Certain Definite Integral Involving Struve and Modified Struve Function in the form of Hypergeometric Function

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Abstract

In this paper we have developed certain definite integral involving Struve and modified struve Function in association with Hypergeometric function.

Key Words : Bessel Function, Hypergeometric Function, Struve function.

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1. Introduction

Yurry A. Brychkov[Brychkov p.221(4.11.5.2-4.11.5.5)] has derived the following formulae

$$\int_0^1 cos^{-1}x\ H_0(ax)dx = \frac{1}{a}[C - Ci(a) + \log\ a], \qquad [|\arg\ a < \pi\ |]. \tag{1.1}$$

$$\int_0^1 \frac{1}{x}\ cos^{-1}x\ H_1(ax)dx = C - 1 - Ci(a) + \log\ a + \frac{1}{a}\ sin\ a, \qquad [|\arg\ a < \pi\ |]. \tag{1.2}$$

$$\int_0^1 cos^{-1}x\ L_0(ax)dx = \frac{1}{a}[Chi(a) - C - \log\ a], \qquad [|\arg\ a < \pi\ |]. \tag{1.3}$$

$$\int_0^1 \frac{1}{x}\ cos^{-1}x\ L_1(ax)dx = 1 - C + Chi(a) - \log\ a - \frac{1}{a}\ sinh\ a, \qquad [|\arg\ a < \pi\ |]. \tag{1.4}$$

Struve functions are solutions of the non-homogeneous Bessel's differential equation:

$$x^2\ \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = \frac{4\left(\frac{x}{2}\right)^{\alpha+1}}{\sqrt{\pi}\ \Gamma(\alpha + \frac{1}{2})} \tag{1.5}$$

and are defined as:

$$H_\alpha(x) = \frac{2\left(\frac{x}{2}\right)^{\alpha}}{\Gamma(\alpha + \frac{1}{2})\Gamma\left(\frac{1}{2}\right)} \int_0^{\frac{\pi}{2}} sin(x\ cos\ \theta)sin^{2\alpha}(\theta)\ d\theta \tag{1.6}$$

Modified Struve function is:

$$L_\alpha(x) = I_{-\alpha}(x) - \frac{2\left(\frac{x}{2}\right)^{\alpha}}{\Gamma(\alpha + \frac{1}{2})\Gamma\left(\frac{1}{2}\right)} \int_0^{\infty} sin(xu)(1 + u^2)^{\alpha - \frac{1}{2}}\ du \tag{1.7}$$

Bessel functions of the first kind, denoted as $J_\alpha(x)$, are solutions of Bessel's differential equation that are finite at the origin ($x = 0$) for integer or positive α, and diverge as x approaches zero for negative non-integer α(See[12]). It is possible to define the function by its Taylor series expansion around $x = 0$.

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\ \Gamma(m + \alpha + 1)} \left(\frac{x}{2}\right)^{2m+\alpha} \tag{1.8}$$

where $\Gamma(z)$ is the gamma function, a shifted generalization of the factorial function to non-integer values. The Bessel function of the first kind is an entire function if α is an integer.

The Bessel functions are valid even for complex arguments x, and an important special case is that of a purely imaginary argument(See[12]). In this case, the solutions to the Bessel equation are called the modified Bessel functions (or occasionally the hyperbolic Bessel functions) of the first and second kind. The first kind of modified Bessel function is defined as

$$I_\alpha(x) = \iota^{-\alpha} J_\alpha(\iota x) = \sum_{m=0}^{\infty} \frac{1}{m!\ \Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha} \tag{1.9}$$

A generalized hypergeometric function ${}_pF_q(a_1, ..., a_p; b_1, ..., b_q; z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)...(k+a_p)}{(k+b_1)(K+b_2)...(k+b_q)(k+1)}\ z. \tag{1.10}$$

Where $k+1$ in the denominator is present for historical reasons of notation[Koepf p.12(2.9)], and the resulting generalized hypergeometric function is written

$${}_pF_q\left[\begin{array}{lll} a_1, a_2, \cdots, a_p & ; & \\ & & z \\ b_1, b_2, \cdots, b_q & ; & \end{array}\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k z^k}{(b_1)_k (b_2)_k \cdots (b_q)_k k!} \tag{1.11}$$

where the parameters $b_1, b_2, \cdots, b_q$ are positive integers.

The ${}_pF_q$ series converges for all finite z if $p \leq q$, converges for $\mid z \mid < 1$ if $p = q+1$, diverges for all z , $z \neq 0$ if $p > q+1$[Luke p.156(3)].

The function ${}_2F_1(a, b; c; z)$ corresponding to $p = 2, q = 1$, is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function [Gauss p.123-162]. To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

In mathematics, the falling factorial or Pochhammer symbol (sometimes called the descending factorial,falling sequential product, or lower factorial) is defined as the polynomial[Steffensen p.8]

$$(x)_n = x(x-1)(x-2).......(x-n+1) = \prod_{k=1}^{n}(x-k+1) = \prod_{k=0}^{n-1}(x-k) \tag{1.12}$$

2. Main Formulae of the Integration

$$\int_0^1 x\ cos^{-1}x\ H_0(ax)dx = \frac{1}{9\pi}\Big[4a\ {}_2F_3\Big(1, 2; \frac{3}{2}, \frac{5}{2}, \frac{5}{2}; -\frac{a^2}{4}\Big)\Big]. \tag{2.1}$$

$$\int_0^1 x\ cos^{-1}x\ L_0(ax)dx = \frac{1}{9\pi}\Big[4a\ {}_2F_3\Big(1, 2; \frac{3}{2}, \frac{5}{2}, \frac{5}{2}; \frac{a^2}{4}\Big)\Big]. \tag{2.2}$$

$$\int_0^1 x^2\ cos^{-1}x\ H_0(ax)dx = \frac{1}{4a^3}\Big[a^2 + 4\ Ci(a) - 4\ \log\ a - 4a\ sin\ a - 8\ cos\ a - 4\gamma + 8\Big]. \tag{2.3}$$

$$\int_0^1 x^2\ cos^{-1}x\ L_0(ax)dx = -\frac{1}{4a^3}\Big[a^2 - 4\ Chi(a) + 4\ \log\ a - 4a\ sinh\ a + 8\ cosh\ a + 4\gamma - 8\Big]. \tag{2.4}$$

$$\int_0^1 x^3\ cos^{-1}x\ H_0(ax)dx = \frac{1}{75\pi}\Big[16a\ {}_3F_4\Big(1,\frac{5}{2},3;\frac{3}{2},\frac{3}{2},\frac{7}{2},\frac{7}{2};-\frac{a^2}{4}\Big)\Big]. \tag{2.5}$$

$$\int_0^1 x^3\ cos^{-1}x\ L_0(ax)dx = \frac{1}{75\pi}\Big[16a\ {}_3F_4\Big(1,\frac{5}{2},3;\frac{3}{2},\frac{3}{2},\frac{7}{2},\frac{7}{2};\frac{a^2}{4}\Big)\Big]. \tag{2.6}$$

$$\int_0^1 x^5\ cos^{-1}x\ H_0(ax)dx = \frac{1}{245\pi}\Big[32a\ {}_3F_4\Big(1,\frac{7}{2},4;\frac{3}{2},\frac{3}{2},\frac{9}{2},\frac{9}{2};-\frac{a^2}{4}\Big)\Big]. \tag{2.7}$$

$$\int_0^1 x^5\ cos^{-1}x\ L_0(ax)dx = \frac{1}{245\pi}\Big[32a\ {}_3F_4\Big(1,\frac{7}{2},4;\frac{3}{2},\frac{3}{2},\frac{9}{2},\frac{9}{2};\frac{a^2}{4}\Big)\Big]. \tag{2.8}$$

$$\int_0^1 x^7\ cos^{-1}x\ H_0(ax)dx = \frac{1}{2835\pi}\Big[256a\ {}_3F_4\Big(1,\frac{9}{2},5;\frac{3}{2},\frac{3}{2},\frac{11}{2},\frac{11}{2};-\frac{a^2}{4}\Big)\Big]. \tag{2.9}$$

$$\int_0^1 x^7\ cos^{-1}x\ L_0(ax)dx = \frac{1}{2835\pi}\Big[256a\ {}_3F_4\Big(1,\frac{9}{2},5;\frac{3}{2},\frac{3}{2},\frac{11}{2},\frac{11}{2};\frac{a^2}{4}\Big)\Big]. \tag{2.10}$$

$$\int_0^1 x^9\ cos^{-1}x\ H_0(ax)dx = \frac{1}{7623\pi}\Big[512a\ {}_3F_4\Big(1,\frac{11}{2},6;\frac{3}{2},\frac{3}{2},\frac{13}{2},\frac{13}{2};-\frac{a^2}{4}\Big)\Big]. \tag{2.11}$$

$$\int_0^1 x^9\ cos^{-1}x\ L_0(ax)dx = \frac{1}{7623\pi}\Big[512a\ {}_3F_4\Big(1,\frac{11}{2},6;\frac{3}{2},\frac{3}{2},\frac{13}{2},\frac{13}{2};\frac{a^2}{4}\Big)\Big]. \tag{2.12}$$

$$\int_0^1 x^{11}\ cos^{-1}x\ L_0(ax)dx = \frac{1}{39039\pi}\Big[2048a\ {}_3F_4\Big(1,\frac{13}{2},7;\frac{3}{2},\frac{3}{2},\frac{15}{2},\frac{15}{2};\frac{a^2}{4}\Big)\Big]. \tag{2.13}$$

$$\int_0^1 x^{13}\ cos^{-1}x\ H_0(ax)dx = \frac{1}{96525\pi}\Big[4096a\ {}_3F_4\Big(1,\frac{15}{2},8;\frac{3}{2},\frac{3}{2},\frac{17}{2},\frac{17}{2};-\frac{a^2}{4}\Big)\Big]. \tag{2.14}$$

$$\int_0^1 x^{13}\ cos^{-1}x\ L_0(ax)dx = \frac{1}{96525\pi}\Big[4096a\ {}_3F_4\Big(1,\frac{15}{2},8;\frac{3}{2},\frac{3}{2},\frac{17}{2},\frac{17}{2};\frac{a^2}{4}\Big)\Big]. \tag{2.15}$$

$$\int_0^1 \frac{1}{x}\ cos^{-1}x\ H_0(ax)dx = \frac{1}{\pi}\Big[2a\ {}_3F_4\Big(1,\frac{1}{2},1;\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2};-\frac{a^2}{4}\Big)\Big]. \tag{2.16}$$

$$\int_0^1 \frac{1}{x}\ cos^{-1}x\ L_0(ax)dx = \frac{1}{\pi}\Big[2a\ {}_3F_4\Big(1,\frac{1}{2},1;\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2};\frac{a^2}{4}\Big)\Big]. \tag{2.17}$$

$$\int_0^1 \frac{1}{x^2}\ cos^{-1}x\ L_1(ax)dx = \frac{1}{3\pi}\Big[2a^2\ {}_3F_4\Big(1,\frac{1}{2},1;\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{5}{2};\frac{a^2}{4}\Big)\Big]. \tag{2.18}$$

$$\int_0^1 cos^{-1}x\ H_1(ax)dx = \frac{1}{27\pi}\Big[4a^2\ {}_2F_3\Big(1,2;\frac{5}{2},\frac{5}{2},\frac{5}{2};-\frac{a^2}{4}\Big)\Big]. \tag{2.19}$$

$$\int_0^1 x\ cos^{-1}x\ H_1(ax)dx = \frac{1}{4a^2}\Big[a^2-4\ Ci(a)+4\ \log\ a+4\ cos\ a+4\gamma-4\Big]. \tag{2.20}$$

$$\int_0^1 x^2\ cos^{-1}x\ H_1(ax)dx = \frac{1}{225\pi}\Big[16a^2\ {}_2F_3\Big(1,3;\frac{3}{2},\frac{7}{2},\frac{7}{2};-\frac{a^2}{4}\Big)\Big]. \tag{2.21}$$

$$\int_0^1 x^3\ cos^{-1}x\ H_1(ax)dx = \frac{1}{32a^4}\Big[3a^4+8a^2+96\ Ci(a)-96\ \log\ a+32(a^2-7)cos\ a-128\ sin\ a-96\gamma+224\Big]. \tag{2.22}$$

$$\int_0^1 x^4\ cos^{-1}x\ H_1(ax)dx = \frac{1}{735\pi}\Big[32a^2\ {}_3F_4\Big(1,4,\frac{7}{2};\frac{3}{2},\frac{5}{2},\frac{9}{2},\frac{9}{2};-\frac{a^2}{4}\Big)\Big]. \tag{2.23}$$

$$\int_0^1 x^6\ cos^{-1}x\ H_1(ax)dx = \frac{1}{8505\pi}\Big[256a^2\ {}_3F_4\Big(1,5,\frac{9}{2};\frac{3}{2},\frac{5}{2},\frac{11}{2},\frac{11}{2};-\frac{a^2}{4}\Big)\Big]. \tag{2.24}$$

$$\int_0^1 x^8\ cos^{-1}x\ H_1(ax)dx = \frac{1}{22869\pi}\Big[512a^2\ {}_3F_4\Big(1,6,\frac{11}{2};\frac{3}{2},\frac{5}{2},\frac{13}{2},\frac{13}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.25)$$

$$\int_0^1 x^{10}\ cos^{-1}x\ H_1(ax)dx = \frac{1}{117117\pi}\Big[2048a^2\ {}_3F_4\Big(1,7,\frac{13}{2};\frac{3}{2},\frac{5}{2},\frac{15}{2},\frac{15}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.26)$$

$$\int_0^1 \frac{1}{x^2}\ cos^{-1}x\ H_1(ax)dx = \frac{1}{3\pi}\Big[2a^2\ {}_3F_4\Big(1,\frac{1}{2},1;\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{5}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.27)$$

$$\int_0^1 cos^{-1}x\ H_2(ax)dx = \frac{-a\ Ci(a)+a\ \log\ a+2\ sin\ a}{a^2}+\frac{a}{12}+\frac{\gamma-2}{a}. \quad (2.28)$$

$$\int_0^1 cos^{-1}x\ L_2(ax)dx = \frac{1}{12a^2}[24\ sinh\ a-a(a^2+12\ Chi(a)-12\log\ a-12\gamma+24]. \quad (2.29)$$

$$\int_0^1 x\ cos^{-1}x\ H_2(ax)dx = \frac{1}{1125\pi}\Big[16a^3\ {}_3F_4\Big(1,\frac{5}{2},3;\frac{3}{2},\frac{7}{2},\frac{7}{2},\frac{7}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.30)$$

$$\int_0^1 x\ cos^{-1}x\ L_2(ax)dx = \frac{1}{1125\pi}\Big[16a^3\ {}_3F_4\Big(1,\frac{5}{2},3;\frac{3}{2},\frac{7}{2},\frac{7}{2},\frac{7}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.31)$$

$$\int_0^1 x^2\ cos^{-1}x\ H_2(ax)dx = \frac{1}{32a^3}\Big[a^4+8a^2-96\ Ci(a)+96\ \log\ a+32a\ sin\ a+128\ cos\ a+96\gamma-128\Big]. \quad (2.32)$$

$$\int_0^1 x^2\ cos^{-1}x\ L_2(ax)dx = -\frac{1}{32a^3}\Big[a^4-8a^2-96\ Chi(a)+96\ \log\ a+32a\ sinh\ a+128\ cosh\ a+96\gamma-128\Big]. \quad (2.33)$$

$$\int_0^1 x^3\ cos^{-1}x\ H_2(ax)dx = \frac{1}{3675\pi}\Big[32a^3\ {}_2F_3\Big(1,4;\frac{3}{2},\frac{9}{2},\frac{9}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.34)$$

$$\int_0^1 x^3\ cos^{-1}x\ L_2(ax)dx = \frac{1}{3675\pi}\Big[32a^3\ {}_2F_3\Big(1,4;\frac{3}{2},\frac{9}{2},\frac{9}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.35)$$

$$\int_0^1 x^4\ cos^{-1}x\ L_2(ax)dx = \frac{1}{288a^5}[-5a^6+27a^4-216a^2+288(a^2+23)a\ sinh\ a-$$
$$-288(7a^2+38)cosh\ a+4320\ Chi(a)-4320\log\ a-4320\gamma+10944]. \quad (2.36)$$

$$\int_0^1 x^5\ cos^{-1}x\ H_2(ax)dx = \frac{1}{42525\pi}\Big[256a^3\ {}_3F_4\Big(1,\frac{9}{2},5;\frac{3}{2},\frac{7}{2},\frac{11}{2},\frac{11}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.37)$$

$$\int_0^1 x^5\ cos^{-1}x\ L_2(ax)dx = \frac{1}{42525\pi}\Big[256a^3\ {}_3F_4\Big(1,\frac{9}{2},5;\frac{3}{2},\frac{7}{2},\frac{11}{2},\frac{11}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.38)$$

$$\int_0^1 x^7\ cos^{-1}x\ H_2(ax)dx = \frac{1}{114345\pi}\Big[512a^3\ {}_3F_4\Big(1,\frac{11}{2},6;\frac{3}{2},\frac{7}{2},\frac{13}{2},\frac{13}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.39)$$

$$\int_0^1 x^7\ cos^{-1}x\ L_2(ax)dx = \frac{1}{114345\pi}\Big[512a^3\ {}_3F_4\Big(1,\frac{11}{2},6;\frac{3}{2},\frac{7}{2},\frac{13}{2},\frac{13}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.40)$$

$$\int_0^1 x^9\ cos^{-1}x\ H_2(ax)dx = \frac{1}{585585\pi}\Big[2048a^3\ {}_3F_4\Big(1,\frac{13}{2},7;\frac{3}{2},\frac{7}{2},\frac{15}{2},\frac{15}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.41)$$

$$\int_0^1 x^9\ cos^{-1}x\ L_2(ax)dx = \frac{1}{585585\pi}\Big[2048a^3\ {}_3F_4\Big(1,\frac{13}{2},7;\frac{3}{2},\frac{7}{2},\frac{15}{2},\frac{15}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.42)$$

$$\int_0^1 x^{11}\ cos^{-1}x\ H_2(ax)dx = \frac{1}{1447875\pi}\Big[4096a^3\ {}_3F_4\Big(1,\frac{15}{2},8;\frac{3}{2},\frac{7}{2},\frac{17}{2},\frac{17}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.43)$$

$$\int_0^1 x^{11}\ cos^{-1}x\ L_2(ax)dx = \frac{1}{1447875\pi}\Big[4096a^3\ {}_3F_4\Big(1,\frac{15}{2},8;\frac{3}{2},\frac{7}{2},\frac{17}{2},\frac{17}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.44)$$

$$\int_0^1 x^{13}\ cos^{-1}x\ L_2(ax)dx = \frac{1}{27895725\pi}\Big[65536a^3\ {}_3F_4\Big(1,\frac{17}{2},9;\frac{3}{2},\frac{7}{2},\frac{19}{2},\frac{19}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.45)$$

$$\int_0^1 x^{15}\ cos^{-1}x\ L_2(ax)dx = \frac{1}{65819325\pi}\Big[131072a^3\ {}_3F_4\Big(1,\frac{19}{2},10;\frac{3}{2},\frac{7}{2},\frac{21}{2},\frac{21}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.46)$$

$$\int_0^1 x^{17}\ cos^{-1}x\ L_2(ax)dx = \frac{1}{3055402235\pi}\Big[524288a^3\ {}_3F_4\Big(1,\frac{21}{2},11;\frac{3}{2},\frac{7}{2},\frac{23}{2},\frac{23}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.47)$$

$$\int_0^1 \frac{1}{x}\ cos^{-1}x\ H_2(ax)dx = \frac{1}{135\pi}\Big[4a^3\ {}_2F_3\Big(1,2;\frac{5}{2},\frac{5}{2},\frac{7}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.48)$$

$$\int_0^1 \frac{1}{x}\ cos^{-1}x\ L_2(ax)dx = \frac{1}{135\pi}\Big[4a^3\ {}_2F_3\Big(1,2;\frac{5}{2},\frac{5}{2},\frac{7}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.49)$$

$$\int_0^1 \frac{1}{x^2}\ cos^{-1}x\ H_2(ax)dx = \frac{1}{9a^2}[a^3(-3\ Ci(a)+3\ \log\ a+3\gamma-4)+3(a^2+1)sin\ a-3a\ cos\ a]. \quad (2.50)$$

$$\int_0^1 \frac{1}{x^2}\ cos^{-1}x\ L_2(ax)dx = -\frac{1}{9a^2}[a^3(-3\ Chi(a)+3\ \log\ a+3\gamma-4)+3(a^2-1)sinh\ a+3a\ cosh\ a]. \quad (2.51)$$

$$\int_0^1 \frac{1}{x^3}\ cos^{-1}x\ H_2(ax)dx = \frac{1}{15\pi}\Big[2a^3\ {}_3F_4\Big(1,\frac{1}{2},1;\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{7}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.52)$$

$$\int_0^1 \frac{1}{x^3}\ cos^{-1}x\ L_2(ax)dx = \frac{1}{15\pi}\Big[2a^3\ {}_3F_4\Big(1,\frac{1}{2},1;\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{7}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.53)$$

$$\int_0^1 cos^{-1}x\ H_3(ax)dx = \frac{1}{7875\pi}\Big[16a^4\ {}_3F_4\Big(1,3,\frac{5}{2};\frac{3}{2},\frac{7}{2},\frac{7}{2},\frac{9}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.54)$$

$$\int_0^1 cos^{-1}x\ L_3(ax)dx = \frac{1}{7875\pi}\Big[16a^4\ {}_3F_4\Big(1,3,\frac{5}{2};\frac{3}{2},\frac{7}{2},\frac{7}{2},\frac{9}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.55)$$

$$\int_0^1 x\ cos^{-1}x\ L_3(ax)dx = \frac{-3a\log\ a-8\ sinh\ a+a\ cosh\ a}{a^3}+\frac{3\ Chi(a)}{a^2}-\frac{a^2}{160}+\frac{7-3\gamma}{a^2}+\frac{1}{12}. \quad (2.56)$$

$$\int_0^1 x^2\ cos^{-1}x\ H_3(ax)dx = \frac{1}{25725\pi}\Big[32a^4\ {}_3F_4\Big(1,4,\frac{7}{2};\frac{3}{2},\frac{9}{2},\frac{9}{2},\frac{9}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.56)$$

$$\int_0^1 x^2\ cos^{-1}x\ L_3(ax)dx = \frac{1}{25725\pi}\Big[32a^4\ {}_3F_4\Big(1,4,\frac{7}{2};\frac{3}{2},\frac{9}{2},\frac{9}{2},\frac{9}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.57)$$

$$\int_0^1 x^3\ cos^{-1}x\ L_3(ax)dx = -\frac{1}{288a^4}[a^6-9a^4+216a^2-288(a^2+23)cosh\ a-4320\log\ a+$$

$$+4320\ Chi(a)+2304a\ sinh\ a-4320\gamma+6624]. \quad (2.58)$$

$$\int_0^1 x^4\ cos^{-1}x\ H_3(ax)dx = \frac{1}{297675\pi}\Big[256a^4\ {}_2F_3\Big(1,5;\frac{3}{2},\frac{11}{2},\frac{11}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.59)$$

$$\int_0^1 x^4\ cos^{-1}x\ L_3(ax)dx = \frac{1}{297675\pi}\Big[256a^4\ {}_2F_3\Big(1,5;\frac{3}{2},\frac{11}{2},\frac{11}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.60)$$

$$\int_0^1 x^6\ cos^{-1}x\ H_3(ax)dx = \frac{1}{800415\pi}\Big[512a^4\ {}_3F_4\Big(1,\frac{11}{2},6;\frac{3}{2},\frac{9}{2},\frac{13}{2},\frac{13}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.61)$$

$$\int_0^1 x^6\ cos^{-1}x\ L_3(ax)dx = \frac{1}{800415\pi}\Big[512a^4\ {}_3F_4\Big(1,\frac{11}{2},6;\frac{3}{2},\frac{9}{2},\frac{13}{2},\frac{13}{2};-\frac{a^2}{4}\Big)\Big]. \tag{2.62}$$

$$\int_0^1 x^8\ cos^{-1}x\ H_3(ax)dx = \frac{1}{4099095\pi}\Big[2048a^4\ {}_3F_4\Big(1,\frac{13}{2},7;\frac{3}{2},\frac{9}{2},\frac{15}{2},\frac{15}{2};-\frac{a^2}{4}\Big)\Big]. \tag{2.63}$$

$$\int_0^1 x^8\ cos^{-1}x\ L_3(ax)dx = \frac{1}{4099095\pi}\Big[2048a^4\ {}_3F_4\Big(1,\frac{13}{2},7;\frac{3}{2},\frac{9}{2},\frac{15}{2},\frac{15}{2};\frac{a^2}{4}\Big)\Big]. \tag{2.64}$$

$$\int_0^1 x^{10}\ cos^{-1}x\ H_3(ax)dx = \frac{1}{10135125\pi}\Big[4096a^4\ {}_3F_4\Big(1,\frac{15}{2},8;\frac{3}{2},\frac{9}{2},\frac{17}{2},\frac{17}{2};-\frac{a^2}{4}\Big)\Big]. \tag{2.65}$$

$$\int_0^1 x^{10}\ cos^{-1}x\ L_3(ax)dx = \frac{1}{10135125\pi}\Big[4096a^4\ {}_3F_4\Big(1,\frac{15}{2},8;\frac{3}{2},\frac{9}{2},\frac{17}{2},\frac{17}{2};\frac{a^2}{4}\Big)\Big]. \tag{2.66}$$

$$\int_0^1 x^{12}\ cos^{-1}x\ H_3(ax)dx = \frac{1}{195270075\pi}\Big[65536a^4\ {}_3F_4\Big(1,\frac{17}{2},9;\frac{3}{2},\frac{9}{2},\frac{19}{2},\frac{19}{2};-\frac{a^2}{4}\Big)\Big]. \tag{2.67}$$

$$\int_0^1 x^{12}\ cos^{-1}x\ L_3(ax)dx = \frac{1}{195270075\pi}\Big[65536a^4\ {}_3F_4\Big(1,\frac{17}{2},9;\frac{3}{2},\frac{9}{2},\frac{19}{2},\frac{19}{2};\frac{a^2}{4}\Big)\Big]. \tag{2.68}$$

$$\int_0^1 x^{14}\ cos^{-1}x\ L_3(ax)dx = \frac{1}{460735275\pi}\Big[1311072a^4\ {}_3F_4\Big(1,\frac{19}{2},10;\frac{3}{2},\frac{9}{2},\frac{21}{2},\frac{21}{2};\frac{a^2}{4}\Big)\Big]. \tag{2.69}$$

$$\int_0^1 x^{16}\ cos^{-1}x\ L_3(ax)dx = \frac{1}{2138781645\pi}\Big[524288a^4\ {}_3F_4\Big(1,\frac{21}{2},11;\frac{3}{2},\frac{9}{2},\frac{23}{2},\frac{23}{2};\frac{a^2}{4}\Big)\Big]. \tag{2.70}$$

$$\int_0^1 x^{18}\ cos^{-1}x\ L_3(ax)dx = \frac{1}{4897902555\pi}\Big[1048576a^4\ {}_3F_4\Big(1,\frac{23}{2},12;\frac{3}{2},\frac{9}{2},\frac{25}{2},\frac{25}{2};\frac{a^2}{4}\Big)\Big]. \tag{2.71}$$

$$\int_0^1 x^{20}\ cos^{-1}x\ L_3(ax)dx = \frac{1}{44365059375\pi}\Big[8388608a^4\ {}_3F_4\Big(1,\frac{25}{2},12;\frac{3}{2},\frac{9}{2},\frac{27}{2},\frac{27}{2};\frac{a^2}{4}\Big)\Big]. \tag{2.72}$$

$$\int_0^1 \frac{1}{x}\ cos^{-1}x\ L_3(ax)dx = \frac{1}{180a^3}[60(a^2-4)sinh\ a+a^3(-3a^2-60\ Chi(a)+60\log\ a+60\gamma-140)+240\ a\ cosh\ a]. \tag{2.73}$$

$$\int_0^1 \frac{1}{x}\ cos^{-1}x\ H_3(ax)dx = \frac{1}{180a^3}[60(a^2+4)sin\ a+a^3(3a^2-60\ Ci(a)+60\ \log\ a+60\gamma-140)-240a\ cos\ a]. \tag{2.74}$$

$$\int_0^1 \frac{1}{x^2}\ cos^{-1}x\ H_3(ax)dx = \frac{1}{945\pi}\Big[4a^4\ {}_2F_3\Big(1,2;\frac{5}{2},\frac{5}{2},\frac{9}{2};-\frac{a^2}{4}\Big)\Big]. \tag{2.75}$$

$$\int_0^1 \frac{1}{x^2}\ cos^{-1}x\ L_3(ax)dx = \frac{1}{945\pi}\Big[4a^4\ {}_2F_3\Big(1,2;\frac{5}{2},\frac{5}{2},\frac{9}{2};\frac{a^2}{4}\Big)\Big]. \tag{2.76}$$

$$\int_0^1 \frac{1}{x^4}\ cos^{-1}x\ H_3(ax)dx = \frac{1}{105\pi}\Big[2a^4\ {}_3F_4\Big(1,1,\frac{1}{2};\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{9}{2};-\frac{a^2}{4}\Big)\Big]. \tag{2.77}$$

$$\int_0^1 cos^{-1}x\ L_4(ax)dx = \frac{8\ sinh\ a}{a^4} - \frac{a^3}{1120} - \frac{8\ cosh\ a}{a^3} + \frac{Chi(a)-\log\ a}{a} + \frac{a}{60} + \frac{\frac{8}{3}-\gamma}{a}. \tag{2.78}$$

$$\int_0^1 x\ cos^{-1}x\ H_4(ax)dx = \frac{1}{231525\pi}\Big[32a^5\ {}_3F_4\Big(1,\frac{7}{2},4;\frac{3}{2},\frac{9}{2},\frac{9}{2},\frac{11}{2};-\frac{a^2}{4}\Big)\Big]. \tag{2.79}$$

$$\int_0^1 x\ cos^{-1}x\ L_4(ax)dx = \frac{1}{231525\pi}\Big[32a^5\ {}_3F_4\Big(1,\frac{7}{2},4;\frac{3}{2},\frac{9}{2},\frac{9}{2},\frac{11}{2};\frac{a^2}{4}\Big)\Big]. \tag{2.80}$$

$$\int_0^1 x^3\ cos^{-1}x\ H_4(ax)dx = \frac{1}{2679075\pi}\Big[256a^5\ {}_3F_4\Big(1,\frac{9}{2},5;\frac{3}{2},\frac{11}{2},\frac{11}{2},\frac{11}{2};-\frac{a^2}{4}\Big)\Big]. \tag{2.81}$$

$$\int_0^1 x^3\ cos^{-1}x\ L_4(ax)dx = \frac{1}{2679075\pi}\Big[256a^5\ {}_3F_4\Big(1,\frac{9}{2},5;\frac{3}{2},\frac{11}{2},\frac{11}{2},\frac{11}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.82)$$

$$\int_0^1 x^5\ cos^{-1}x\ H_4(ax)dx = \frac{1}{7203735\pi}\Big[512a^5\ {}_2F_3\Big(1,6;\frac{3}{2},\frac{13}{2},\frac{13}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.83)$$

$$\int_0^1 x^5\ cos^{-1}x\ L_4(ax)dx = \frac{1}{7203735\pi}\Big[512a^5\ {}_2F_3\Big(1,6;\frac{3}{2},\frac{13}{2},\frac{13}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.84)$$

$$\int_0^1 x^7\ cos^{-1}x\ H_4(ax)dx = \frac{1}{36891855\pi}\Big[2048a^5\ {}_3F_4\Big(1,7,\frac{13}{2};\frac{3}{2},\frac{11}{2},\frac{15}{2},\frac{15}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.85)$$

$$\int_0^1 x^7\ cos^{-1}x\ L_4(ax)dx = \frac{1}{36891855\pi}\Big[2048a^5\ {}_3F_4\Big(1,7,\frac{13}{2};\frac{3}{2},\frac{11}{2},\frac{15}{2},\frac{15}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.86)$$

$$\int_0^1 x^9\ cos^{-1}x\ H_4(ax)dx = \frac{1}{91216125\pi}\Big[4096a^5\ {}_3F_4\Big(1,8,\frac{15}{2};\frac{3}{2},\frac{11}{2},\frac{17}{2},\frac{17}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.87)$$

$$\int_0^1 x^9\ cos^{-1}x\ L_4(ax)dx = \frac{1}{91216125\pi}\Big[4096a^5\ {}_3F_4\Big(1,8,\frac{15}{2};\frac{3}{2},\frac{11}{2},\frac{17}{2},\frac{17}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.88)$$

$$\int_0^1 x^{11}\ cos^{-1}x\ H_4(ax)dx = \frac{1}{1757430675\pi}\Big[65536a^5\ {}_3F_4\Big(1,9,\frac{17}{2};\frac{3}{2},\frac{11}{2},\frac{19}{2},\frac{19}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.89)$$

$$\int_0^1 x^{11}\ cos^{-1}x\ L_4(ax)dx = \frac{1}{1757430675\pi}\Big[65536a^5\ {}_3F_4\Big(1,9,\frac{17}{2};\frac{3}{2},\frac{11}{2},\frac{19}{2},\frac{19}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.90)$$

$$\int_0^1 x^{13}\ cos^{-1}x\ L_4(ax)dx = \frac{1}{41466174775\pi}\Big[131072a^5\ {}_3F_4\Big(1,10,\frac{19}{2};\frac{3}{2},\frac{11}{2},\frac{21}{2},\frac{21}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.91)$$

$$\int_0^1 x^{15}\ cos^{-1}x\ L_4(ax)dx = \frac{1}{19249034805\pi}\Big[524288a^5\ {}_3F_4\Big(1,11,\frac{21}{2};\frac{3}{2},\frac{11}{2},\frac{23}{2},\frac{23}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.92)$$

$$\int_0^1 x^{17}\ cos^{-1}x\ L_4(ax)dx = \frac{1}{44081122995\pi}\Big[1048576a^5\ {}_3F_4\Big(1,11,\frac{21}{2};\frac{3}{2},\frac{11}{2},\frac{23}{2},\frac{23}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.93)$$

$$\int_0^1 \frac{1}{x}\ cos^{-1}x\ L_4(ax)dx = \frac{1}{70875\pi}\Big[16a^5\ {}_3F_4\Big(1,3,\frac{5}{2};\frac{3}{2},\frac{7}{2},\frac{7}{2},\frac{11}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.94)$$

$$\int_0^1 \frac{1}{x^3}\ cos^{-1}x\ H_4(ax)dx = \frac{1}{8505\pi}\Big[4a^5\ {}_2F_3\Big(1,2;\frac{5}{2},\frac{5}{2},\frac{11}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.95)$$

$$\int_0^1 \frac{1}{x^3}\ cos^{-1}x\ L_4(ax)dx = \frac{1}{8505\pi}\Big[4a^5\ {}_2F_3\Big(1,2;\frac{5}{2},\frac{5}{2},\frac{11}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.96)$$

$$\int_0^1 \frac{1}{x^5}\ cos^{-1}x\ H_4(ax)dx = \frac{1}{945\pi}\Big[2a^5\ {}_3F_4\Big(1,1,\frac{1}{2};\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{11}{2};-\frac{a^2}{4}\Big)\Big]. \quad (2.97)$$

$$\int_0^1 \frac{1}{x^5}\ cos^{-1}x\ L_4(ax)dx = \frac{1}{945\pi}\Big[2a^5\ {}_3F_4\Big(1,1,\frac{1}{2};\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{11}{2};\frac{a^2}{4}\Big)\Big]. \quad (2.98)$$

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