# The Path Induced Edge-to-Vertex Monophonic Number of Graphs

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***ABSTRACT--*** *Let $G$ be a connected graph with at least three vertices. An edge-to-vertex monophonic set is called a path induced edge-to-vertex monophonic set of $G$ if $< M >$ contains a Hamiltonian path. The minimum cardinality of a path induced edge-to-vertex monophonic set is called the path induced edge-to-vertex monophonic number of $G$ and is denoted by $pim_{ev}(G)$. The minimum path induced edge-to-vertex monophonic set with $|M| = pim_{ev}(G)$ is called a minimum path induced edge-to-vertex monophonic number of $G$ or $pim_{ev}$-set of $G$. Some general properties satisfied by this concept are studied. For every pair $a$ and $b$ of integers with $4 \leq a \leq b$, there exists a connected path induced edge-to-vertex geodetic graph $G$ such that $pim_{ev}(G) = a$ and $pig_{ev}(G) = b$.*

**Keywords:** *monophonic path, monophonic number, path induced edge-to-vertex monophonic number.*

**AMS Subject Classification: 05C12.**

## 1. INTRODUCTION

By a *graph* $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$ respectively. We consider connected graph with at least three vertices. For basic graph theoretic terminology we refer to Harary [2]. For two vertices $u$ and $v$ in a connected graph $G$, the *distance* $d(u, v)$ is the length of a shortest $u - v$ path in $G$. An $u - v$ path of length $d(u, v)$ is called an $u - v$ *geodesic*. For a vertex $v$ of $G$, the *eccentricity* $e(v)$ is the distance between $v$ and a vertex farthest from $v$. The minimum eccentricity among the vertices is the *radius*, *rad* $G$ and the maximum eccentricity is the *diameter*, *diam* $G$ of $G$. For subsets $A$ and $B$ of $V(G)$, the *distance* $d(A, B)$ is defined as $d(A, B) = min\{d(x, y) : x \in A, y \in B\}$. An $u - v$ path of length $d(A, B)$ is called an $A - B$ *geodesic* joining the sets $A$, $B$ where $u \in A$ and $v \in B$. A vertex $x$ is said to lie on an $A - B$ *geodesic* if $x$ is a vertex of an $A - B$ *geodesic*. For $A = (u, v)$ and $B = (z, w)$ with $uv$ and $zw$ edges, we write an $A - B$ geodesic as $uv - zw$ *geodesic* and $d(A, B)$ as $d(uv, zw)$. The *maximum degree* of G, denoted by $\Delta(G)$, is given by $\Delta(G) = max\{degG(v): v \in V(G)\}$, $N(v) = \{u \in V(G): uv \in E(G)\}$ is called the *neighborhood* of the vertex $v$ in $G$. A vertex $v$ is an *extreme vertex* of a graph $G$ if the sub graph induced by its neighbors is complete. An edge $e$ of a graph $G$ is called an *extreme edge* of $G$, if one of its ends is an extreme vertex of $G$. A *chord* of a path $u_0, u_1, u_2, \ldots u_h$ is an edge $u_i u_j$, with $j \geq i + 2$. An $u - v$ path is called a *monophonic path* if it is a chordless path. A *monophonic set* of $G$ is a set $M \subseteq V$ such that every vertex of $G$ lies on a monophonic path joining some pair of vertices in $M$. A set $S \subseteq E$ is called an *edge-to-vertex monophonic set* if every vertex of $G$ lies on a monophonic path between two vertices in $V(S)$. The *edge-to-vertex monophonic number* $m_{ev}(G)$ of $G$ is the minimum cardinality of its edge-to-vertex monophonic sets and any edge-to-vertex monophonic set of cardinality $m_{ev}(G)$ is an $m_{ev}$- set of $G$. A monophonic set $M \subseteq V$ is called a *path induced monophonic set* of $G$ if $< M >$ has a Hamiltonian path. The minimum cardinality of a path induced monophonic set is called *path induced monophonic number* of $G$, denoted by $pim(G)$. A path induced monophonic set with $|M| = pim(G)$ is called a minimum path induced monophonic number of $G$ or $pim$-set of $G$.

The following theorems are used in sequel.

**Theorem 1.1 [3].** If $v$ is an extreme vertex of a connected graph $G$, then every edge-to-vertex monophonic set of $G$ contains at least one extreme edge incident with $v$.

**Theorem 1.2 [5].** Every end-edge of a connected graph $G$ belongs to every edge-to-vertex geodetic set of $G$.

**The Path Induced Edge-to-Vertex Monophonic Graphs**

**Definition 2.1.** Let $G$ be a connected graph with at least three vertices. An edge-to-vertex monophonic set is called a *path induced edge-to-vertex monophonic set* of $G$ if $< M >$ contains a Hamiltonian path. The minimum cardinality of a path induced edge-to-vertex monophonic set is called the *path induced edge-to-vertex monophonic number* of $G$ and is denoted by $pim_{ev}(G)$. The minimum path induced edge-to-vertex monophonic set with $|M| = pim_{ev}(G)$ is called a *minimum path induced edge-to-vertex monophonic number* of $G$ or $pim_{ev}$-set of $G$.

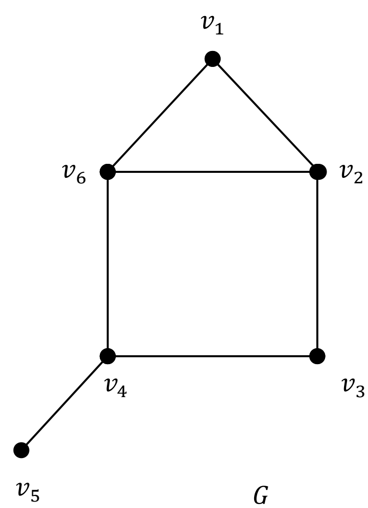



Figure 2.1

**Example 2.2.** For the graph $G$ given in Figure 2.1, $M_1 = \{v_1v_2, v_4v_5\}$ is an edge-to-vertex monophonic set of $G$, so that $m_{ev}(G) = 2$. Since $< M_1 >$ has no Hamiltonian path, $M_1$ is not a path induced edge-to-vertex monophonic set of $G$ and so $pim_{ev}(G) \geq 3$. Let $M_2 = \{v_1v_2, v_4v_5, v_4v_6\}$. Then $M_2$ is a $pim_{ev}$-set of $G$ so that $pim_{ev}(G) = 3$.

**Remark 2.3.** There can be more than one $pim_{ev}$-set of $G$. For the graph $G$ given in Figure 2.2, $M_1 = \{v_1v_2, v_2v_3, v_3v_4\}$ and $M_2 = \{v_1v_2, v_2v_5, v_4v_5\}$ are two $pim_{ev}$-sets of $G$.

**Remark 2.4.** The path induced edge-to-vertex monophonic set does not exist for all connected graphs. For the graph $G$ given in Figure 2.3, there is no path induced edge-to-vertex monophonic set.

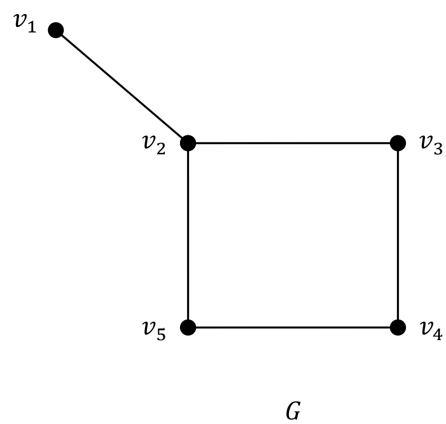



Figure 2.2

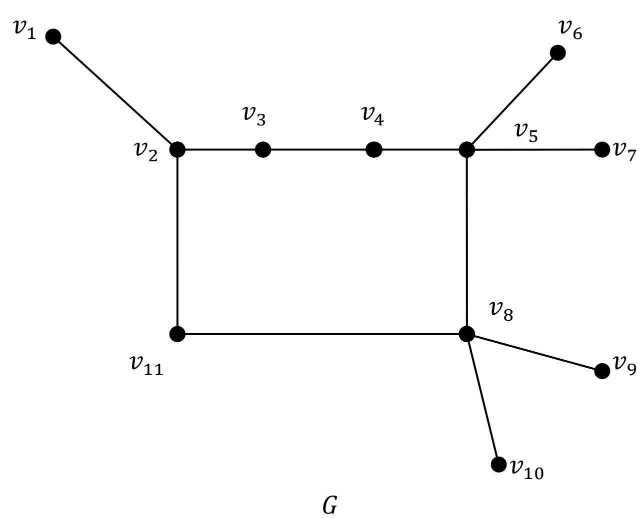



Figure 2.3

**Observation 2.5.**

(i) Each end edge of $G$ belongs to every path induced edge-to-vertex monophonic set of $G$.

(ii) Let $G$ be a non-trivial tree, then $G$ has a path induced edge-to-vertex monophonic set if and only if $G$ is a path.

(iii) The cycle $G = C_p$ $(p \geq 3)$, has a path induced edge-to-vertex monophonic set.

(iv) The complete graph $G = K_p, (p \geq 3)$, has a path induced edge-to-vertex monophonic set.

(v) For the complete bipartite graph $G = K_{m,n}(2 \leq m \leq n)$, has a path induced edge-to-vertex monophonic set if and only if $m = n$. ■

**Definition 2.6.** A graph $G$ is said to be a *path induced edge-to-vertex monophonic graph* if it has a path induced edge-to-vertex monophonic set.

**Theorem 2.7.** Let $G$ be a connected graph. Then $G$ is a path induced edge-to-vertex monophonic graph if and only if $G$ contains at most two end edges and also if $G$ contains two end edges, then they are not adjacent.

**Proof.** Let $G$ be a path induced edge-to-vertex monophonic graph. We prove that $G$ contains at most two end edges. On the contrary $G$ contains at least 3 end edges. Let $M$ be a path induced edge-to-vertex monophonic set of $G$. Then by Observation 2.1(i) $M$ contain at least 3 end edges. Hence it follows that $< M >$ contains no Hamiltonian path, Which is a contradiction. Therefore $G$ contains at most two end edges.

Conversely, Let $G$ contains at most two end edges and they are not adjacent. Let $M \subseteq E(G)$ be an edge-to-vertex monophonic set of $G$. By Observation 2.1 (i) $M$ contains each end edges. Let $M' \subseteq E(G)$ and $M' = M \cup Z$, where $Z \subseteq E(Z)$ such that $< M' >$ has a Hamiltonian path. Therefore $G$ is a path induced edge-to-vertex monophonic set of $G$. ■

**Theorem 2.8.** If $v$ is an extreme vertex of a path induced edge-to-vertex monophonic graph $G$, then every path induced edge-to-vertex monophonic set contains at least one extreme edge is incident with $v$.

**Proof.** Since every path induced edge-to-vertex monophonic set of $G$, is a edge-to-vertex monophonic set of $G$, the result follows from Theorem 1.1. ■

**Corollary 2.9.** Each end edge of a path induced edge-to-vertex monophonic graph $G$ belongs to every path induced edge-to-vertex monophonic set of $G$.

**Proof.** This follows from Theorem 2.8. ■

**Theorem 2.10.** Let $G$ be a connected non-trivial graph of size $q$. If $G$ contains a spanning path, then $G$ is a path induced edge-to-vertex monophonic graph.

**Proof.** Since $G$ contains a spanning path $P$, $P$ is a Hamiltonian path of $G$. Hence it follows that $V(P)$ is a path induced edge-to-vertex monophonic set of $G$ so that $G$ is a path induced edge-to-vertex monophonic graph. ■

**Theorem 2.11.** Let $G$ be a path induced edge-to-vertex monophonic graph with cut-edges and let $M$ be a path induced edge-to-vertex monophonic set of $G$. If $e$ is any cut-edge of $G$, then every component of $G - e$ contains an element of $M$.

**Proof.** Let $e = uv$ and $G_1, G_2$ are the only two components of $G - e$ such that $u \in V(G_1)$ and $v \in V(G_2)$. Let $M$ be a path induced edge-to-vertex monophonic set of $G$ and $P$ be a Hamiltonian path in $< M >$. On the contrary suppose that $M$ contains no elements of $G_1$. Let $x$ belongs to $V(G_1)$ such that $x \neq u$. Then $x$ lies on a monophonic path $P'$ joining two edges of $M$. Since $u$ is a cut-vertex of $G$, $u$ appear twice in $P'$. And so $P'$ is not a path, which is a contradiction. Therefore every component of $G - e$ contains an element of $M$. ■

**Theorem 2.12.** Every cut-edge of a path induced edge-to-vertex monophonic set of $G$ belongs to every path induced edge-to-vertex monophonic set of $G$.

**Proof.** Let $e$ be any cut-edge of $G$ and let $G_1, G_2$ be the components of $G - \{e\}$. Let $M$ be a path induced edge-to-vertex monophonic set of $G$. Then by Theorem 2.11, $M$ contains at least one element from each $G_i$ $(1 \leq i \leq 2)$. Since $< M >$ contains a Hamiltonian path, it follows that $e \in M$. ■

In the following we determine the path induced edge-to-vertex monophonic number of some standard graphs.

**Theorem 2.13.** For a path $G = P_p(p \geq 3), pim_{ev}(G) = q$.

**Proof.** This follows from Observation 2.5(i) and Theorem 2.12. ■

**Theorem 2.14.** For the complete graph $G = K_p (p \geq 3)$, $pim_{ev}(G) = p - 1$.

**Proof.** Let $K_p : v_1, v_2, v_3, \ldots, v_p$ be a complete graph. Let $M = \{v_1v_2, v_2v_3, \ldots, v_{p-1}v_p\}$. Then every vertex of $G$ is incident with an element of $M$. Therefore $M$ is a path induced edge-to-vertex monophonic set of $G$. Since $< M >$ contains the Hamiltonian path $P: v_1, v_2, v_3, \ldots, v_p$, $M$ is a path induced edge-to-vertex monophonic set of $G$. Therefore $pim_{ev}(G) \leq p - 1$. We prove that $pim_{ev}(G) = p - 1$. Suppose that $pim_{ev}(G) \geq p$. Then there exists a $pim_{ev}$-set $M'$such that $|M'| \geq p$. Let $e = uv$ be an edge of $M'$ such that $e' = ux$ and $e'' = vy$ belongs to $M'$. Then $M_1 = M' - e$ is a $pim_{ev}$-set of $G$ so that $pim_{ev}(G) \leq p - 1$,. which is a contradiction.

Therefore $pim_{ev}(G) = p - 1$. ■

**Theorem 2.15.** For the cycle $G = C_p (p \geq 4), pim_{ev}(G) = 2$.

**Proof.** Let $e$ and $f$ be two adjacent edges of $C_p$. Then $M = \{e, f\}$ is a $pim_{ev}$-set of $G$ so that $pim_{ev}(G) = 2$. ■

**Theorem 2.16.** For the complete bipartite graph $G = K_{m,n}$ $(2 \leq m \leq n)$, $pim_{ev}(G) = 3$.

**Proof.** Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two bipartite set of $G$.

Let $M = \{x_1y_1, x_2y_2\}$. Then $M$ is a path induced edge-to-vertex monophonic set of $G$. Since $< M >$ has no Hamiltonian path, $M$ is not a path induced edge-to-vertex monophonic set of $G$ and so that $pim_{ev}(G) \geq 3$. Let $M' = \{x_1y_1, x_2y_2, x_1y_2\}$. Then $M'$ is a path induced edge-to-vertex monophonic set of $G$ so that $pim_{ev}(G) = 3$. ■

**Theorem 2.17.** For the wheel $G = W_p = K_1 + C_{p-1}, (p \geq 4)$, $pim_{ev}(G) = 2$.

**Proof.** Let $C_{p-1}$: $v_1, v_2, v_3, \ldots, v_{p-1}, v_1$ be a cycle of length $p - 1$. Then $M = \{v_1v_2, v_2v_3\}$ is a path induced edge-to-vertex monophonic set of $G$ so that $pim_{ev}(G) = 2$. ■

**Theorem 2.18.** For a path induced edge-to-vertex monophonic graph of size $q$, $2 \leq pim_{ev}(G) \leq q$.

**Proof.** Any path induced edge-to-vertex monophonic set need at least two vertices. Therefore $pim_{ev}(G) \geq 2$. Since $G$ is a path induced edge-to-vertex monophonic graph, $M = V$ is a path induced edge-to-vertex monophonic set and so $pim_{ev}(G) \leq q$. Thus $2 \leq pim_{ev}(G) \leq q$. ■

**Remark 2.19.** The bounds in Theorem 2.18 are sharp. For the graph $G = C_p$ $(p \geq 4)$, $pim_{ev}(G) = 2$ and for $G = P_p$ $(p \geq 3)$, $pim_{ev}(G) = q$. Also the bounds in Theorem 2.18 can be strict. For the graph $G$ given in Figure 2.1, $pim_{ev}(G) = 3$, $q = 7$. Thus $2 < pim_{ev}(G) < q$.

**Theorem 2.20.** Let $G$ be a path induced edge-to-vertex monophonic graph of size $q \geq 3$ which is not a path. Then $pim_{ev}(G) \leq q - 1$.

**Proof.** Since $G$ is not a path, $G$ contains at least one cycle. Let $C$ be a cycle in $G$ and $e \in E(G)$. Then $M = E(G) - e$ is a path induced edge-to-vertex monophonic set of $G$ and so $pim_{ev}(G) \leq q - 1$. ■

**Remark 2.21.** The bound in Theorem 3.20 is sharp for the graph $G = K_3$ and so $pim_{ev}(G) = 2 = q - 1$.

**Theorem 2.22.** Let $G$ be a path induced edge-to-vertex monophonic graph of size $q \geq 3$. Then $pim_{ev}(G) = q$ if and only if $G$ is the path $P_p$.

**Proof.** Let $G = P_p$. Then by Theorem 2.13, $pim_{ev}(G) = q$. Conversely let $pim_{ev}(G) = q$. We prove that $G$ is the path $P_p$. On the contrary suppose $G$ is not the path $P_p$, then by Theorem 2.20, $pim_{ev}(G) \leq q - 1$ , which is a contradiction. Therefore $G$ is a path.

Denote $\Im$ by the two classes of graphs given in Figure 2.4 (a) and 2.4 (b)

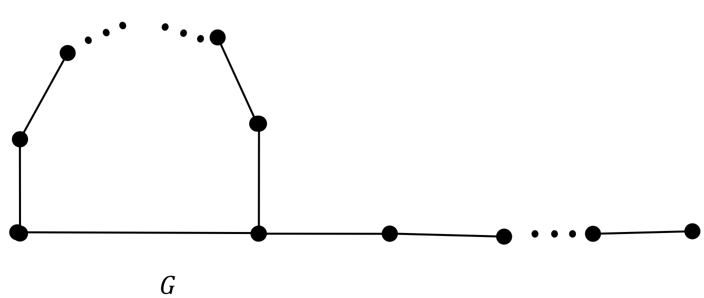
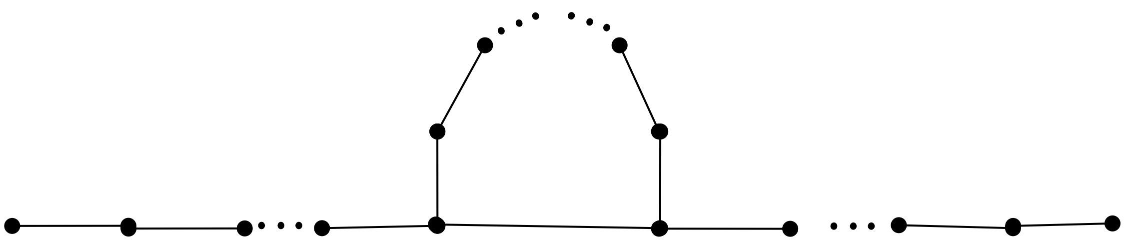



Figure 2.4 (a)



$G$

Figure 2.4 (b)

**Theorem 2.23.** Let $G$ be a path induced edge-to-vertex monophonic graph of size $q \geq 4$. Then $pim_{ev}(G) = q - 1$ if and only if $G$ is the class of graphs given in family $\mathfrak{I}$ of Figure 2.4(a) and 2.4(b).

**Proof.** Let $G$ be the graph given in Figure 2.4(a) . It is easily verified that $pim_{ev}(G) = q - 1$. Conversely let $pim_{ev}(G) = q - 1$. First we prove that $G$ is unicycle. If not let $C_1$ and $C_2$ be the two cycles in $G$. Let $v$ be a common vertex of $C_1$ and $C_2$. Let $x$ be a vertex in $C_1$ and $y$ be a vertex in $C_2$ such that $vx, vy \in E(G)$. Then $M = E(G) - \{vx, vy\}$ is a path induced edge-to-vertex monophonic set and so $pim_{ev}(G) \leq q - 2$, which is a contradiction. Then $C_1$ and $C_2$ has no vertex in common. Let $C_1$ and $C_2$ have a common edge $uv$. Let $x \in C_1$ and $y \in C_2$ such that $ux, uy \in E(G), u, v \neq x$. Then $M_1 = E(G) - \{ux, vy\}$ is a path induced edge-to-vertex monophonic set of $G$ so that $pim_{ev}(G) \leq q - 2$, which is a contradiction. Therefore $C_1$ and $C_2$ has no common edge. Suppose that $C_1$ and $C_2$ are connected by a path $u_0, u_1, \ldots, u_n$ such that $u_0 \in C_1$ and $u_n \in C_2$. Let $x \in C_1$ such that $u_0x \in E(G)$ and $y \in C_2$ such that $u_ny \in E(G)$. Then $M_2 = E(G) - \{u_0x,\ u_ny\}$ is a path induced edge-to-vertex monophonic set of $G$ so that $pim_{ev}(G) \leq q - 2$, which is a contradiction. Therefore $C_1$ and $C_2$ are not connected by a path. Therefore $G$ is a unicyclic graph. Let $C(G)$ denotes the length of the longest cycle in $G$. Since $G$ is a unicyclic graph, $G$ contains either one or two vertices of degree 3.

Case (i) Let $G$ contains only one vertex of degree 3, say $u$. Since $G$ is a unicyclic graph, $G$ contains exactly one end edge and all the edges in $E(G) - E(C)$ are cut edges except the end edge.

Case (ii) Suppose that $C$ contains two vertices $u$ and $v$ such that $\deg(u) = \deg(v) = 3$. Since $pim_{ev}(G) = q - 1, uv \in E(G)$. Also since $q \geq 4, E(G) - E(C)$ contains exactly two end edges and all the remaining edges are cut edges. Therefore $G$ is the family $\mathfrak{I}$ of graph given in Figure 2.4 (b). So we have done. ∎

**Theorem 2.24.** Let $G$ be a path induced edge-to-vertex geodetic graph. Then $G$ is a path induced edge-to-vertex monophonic graph.

**Proof.** Let $G$ be a path induced edge-to-vertex geodetic graph. Let $S$ be a path induced geodetic set of $G$. Since every geodetic path is a monophonic path, $S$ is a path induced edge-to- vertex monophonic set of $G$. Therefore $G$ is a path induced edge-to-vertex monophonic graph. ■

**Theorem 2.25.** Let $G$ be a path induced edge-to-vertex geodetic graph. Then $2 \leq pim_{ev}(G) \leq pig_{ev}(G) \leq q$.

**Proof.** Every path induced edge-to-vertex monophonic set has at least two edges and so $pim_{ev}(G) \geq 2$. Since every path induced edge-to-vertex monophonic set is a path induced edge-to-vertex geodetic set and so $pim_{ev}(G) \leq pig_{ev}(G)$. Also the set of all edges of $G$ is a path induced edge-to-vertex geodetic set of $G$ so that $pig_{ev}(G) \leq q$. ■

**Remark 2.26.** The bounds in Theorem 2.25 are sharp for the graph $G = C_4$, $pim_{ev}(G) = 2$ and for the graph $G = K_4$, $pim_{ev}(G) = 3 = pig_{ev}(G)$. For the path $G = P_p$, $pim_{ev}(G) = q$. The inequalities in Theorem 3.25 can be strict. For the graph $G$ given in Figure 2.1, $pim_{ev}(G) = 3$, $pig_{ev}(G) = 4$ and $q = 7$ so that $2 < pim_{ev}(G) < pig_{ev}(G) < q$.

**Theorem. 2.27.** For every pair $a$ and $b$ of integers with $4 \leq a \leq b$, there exists a connected path induced edge-to-vertex geodetic graph $G$ such that $pim_{ev}(G) = a$ and $pig_{ev}(G) = b$.

**Proof:** Let $P_a: u_0, u_1, u_2, \ldots, u_a$ be a path of size $a$ and $Q_{b-a}$: $v_0, v_1, v_2, \ldots, v_{b-a}$ be a path of size $b - a$. Let $G$ be the graph obtained from $P_a$ *and* $Q_{b-a}$ by introducing the edges $v_0u_1$ and $v_{b-a}u_3$. The graph $G$ is shown in Figure 2.5.

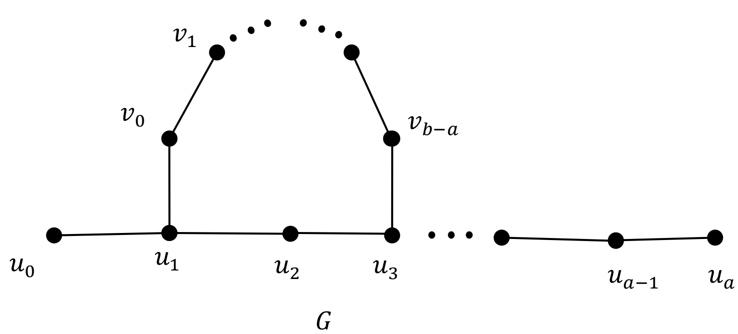



Figure 2.5

First we show that $pim_{ev}(G) = a$. Let $M = \{u_0u_1, u_3u_4, \dots, u_{a-1}u_a\}$ be the set of end edges and cut-edges of $G$. Then by Corollary 2.9 and Theorem 2.12 $M$ is a subset of every path induced edge-to-vertex monophonic set of $G$ and so $pim_{ev}(G) \geq a - 2$. Since $J_{ee}[M] \neq V$, $M$ is not a path induced edge-to-vertex monophonic set of $G$. Also $J_{ee}[M \cup \{e\}] \neq V$ where $e \notin M$, and so $pim_{ev}(G) \geq a$. Now $M \cup \{u_1u_2, u_2u_3\}$ is a path induced edge-to-vertex monophonic set of $G$ so that $pim_{ev}(G) = a$. Next, we prove that $pig_{ev}(G) = b$. Let $Z = \{u_0u_1, u_3u_4, \dots, u_{a-1}u_a\}$ be the set of end edges and cut-edges of $G$. By Theorem 1.2 , $Z$ is a subset of every path induced edge-to-vertex geodetic set of $G$. Since $I_{ee}[Z] \neq V, Z$ is not a path induced edge-to-vertex geodetic set of $G$. It is easily observed that every path induced edge-to-vertex geodetic set of $G$ contains $Z_1 = \{u_1v_0, u_3v_{b-a}\} \cup V(Q_{b-a})$ and so $pim_{ev}(G) \geq a - 2 + b - a + 2 = b$. Since $S = Z \cup Z_1$ is a path induced edge-to-vertex geodetic set of $G$, we have $pim_{ev}(G) = b$. Also, since $G$ contains a path induced edge-to-vertex geodetic set , $G$ is a path induced edge-to-vertex geodetic graph. ■

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