \beta \prod - \alpha C(u+v-C)$,then,we have

$$\rho(C^{\alpha,\beta}) = (1+\alpha-\beta)\rho(C) + 6\alpha \iint_{I^2} (u^2 + 2uv - 4uC) C_u\, dudv . \quad (13)$$

Proof. Now, combining (4) and (12) we get

$$\rho(C^{\alpha,\beta}) = 12 \iint_{I^2} ((1+\alpha-\beta)C + \beta \prod - \alpha C(u+v-C))\,dudv - 3$$
$$= 12 \iint_{I^2} (1+\alpha-\beta)C dudv - 3 + 12 \iint_{I^2} \beta \prod dudv - 12 \iint_{I^2} \alpha C(u+v-C)\,dudv \;. (14)$$

The integral (14) can be writen as the sum of three terms, for the first term we have

$$12 \iint_{I^2} (1+\alpha-\beta) C\, dudv - 3 = (1+\alpha-\beta)(12 \iint_{I^2} C\, dudv - 3 + 3) - 3$$
$$= (1+\alpha-\beta)\rho(C) + 3(\alpha-\beta) . \quad (15)$$

For second term, then

$$12 \iint_{I^2} \beta \prod dudv = 3\beta . \quad (16)$$

For third integral

$$\iint_{I^2} (u+v-C) C\, dudv = -\iint_{I^2} C^2 dudv + \iint_{I^2} uC\, dudv + \iint_{I^2} vC\, dudv , \quad (17)$$

where

$$-\iint_{I^2} C^2 dudv = -\int_0^1 (\int_0^1 C^2 du) dv ,$$

$$\int_0^1 C^2 du = [uC^2]_0^1 - \int_0^1 2uCC_u du = v^2 - 2\int_0^1 uCC_u du ,$$

then

$$-\iint_{I^2} C^2 du dv = -\int_0^1 (v^2 - 2\int_0^1 uCC_u du) dv = -\frac{1}{3} + 2\iint_{I^2} uCC_u du dv . \tag{18}$$

moreover

$$\iint_{I^2} uCdudv = \int_0^1 (\int_0^1 uCdu) dv ,$$

$$\int_0^1 uCdu = [\frac{u^2}{2}C]_0^1 - \int_0^1 \frac{u^2}{2} C_u du = \frac{v}{2} - \frac{1}{2}\int_0^1 u^2 C_u du ,$$

hence

$$\iint_{I^2} uCdudv = \int_0^1 (\frac{v}{2} - \frac{1}{2}\int_0^1 u^2 C_u du) dv = \frac{1}{4} - \frac{1}{2}\iint_{I^2} u^2 C_u dudv . \tag{19}$$

Similarly,

$$\iint_{I^2} vCdudv = \int_0^1 v(\int_0^1 Cdu) dv ,$$

$$\int_0^1 Cd u = [uC]_0^1 - \int_0^1 uC_u du = v - \int_0^1 uC_u du ,$$

$$\int_0^1 v(\int_0^1 Cdu) dv = \int_0^1 (v^2 - \int_0^1 uvC_u du) dv ,$$

thus

$$\iint_{I^2} vCdudv = \frac{1}{3} - \iint_{I^2} uvC_u dudv . \tag{20}$$

By giving (18), (19), (20) to (17) we get

$$\iint_{I^2} (u + v - C)Cdudv = 1/4 - 1/2\iint_{I^2} (u^2 + 2uv - 4uC)C_u dudv . \tag{21}$$

Finally, combining (15), (16) and (21) we get that (13) holds.

Example I. In the previous discussion, the formulas of calculations of the measures of association for the two-parameter composite perturbation of copulas is given. Next, the measures of association for different copulas and their perturbations will be considered:

For Product copula Π ,combining (4), we get the two-parameter composite perturbation $\Pi^{\alpha,\beta}$ .By theorem 4.1 and 4.2,It is not difficult to calculate its Kendall's tau and Spearman's. Thus

$$\tau(\Pi^{\alpha,\beta}) = 0.222\alpha \in [-0.222, 0.222],$$

$$\rho(\Pi^{\alpha,\beta}) = 0.333\alpha \in [-0.333, 0.333],$$

where $\alpha \in [-1,1]$.

However, for $\alpha = \beta$,then the two-parameter composite perturbation (4) is transformed into the one-parameter perturbation (5),therefore, whose Kendall's tau and Spearman's rho are

$$\tau(\Pi^{\beta}) = 0.222\beta \in [0, 0.222]$$

and

$$\rho(\Pi^{\beta}) = 0.333\beta \in [0, 0.333]$$

where $\beta \in [0,1]$.

Similarly, for $FGM(u,v) = uv + \theta uv(1-u)(1-v)$,where $\theta \in [-1,1]$,then results of calculations of the measures of association for two-parameter composite perturbation $FGM^{\alpha,\beta}$ are in TABLE I.

TABLE II

COEFFICIENTS OF MEASURES OF ASSOCIATION FOR $FGM^{\alpha,\beta}$

θ	$\tau(FGM^{\alpha,\beta})$	$\rho(FGM^{\alpha,\beta})$
-1	$-0.002\alpha^2 + 0.124\alpha + 0.222\beta - 0.004\alpha\beta - 0.222$	$0.18\alpha + 0.333\beta - 0.333$
-0.5	$-0.002\alpha^2 + 0.17\alpha + 0.111\beta - 0.001\alpha\beta - 0.111$	$0.253\alpha + 0.167\beta - 0.167$
0	0.222α	0.333α
0.5	$0.003\alpha^2 + 0.281\alpha - 0.111\beta - 0.001\alpha\beta + 0.111$	$0.42\alpha - 0.167\beta + 0.167$
1	$0.007\alpha^2 + 0.347\alpha - 0.222\beta - 0.004\alpha\beta + 0.222$	$0.513\alpha - 0.333\beta + 0.333$

As $\alpha \in [-1,1], \beta \in [0,1]$ and $\alpha \le \beta$, for $\theta = -1$, $\alpha = -1$, $\beta = 0$,then, the minimum values of $\tau(FGM^{\alpha,\beta})$ is -0.348; however, for $\theta = 1, \alpha = 1, \beta = 1$,The minimum values we can get $\tau(FGM^{\alpha,\beta})$ is 0.35.therefore, the range of $\tau(FGM^{\alpha,\beta})$ is $[-0.348, 0.35]$ .Similarly, $\rho(FGM^{\alpha,\beta}) \in [-0.513, 0.513]$. Moreover, for $\rho(FGM^{\alpha,\beta})$,When the values of θ is from -1 to 1, the influence of the perturbation parameter α to ρ is gradually increasing, because the coefficient of α is gradually increasing from 0.18 to 0.513. However, for $\theta \in [-1,0]$, that influence of the perturbation parameter β to ρ is decreasing for increasing values of θ , ,and for $\theta \in [0,1]$, the influence of the perturbation parameter β to ρ is negative, but increasing for increasing values of θ .

For $\alpha = 0$, $\beta = 0$, then $FGM^{0,0} = FGM$, where $\tau(FGM) \in [-0.222, 0.222]$ and $\rho(FGM) \in [-0.333, 0.333]$.

For $\alpha = \beta \in [0,1]$, then the equation (4) is equivalent to the equation (5), i.e., one-parameter perturbation of copulas. Where Kendall's tau and Spearman's rho are in TABLE II,

TABLE II

COEFFICIENTS OF MEASURES OF ASSOCIATION FOR FGM^{β}

θ	$\tau(FGM^{\beta})$	$\rho(FGM^{\beta})$
-1	$-0.222 + 0.35\beta - 0.007\beta^2$	$-0.333+0.513\beta$
-0.5	$-0.112 + 0.285\beta - 0.003\beta^2$	$-0.167 + 0.42\beta$
0	0.22β	0.33β
0.5	$0.113 + 0.175\beta + 0.002\beta^2$	$0.167 + 0.253\beta$
1	$0.217 + 0.12\beta + 0.003\beta^2$	$0.333 + 0.18\beta$

By TABLE II, for the Kendall's tau of FGM^{β} , when θ is -1, the unary quadratic function opens downward and the range of the perturbation parameter β is on the left side of the symmetry axis, i.e., $\beta \in [0,1]$, and the symmetry axis is $\beta = -0.35/(2\times(-0.007)) = 25$, thus,which clearly is an increasing function in interval $[0,1]$. We can get minimum -0.222 for $\beta = 0$. However, for $\theta = 1$, Similarly, we can attain maximum 0.34 for $\beta = 1$. Therefore, for $\theta \in [-1,1]$, then $\tau(FGM^{\beta}) \in [-0.222, 0.34]$. Consequently, for Spearman's rho of FGM^{β}, it is not difficult to obtain $\rho(FGM^{\beta}) \in [-0.333, 0.513]$ for $\theta \in [-1,1]$ and $\beta \in [0,1]$.

In summary, for a fixed Copula C, The two-parameter composite perturbation of copulas and the one-parameter perturbation of copulas both broaden the range of the Kendall's tau and Spearman's rho. However, the two-parameter composite perturbation of copulas is wider than that of the one-parameter perturbation of copulas. This can be seen as the superiority of the two-parameter composite perturbation of copulas.

Example II. In the paper(e.g.[12]), a general FGM copula was introduced, where

$$C_{\theta}(u,v) = uv + \theta u^{a} v^{b} (1-u^{m})^{c} (1-v^{n})^{d}.$$

the general FGM copula is used to describe the correlation between the ditch bed ratio and the water level difference in the debris flow geomorphology. Although this general FGM copula broadens the range of the coefficients of measures of association, the accuracy of fitting empirical data is not guaranteed. Compared to perturbations of copulas, it's is a big disadvantage. The perturbation of copulas not only broaden the range of dependence measures to a certain degree, but also increase the accuracy of fitting real data by adding the perturbations. Next take the temperature data as an example to compare the above copulas.

Since copula is rarely used in environmental meteorology, thus studying the correlation of temperatures in different regions can help travel between the two cities. Tianjin and Shanghai are two major cities in the north and the south, and exchanges are becoming more frequent in all aspects. Hence it is quite meaningful to study the correlation of temperatures for Tianjin and Shanghai. According to people's habits of temperature, we selected the daily maximum temperature data (from the weather website) of Tianjin and Shanghai from June 1 to August 31, 2017 for a total of 92 days.

By mathematical calculation software, we can get Kendall's tau and Spearman's rho of the sample data, where $\hat{\tau} = 0.158$, $\hat{\rho} = 0.214$. Since the relationship between $\hat{\tau}$ and $\hat{\rho}$ is approximately $3\hat{\tau} = 2\hat{\rho}$, thus the FGM family(e.g.[1]) as appropriate copulas will be considered. For the FGM family, it's not difficult to obtain $\tau(FGM) = 2\theta/9$ and $\rho(FGM) = \theta/3$. Let $\hat{\rho} = \rho(FGM)$, then we can conclude that $\hat{\theta} = 3\times 0.214 = 0.642$, thus an appropriate FGM copula for fitting temperature data is

$$FGM_{\theta}(u,v) = uv + 0.642\, uv(1-u)(1-v),$$

which is also a general FGM copula for $a = b = c = d = m = n = 1$ (e.g.[12])

However, when $a = b = c = d = 2$, $m = n = 1$, similarly, let $\hat{\rho} = \rho(FGM_{\theta})$, then $\hat{\theta} = 75 \times 0.214 = 16.05$ (e.g[12]), thus corresponds to a general FGM copula is

$$FGM_{\theta}(u,v) = uv + 16.05\, u^{2} v^{2} (1-u)^{2} (1-v)^{2},$$

For the one-parameter perturbation FGM^{β}, If $\hat{\theta} = 0.642$, then the perturbation parameter $\beta = 0$ by $\hat{\rho} = \rho(FGM^{\beta})$, i.e., $FGM^{\beta} = FGM$. In other words, the perturbations of copulas do not work. In order to study the effect of fitting data of the summer maximum temperature of Tianjin and Shanghai for the original FGM adding the perturbations. Let $\theta = 0.5$ of FGM, on this basis, add the perturbations. For $\theta = 0.5$, then $\rho(FGM^{\beta})$ is given by TABLE II, thus

$$\hat{\rho} = \rho(FGM^{\beta}) = 0.167 + 0.253\beta = 0.214, \tag{22}$$

Then $\beta = 0.186$. Consequently, the appropriate FGM^{β} is given, i.e.,

$$FGM^{\beta}(u,v) = C(u,v) + 0.186(u - C)(v - C),$$

where C denotes FGM family with $\theta = 0.5$.

Next, for the two-parameter composite perturbation $FGM^{\alpha,\beta}$, similarly, let $\theta = 0.5$ of FGM, and add the perturbations. The $\tau(FGM^{\alpha,\beta})$ and $\rho(FGM^{\alpha,\beta})$ of the two-parameter composite perturbation $FGM^{\alpha,\beta}$ are given by Table 1 for $\theta = 0.5$. If $\hat{\tau} = \tau(FGM^{\alpha,\beta})$ and $\hat{\rho} = \rho(FGM^{\alpha,\beta})$, then for $0 \le \beta \le 1, -1 \le \alpha \le 1, 0 \le \beta - \alpha \le 1$, there are no solutions about α and β. Thus, we only need to consider

$$\hat{\rho} = \rho(FGM^{\alpha,\beta}) = 0.42\alpha - 0.167\beta + 0.167 = 0.214, \tag{23}$$

with α, β satisfying formula (23). Hence, three pairs (α, β) of perturbation parameters are obtained by (23), i.e., $(0.51, 1)$, $(0.43, 0.8)$, $(0.35, 0.6)$. Finally, combining the three pairs (α, β) with (4) we get three models of the two-parameter composite perturbation $FGM^{\alpha,\beta}$.

In previous discussion, the FGM, the general FGM, the one-parameter perturbation FGM^{β} and the two-parameter composite perturbation $FGM^{\alpha,\beta}$ respectively fit the data of maximum temperature of Tianjin and Shanghai in summer. In order to select the best one from these six models, by the method of the squared Euclidean distance, we use the six theoretical copula to subtract the empirical copula respectively and then take the square. Where the six calculated squared Euclidean distances are in TABLE III.

TABLE III

THE SQUARED EUCLIDEAN DISTANCE BETWEEN THE THEORETICAL COPULA AND THE EMPIRICAL COPULA

Copula	$FGM_{0.642}$	$FGM_{16.05}$	$FGM^{0.186}$	$FGM^{0.35,0.6}$	$FGM^{0.43,0.8}$	$FGM^{0.51,1}$
Distance	0.0223	0.0288	0.0221	0.0259	0.0228	0.0219

Because the smaller the squared Euclidean distance, the better the fitting effect. By Table 3, for $\alpha = 0.51, \beta = 1$, the two-parameter composite perturbation $FGM^{\alpha,\beta}$ has the best effect on the fitting of the highest temperature data in Tianjin and Shanghai, and the model is optimal. Therefore, the two-parameter composite perturbation $FGM^{\alpha,\beta}$ is superior to the one-parameter perturbation FGM^{β} and the general FGM copula in data modeling.

V. CONCLUSION

By adding perturbations to the original copula, not only can the diversity of the copula form be expanded, but also the accuracy of fitting the empirical data can be further improved. This paper introduces the two-parameter composite perturbation of copulas and proves its some properties. Moreover, it can easily be checked that its special form is the one-parameter perturbation of copulas. Compared with the one-parameter perturbation, the two-parameter composite perturbation of copulas not only has invariance, but also broadens the range of the coefficients of measures of association, which improves the adaptability of copula in describing the characteristics of different random vectors. More importantly, in terms of fitting an appropriate copula to real data, it is better than the original copula and the one-parameter perturbation of copulas.

REFERENCES

[1] R.B.Nelsen,An Introdution to Copulas .2nd ed. New York: Springer, 2006.

[2] B.De Baets,H.De Meyer,Orthogonal grid constructions of copulas[J]. IEEE Transactions on Fuzzy Systems. 15(6): 1053-1062, 2007.

[3] F.Durante,A.Kolesárová, R.Mesiar,et al,Copulas with given diagonal sections: novel constructions and applications. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems. 15(04): 397-410, 2007.

[4] F.Durante, A.Kolesárová, R.Mesiar, et al. Semilinear copulas. Fuzzy Sets and Systems. 159(1): 63-76, 2008.

[5] F.Durante, S.Saminger-Platz, P.Sarkoci, On patchwork techniques for 2-increasing aggregation functions and copulas.//Soft Methods for Handling Variability and Imprecision. Springer, Berlin, Heidelberg. 349-356, 2008.

[6] F.Durante,J. F.Sánchez,M.Ú.Flores,Bivariate copulas generated by perturbations. Fuzzy Sets and Systems. 228:137-144, 2013.

[7] R.Mesiar,M.Komorníková,Jozef Komorník,Perturbation of bivariate copulas.Fuzzy Sets and Systems. 268: 127-140, 2015.

[8] J.Komorník,M.Komorníková,J.Kalická,Dependence measures for perturbations of copulas. Fuzzy Sets and Systems. 324: 100-116, 2017.

[9] J.Komorník,M.Komorníková,J.Kalická,N.Cuong,Tail dependence of enriched class of perturbed copulas with modelling applications, in:Lidia Filus,Teresa Oliveira,Christos H.Skiadas(Eds.),Stochastic Modeling,Data Analysis &Statistical Applications. pp.53-63, 2016.

[10] A.Kolesárová,G.Mayor,and R.Mesiar,Quadratic constructions of copulas. Information Sciences. 310: 69-76, 2015.

[11] R.L.Plackett,A class of bivariate distributions. Am.Stat.Assoc. 60: 516-522, 1965.

[12] F.X.Xu,Y.Q.Dong,Correlation of a new class of general FGM copulas and their applications. Chinese Journal of Engineering Mathematics. 28(04):475-481, 2011.

[13] A.Dolati,M.Ubeda-Flores, Constructing copulas by means of pairs of order statistics. Kybernetika. 45(6) :992-1002, 2009.

[14] H.Joe,Multivariate models and multivariate dependence concepts. Chapman and Hall/CRC, 1997.

[15] Patton, J.Andrew, Modelling asymmetric exchange rate dependence. International economic review. 47(2): 527-556, 2006.