On the deformation and Static buckling of a toroidal shell segment using a two – term Fourier series imperfections

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Abstract

This paper is concerned with analytical determination of the deformation of an imperfect finite toroidal shell segment pressurized by a static load. The continuously differentiable imperfection is assumed in the form of a two – term Fourier series expansion and the boundary conditions are assumed simply – supported. The formulation contains a small parameter depicting the amplitude of imperfection while the normal displacement and Airy stress function are the ones that are first asymptotically determined. A simple mathematical formula for evaluating the static buckling load is finally determined, and the formula which expressly contains the two Fourier coefficients, amply show – cases the contribution, significance and relevance of these two Fourier coefficients compared to an earlier result which had only one Fourier coefficient.

Keywords: *Static Buckling, Asymptotics and Perturbation technique, Fourier Series Expansions, toroidal Shell segments.*

I. INTRODUCTION

Investigations into imperfection – sensitivity as well as analyses of static and dynamic stability of elastic toroidal shell segments don't seem to attract as much attention as analyses of the most commonly used elastic structures such as columns, plates and even cylindrical shells. Yet, toroidal shell segments are not less imperfection – sensitive than these other structures. One of the early studies on the subject matter was undertaken by Stein and Mc Elman [1], while later on, Hutchinson [2] investigated the initial post buckling behavior of toroidal shells. In yet another study, Oyesanya [3] investigated asymptotic analysis of imperfection – sensitivity of toroidal shell segment and further extended his study [4] to investigate the influence of extra terms on asymptotic analysis of imperfection – sensitivity of toroidal shell segments with random imperfection.

This investigation owes its genesis to a recent study by Ette et al. [5], where, among other things, they assumed that the imperfection of a toroidal shell segment can be taken as a one – term Fourier series expansion. While such assumption easily ensures a relatively quick solution of the problem posed , it is our contention that a bigger picture and a much more mathematical appraisal of the problem can be achieved by assuming a two – term Fourier series expansion of the imperfection. This formulation contains a small positive parameter depicting the amplitude of the imperfection and on which asymptotic series expansions of the displacement and Airy stress function are initiated, assuming that the toroidal shell segment is trapped by a static load. Such a mathematical procedure is, of course, not new for, we have extant and relatively current precedents in several earlier investigations such as those in Amazigo [6, 7], Ozoigbo and Ette [8], Bassey et al. [9], Udo – Akpan and Ette [10] and Amazigo and Ette [11], among others.

Techniques and procedures similar to the one enunciated here have also been previously harnessed to study buckling of other elastic materials with a view to determining either their imperfection sensitivity or their static or dynamic stability when subjected to either a static load or a dynamic load. In this respect, mention must be made of investigations by Kolakowski and mania [12], magnucki et al. [13] and Ganaparthi et al. [14]. Mention must also be made of the study by Fan et al. [15], who made an analytical research on dynamic buckling of thin cylindrical shells with thickness variation under axial pressure, as well as Evkin and Lykhachova [16], who studied energy barrier as a criterion for stability estimation of spherical shell under uniform external pressure.

II. MATHEMATICAL FORMULATION OF THE PROBLEM

As in [3, 4], the normal displacement W(X, Y) and Airy stress function F(X, Y) of a toroidal shell segment of Length L, impacted on by a static load P, satisfy the following Karman – Donnell equation of equilibrium and compatibility equation

$$D\nabla^4 W + \frac{1}{a}F_{,XX} + \frac{1}{b}F_{,YY} + P\left[\frac{1}{2}(W+\bar{W})_{,XX} + \left(1-\frac{1}{2}\frac{a}{b}\right)(W+\bar{W})_{,YY}\right] = \hat{S}(W+\bar{W},F) \quad (1)$$

$$\frac{1}{Eh}\nabla^4 F - \frac{1}{a}W_{,XX} - \frac{1}{b}F_{,YY} = -\frac{1}{2}\hat{S}(W+\bar{W},W) \quad (2)$$

$$0 < X < L, \qquad 0 < Y < a \quad (3)$$

$$W = W_{,XX} = F = F_{,XX} = 0 \; at \; X = 0, L \quad (4)$$

where, X and Y are the axial and circumferential spatial variables, E is the Young's modulus, h is the thickness, a and b are the radii, D is the bending stiffness given by $D = \frac{Eh^3}{12(1-\vartheta^2)}$, where ϑ is the Poisson's ratio, $\bar{W}$ is a continuously differentiable imperfection function of X and Y, the parameter ∇^4 is a two – dimensional biharmonic operator defined by $\nabla^4 \equiv \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)^2$, while $\hat{S}$ is a symmetric bilinear differential functional defined as

$$\hat{S}(P,Q) = P_{,XX}Q_{,YY} + P_{,YY}Q_{,YY} - 2P_{,XY}Q_{,XY} \quad (5)$$

III. NONDIMENSIONALIZATION OF THE RELEVANT EQUATIONS

As in Lockhart and Amazigo [17], we shall now introduce the following quantities

$$x = \frac{\pi X}{L}, \quad y = \frac{2\pi Y}{a}, \quad \epsilon\bar{w} = \frac{\bar{W}}{h}, \quad w = \frac{W}{h}, \quad \lambda = \frac{L^2 aP}{\pi^2 D}, \quad \xi = \frac{L^2}{(\pi a)^2},$$

$$A = \frac{L^2\sqrt{12(1-\vartheta^2)}}{\pi^2 ah}, \quad H = \frac{h}{a}, \quad K(\xi) = -\left(\frac{A}{1+\xi}\right)^2, 0 < \epsilon \ll 1,$$

$$S(p,q) = p_{,xx}q_{,yy} + f_{,yy}q_{,xx} - 2p_{,xy}q_{,xy}$$

We note that λ is a nondimensional load parameter and ϵ is a small amplitude of the imperfection. Similar to [17], we shall neglect the boundary layer effect by assuming that the pre – buckling deflection is constant, and so, we assume

$$F = \frac{P}{2}\left(X^2 + \frac{1}{2}\alpha Y^2\right) + \left(\frac{E^2h^2L^2}{\pi^2 a(1+\xi)}\right)f \quad (6)$$

$$W = \frac{Pa^2(1-\vartheta\alpha)}{Eh} + hw \quad (7)$$

where the parameter α takes the value $\alpha = 1$ if pressure contributes to axial stress through end plates, but $\alpha = 0$ if pressure acts laterally. Substituting the nondimensional quantities into (1) – (4) gives

$$\overline{\nabla}^4 w - K(\xi)\left(f_{,xx} + \xi r f_{,yy}\right) + \lambda\left[\frac{\alpha}{2}(w + \epsilon\overline{w})_{,xx} + \xi\left(1 - \frac{\alpha}{2}\right)(w + \epsilon\overline{w})_{,yy}\right] = -K(\xi)HS(f, w + \epsilon\overline{w}) \qquad (8)$$

$$\overline{\nabla}^4 f = (1+\xi)^2\left(w_{,xx} + \xi r w_{,yy}\right) = -\frac{1}{2}H(1+\xi)^2 S(w + \epsilon\overline{w}, w) \qquad (9)$$

$$0 < x < \pi, \qquad 0 < y < 2\pi \qquad (10a)$$

$$w = w_{,xx} = f = f_{,xx} = 0 \ at \ x = 0, \pi \qquad (10b)$$

where, $r = \frac{a}{b}$, $\quad \overline{\nabla}^4 \equiv \left(\frac{\partial^2}{\partial X^2} + \xi \frac{\partial^2}{\partial Y^2}\right)^2$

A subscript following a comma indicates partial differentiation and we have assumed simply – supported boundary conditions as in (10b).

IV. CLASSICAL BUCKLING LOAD

This is defined as the load required to statically buckle the linear perfect structure and the relevant equations are

$$\overline{\nabla}^4 w - K(\xi)\left(f_{,xx} + \xi r f_{,yy}\right) + \lambda\left[\frac{\alpha}{2} w_{,xx} + \xi\left(1 - \frac{\alpha}{2}\right) w_{,yy}\right] = 0 \qquad (11)$$

and

$$\overline{\nabla}^4 f = (1+\xi)^2\left(w_{,xx} + \xi r w_{,yy}\right) = 0 \qquad (12)$$

$$w = w_{,xx} = f = f_{,xx} = 0 \ at \ x = 0, \pi \qquad (13)$$

Based on the boundary conditions, the solution to (11) – (13) will be of the form

$$(w, f) = (a_{mk}, b_{mk}) sinmx sin(ky + \emptyset_{mk}) \qquad (14)$$

where $\emptyset_{mk}$ is an inconsequential phase and $(a_{mk}, b_{mk}) \neq (0, 0)$. Substitution of (14) into (12) and simplification easily gives

$$b_{mk} = \frac{-(1+\xi)^2 m^2 a_{mk}}{(m^2 + \xi k^2)^2 + (1+\xi)^2 \xi r k^2} \qquad (15)$$

while the result of substituting (15) into (11) and subsequent simplification gives

$$(m^2 + \xi k^2)^2 - \lambda\left\{\frac{\alpha m^2}{2} + \xi k^2\left(1 - \frac{\alpha}{2}\right)\right\} - \frac{K(\xi)(m^2 + \xi k^2 r)(1+\xi)^2}{(m^2 + \xi k^2)^2 + (1+\xi)^2 \xi r k^2} = 0 \qquad (16)$$

Batdorf as cited in [17], had assumed that k varies continuously and so, determined the classical buckling load λ_c using the condition $\frac{d\lambda}{dk} = 0$. If $k = n$ is the value of k at buckling, then, (16) easily yields

$$\lambda_c = \frac{(m^2 + \xi n^2)^2 - \frac{K(\xi)(1+\xi)^2(m^2 + \xi n^2 r)}{(m^2 + \xi n^2)^2 + (1+\xi)^2 \xi r n^2}}{\frac{\alpha m^2}{2} + \left(1 - \frac{\alpha}{2}\right)\xi r n^2} \qquad (17)$$

By substituting for $K(\xi)$ and taking $\zeta = \xi n^2$ as well as taking $m = 1$, the classical buckling load, λ_c from (17), becomes

$$\lambda_c = (1+\zeta)^2 + \frac{\dfrac{A^2(1+\zeta r)}{(1+\zeta)^2+(1+\xi)^2\zeta r}}{\dfrac{\alpha}{2}+\left(1-\dfrac{\alpha}{2}\right)\zeta \mathrm{r}} \tag{18}$$

In this case, the displacement w and Airy stress function f become

$$(w,f) = \left(1, \quad \frac{-(1+\xi)^2}{(1+\zeta)^2+(1+\xi)^2\zeta r}\right) a_{1n} sinxsin(ny+\emptyset_{1n}) \tag{19}$$

V. DISPLACEMENT AND AIRY STRESS FUNCTION OF THE NONLINEAR PROBLEM

By exploiting the relative smallness of ϵ relative to unity, it is convenient to assume the asymptotic series

$$\begin{pmatrix} w \\ f \end{pmatrix} = \sum_{i=1}^{\infty} \begin{pmatrix} w^{(i)} \\ f^{(i)} \end{pmatrix} \epsilon^i \tag{20}$$

Substituting (20) into (8) – (10a, b) yields

$$O(\epsilon) \begin{cases} \bar{\nabla}^4 w^{(1)} - K(\xi)\left(f^{(1)}_{,xx} + \xi r f^{(1)}_{,yy}\right) + \lambda\left[\frac{\alpha}{2}\left(w^{(1)}+\epsilon\bar{w}\right)_{,xx} + \xi\left(1-\frac{\alpha}{2}\right)\left(w^{(1)}+\epsilon\bar{w}\right)_{,yy}\right] \\ = -K(\xi)HS\left(f, w^{(1)}+\epsilon\bar{w}\right) \qquad (21) \\ \bar{\nabla}^4 f^{(1)} - (1+\xi)^2\left(w^{(1)}_{,xx} + \xi r w^{(1)}_{,yy}\right) = 0 \qquad (22) \end{cases}$$

$$O(\epsilon^2) \begin{cases} \bar{\nabla}^4 w^{(2)} - K(\xi)\left(f^{(2)}_{,xx} + \xi r f^{(2)}_{,yy}\right) + \lambda\left[\frac{\alpha}{2}\left(w^{(1)}+\bar{w}\right)_{,xx} + \xi\left(1-\frac{\alpha}{2}\right) w^{(1)}_{,yy}\right] \\ = -K(\xi)H\left[S\left(f^{(1)}, w^{(1)}\right) + S\left(f^{(1)}, \bar{w}\right)\right] \qquad (23) \\ \bar{\nabla}^4 f^{(2)} - (1+\xi)^2\left(w^{(2)}_{,xx} + \xi r w^{(2)}_{,yy}\right) = -\frac{1}{2}H(1+\xi)\left[S\left(w^{(1)}, w^{(1)}\right) + S\left(w^{(1)}, \bar{w}\right)\right] \qquad (24) \end{cases}$$

$$O(\epsilon^3) \begin{cases} \bar{\nabla}^4 w^{(3)} - K(\xi)\left(f^{(3)}_{,xx} + \xi r f^{(3)}_{,yy}\right) + \lambda\left[\frac{\alpha}{2} w^{(3)}_{,xx} + \xi\left(1-\frac{\alpha}{2}\right) w^{(3)}_{,yy}\right] \\ = -K(\xi)H\left[S\left(f^{(1)}, w^{(2)}\right) + S\left(f^{(2)}, w^{(1)}\right) + S\left(f^{(2)}, \bar{w}\right)\right] \qquad (25) \\ \bar{\nabla}^4 f^{(3)} - (1+\xi)^2\left(w^{(3)}_{,xx} + \xi r w^{(3)}_{,yy}\right) = -\frac{1}{2}H(1+\xi)^2\left[S\left(w^{(1)}, w^{(2)}\right) + S\left(w^{(2)}, w^{(1)}\right) + S\left(w^{(2)}, \bar{w}\right)\right] \qquad (26) \end{cases}$$

etc.

$$w^{(i)} = w^{(i)}_{,xx} = f^{(i)} = f^{(i)}_{,xx} = 0, at\ x = 0, \pi\ , \qquad i = 1, 2, 3, \ldots \tag{27}$$

Based on the boundary conditions (27), the continuously – differentiable imperfection function will be taken as

$$\bar{w}(x,y) = \left(\bar{a}\cos ny + \bar{b}\sin ny\ \right)\sin mx \tag{28}$$

where $\bar{a}$ and $\bar{b}$ are Fourier coefficients. All along, we shall let the solution of (21) – (27) be in the form

$$\begin{pmatrix} w^{(i)} \\ f^{(i)} \end{pmatrix} = \sum_{p=1,q=1}^{\infty} \left[\begin{pmatrix} w_1^{(i)} \\ f_1^{(i)} \end{pmatrix} cosqy + \begin{pmatrix} w_2^{(i)} \\ f_2^{(i)} \end{pmatrix} sinqy\right] sinpx \tag{29}$$

Thus, assuming (29), the following general expansion will automatically hold for any $w^{(i)}$ and $f^{(i)}$, I = 1, 2, 3, …

$$\bar{\nabla}^4 f^{(i)} - (1+\xi)^2\left(w^{(i)}_{,xx} + \xi r w^{(i)}_{,yy}\right) \equiv \sum_{p=1,q=1}^{\infty}\left[\left\{(p^2+q^2\xi)^2 f_1^{(i)} + (1+\xi)^2(q^2 r\xi - p^2)w_1^{(i)}\right\} sinpxcosqy + \left\{(p^2+q^2\xi)^2 f_2^{(i)} + (1+\xi)^2(q^2 r\xi - p^2)w_2^{(i)}\right\} sinpxsinqy\right] \quad (30)$$

In the same way, the following expansion also holds for any $w^{(i)}$ and $f^{(i)}$, I = 1, 2, 3, …

$$\bar{\nabla}^4 w^{(i)} - K(\xi)\left(f^{(i)}_{,xx} + \xi r f^{(i)}_{,yy}\right) + \lambda\left[\frac{\alpha}{2} w^{(i)}_{,xx} + \xi r\left(1-\frac{\alpha}{2}\right) w^{(i)}_{,yy}\right] \equiv \sum_{p,q=1}^{\infty}\left[\left\{(p^2+q^2\xi)^2 w_1^{(i)} + (p^2 K(\xi) - q^2 r\xi) f_1^{(i)} - \lambda\left(\frac{\alpha p^2}{2} + \left(1-\frac{\alpha}{2}\right)\xi q^2 w_1^{(i)}\right)\right\} sinpxcosqy + \left\{(p^2+q^2\xi)^2 w_2^{(i)} + (p^2 K(\xi) - q^2 r\xi) f_2^{(i)} - \lambda\left(\frac{\alpha p^2}{2} + \left(1-\frac{\alpha}{2}\right)\xi q^2 w_2^{(i)}\right)\right\} sinpxsinqy\right] \quad (31)$$

Integration with respect to x will be from $x = 0$ to $x = \pi$, while integration with respect to y will be $0 \; to \; 2\pi$.

Solution of Equations of Order ϵ

The next procedure is to substitute (29) into (22) while noting (30). Multiplying through by $cosnysinmx$, and next, by $sinnysinmx$, it becomes evident that for $p = m, q = n$, the resultant equations are

$$f_1^{(1)} = \frac{-(1+\xi)^2(n^2 r\xi - m^2)w_1^{(1)}}{(m^2+\xi n^2)^2}, \quad f_2^{(1)} = \frac{-(1+\xi)^2(n^2 r\xi - m^2)w_2^{(1)}}{(m^2+\xi n^2)^2} \quad (32)$$

Substituting (31) into (21) but noting the imperfection $\bar{w}$ which is not captured in (31), and first multiplying through by $cosnysinmx$ and next by $sinnysinmx$, and for $p = m, q = n$ in each case, the following are derived, after substituting for $K(\xi)$ and for $f_1^{(1)}$ and $f_2^{(1)}$ from (32).

$$w_1^{(1)} = \frac{\lambda\bar{a}\left\{\frac{\alpha m^2}{2} + \xi n^2\left(1-\frac{\alpha}{2}\right)\right\}}{\varphi_1}, \quad w_2^{(1)} = \frac{\lambda\bar{b}\left\{\frac{\alpha m^2}{2} + \xi n^2\left(1-\frac{\alpha}{2}\right)\right\}}{\varphi_1} \quad (33a)$$

where,

$$\varphi_1 = \left[(m^2+\xi n^2)^2 + \left\{\left(\frac{mA}{1+\xi}\right)^2 + n^2 r\xi\right\}(1+\xi)^2\left\{\frac{n^2 r\xi - m^2}{(m^2+\xi n^2)^2}\right\} - \lambda\left\{\frac{\alpha m^2}{2} + \xi n^2\left(1-\frac{\alpha}{2}\right)\right\}\right] \quad (33b)$$

So far, it is seen that

$$\begin{pmatrix} w^{(1)} \\ f^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ -\varphi_0 \end{pmatrix}\left(w_1^{(1)}\cos ny + w_2^{(1)} sinny\right) sinmx \quad (34a)$$

where,

$$\varphi_0 = (1+\xi)^2\left\{\frac{n^2 r\xi - m^2}{(m^2+\xi n^2)^2}\right\} \quad (34b)$$

Solution of Equations of Order ϵ^2

Following (29) and for $i = 2$, the solution of this order will generally be of the form

$$\begin{pmatrix} w^{(2)} \\ f^{(2)} \end{pmatrix} = \sum_{p,q=1}^{\infty} \left[\begin{pmatrix} w_1^{(2)} \\ f_1^{(2)} \end{pmatrix} cosqy + \begin{pmatrix} w_2^{(2)} \\ f_2^{(2)} \end{pmatrix} sinqy \right] sinpx \qquad (35)$$

After simplifying the right hand sides of (23) and (24) and substituting therein, the following are easily derived

$$\begin{aligned} &\overline{\nabla}^4 w^{(2)} - K(\xi)\left(f_{,xx}^{(2)} + \xi r f_{,yy}^{(2)}\right) + \lambda\left[\frac{\alpha}{2} w_{,xx}^{(2)} + \xi\left(1 - \frac{\alpha}{2}\right) w_{,yy}^{(2)}\right] \\ &\quad = -K(\xi) H (mn)^2 \varphi_0 \left[\left({w_1^{(1)}}^2 + {w_2^{(1)}}^2 + \bar{a} w_1^{(1)} + \bar{b} w_2^{(1)}\right) cos2mx \right. \\ &\quad + \left({w_2^{(1)}}^3 - {w_1^{(1)}}^2 + \bar{b} w_2^{(1)} - \bar{a} w_1^{(1)}\right) cos2ny \\ &\quad \left. - \left(2 w_1^{(1)} w_2^{(1)} + \bar{a} w_2^{(1)} + \bar{b} w_1^{(1)}\right) sin2ny \right] \qquad (36) \end{aligned}$$

$$\begin{aligned} &\overline{\nabla}^4 f^{(2)} - (1+\xi)^2\left(w_{,xx}^{(2)} + \xi r w_{,yy}^{(2)}\right) \\ &\quad = -\frac{1}{2} H (1+\xi)^2 (mn)^2 \left[-\left({w_1^{(1)}}^2 + {w_2^{(1)}}^2 + \bar{a} w_1^{(1)} + \bar{b} w_2^{(1)}\right) cos2mx \right. \\ &\quad + \left(-{w_2^{(1)}}^2 + {w_1^{(1)}}^2 - \bar{b} w_2^{(1)} + \bar{a} w_1^{(1)}\right) cos2ny \\ &\quad \left. + \left(2 w_1^{(1)} w_2^{(1)} + \bar{a} w_2^{(1)} + \bar{b} w_1^{(1)}\right) sin2ny \right] \qquad (37) \end{aligned}$$

By assuming (30) and (35) and substituting same into (37) and further multiplying the resultant equation, first by $cos2nysinmx$ and next by $sin2nysinmx$, and for $q = 2n$ and $p = m$ in each case, the resultant values are respectively given as

$$f_1^{(2)} = \frac{-(1+\xi)^2(4n^2 r\xi - m^2) w_1^{(2)} + \frac{2H(1+\xi)mn^2}{\pi}\left({w_1^{(1)}}^2 - {w_2^{(1)}}^2 + \bar{a} w_1^{(1)} - \bar{b} w_2^{(1)}\right)}{(m^2 + 4n^2\xi)^2} \qquad (38)$$

and

$$f_2^{(2)} = \frac{-(1+\xi)^2(4n^2 r\xi - m^2) w_2^{(2)} + \frac{2H(1+\xi)mn^2}{\pi}\left(2 w_1^{(1)} w_2^{(1)} - \bar{a} w_2^{(1)} + \bar{b} w_1^{(1)}\right)}{(m^2 + 4n^2\xi)^2} \qquad (39)$$

Next, substituting into (36), using (31) for $i = 2$, and further multiplying the resultant equation first by $cos2nysinmx$ and next by $sin2nysinmx$, it becomes obvious that for $p = m, q = 2n$, the results are respectively given as, (after substituting for $f_1^{(2)}$)

$$\begin{aligned} &\left[(m^2 + 4n^2\xi)^2 + \frac{\left(\left(\frac{Am}{(1+\xi)}\right)^2 + 4n^2\xi r\right)(1+\xi)^2(4n^2 r\xi - m^2)}{(m^2 + 4n^2\xi)^2} - \lambda\left\{\frac{\alpha m^2}{2} + 4\left(1 - \frac{\alpha}{2}\right)\xi n^2\right\}\right] w_1^{(2)} \\ &\quad = \frac{-2H(1+\xi)^2 mn^2 \left({w_1^{(1)}}^2 - {w_2^{(1)}}^2 + \bar{a} w_1^{(1)} + \bar{b} w_2^{(1)}\right)}{(m^2 + 4n^2\xi)^2} \\ &\quad + \frac{4\pi H mn^2 \varphi_0 A^2}{(1+\xi)^2}\left({w_2^{(1)}}^2 - {w_1^{(1)}}^2 + \bar{b} w_2^{(1)} - \bar{a} w_1^{(1)}\right) \qquad (40) \end{aligned}$$

and

$$\left[(m^2+4n^2\xi)^2+\frac{\left(\left(\frac{Am}{(1+\xi)}\right)^2+4n^2\xi r\right)(1+\xi)^2(4n^2r\xi-m^2)}{(m^2+4n^2\xi)^2}-\lambda\left\{\frac{\alpha m^2}{2}+4\left(1-\frac{\alpha}{2}\right)\xi n^2\right\}\right]w_2^{(2)}$$
$$=\frac{-2H(1+\xi)^2mn^2}{\pi(m^2+4n^2\xi)^2}\left(2w_1^{(1)}w_2^{(1)}+\bar{a}w_2^{(1)}+\bar{b}w_1^{(1)}\right)$$
$$-\frac{4\pi Hmn^2\varphi_0A^2}{(1+\xi)^2}\left(2w_1^{(1)}w_2^{(1)}+\bar{a}w_2^{(1)}+\bar{b}w_1^{(1)}\right) \qquad (41)$$

A further simplification of (40) and (41) yields

$$w_1^{(2)}=\left(\frac{\varphi_3+\varphi_4}{\varphi_2}\right)\left(w_2^{(1)^2}-w_1^{(1)^2}+\bar{b}w_2^{(1)}-\bar{a}w_1^{(1)}\right) \qquad (42)$$

where,

$$\varphi_3=\frac{2H(1+\xi)^2mn^2}{\pi(m^2+4n^2\xi)^2},\qquad \varphi_4=\frac{4\pi Hmn^2\varphi_0A^2}{(1+\xi)^2} \qquad (43)$$

$$\varphi_2=\left[(m^2+4n^2\xi)^2+\frac{\left(\left(\frac{Am}{(1+\xi)}\right)^2+4n^2\xi r\right)(1+\xi)^2(4n^2r\xi-m^2)}{(m^2+4n^2\xi)^2}\right.$$
$$\left.-\lambda\left\{\frac{\alpha m^2}{2}+4\left(1-\frac{\alpha}{2}\right)\xi n^2\right\}\right] \qquad (44)$$

and

$$w_2^{(2)}=\left(\frac{\varphi_3+\varphi_4}{\varphi_2}\right)\left(2w_1^{(1)}w_2^{(1)}+\bar{a}w_2^{(1)}+\bar{b}w_1^{(1)}\right) \qquad (45)$$

Thus, at this order of perturbation, we get

$$\begin{pmatrix}w^{(2)}\\f^{(2)}\end{pmatrix}=\left[\begin{pmatrix}w_1^{(2)}\\f_1^{(2)}\end{pmatrix}cos2ny+\begin{pmatrix}w_2^{(2)}\\f_2^{(2)}\end{pmatrix}sin2ny\right]sinmx \qquad (46)$$

Solution of Equations of Order ϵ^3

From (29), it is expected that for $i=3$, the result of this order will be of the form

$$\begin{pmatrix}w^{(3)}\\f^{(3)}\end{pmatrix}=\sum_{p,q=1}^{\infty}\left[\begin{pmatrix}w_1^{(3)}\\f_1^{(3)}\end{pmatrix}cosqy+\begin{pmatrix}w_2^{(3)}\\f_2^{(3)}\end{pmatrix}sinqy\right]sinpx \qquad (47)$$

Substituting on the right hand sides of (25) and (26) yields

$$\bar{\nabla}^4 w^{(3)} - K(\xi)\left(f_{,xx}^{(3)} + \xi r f_{,yy}^{(3)}\right) + \lambda\left[\frac{\alpha w_{,xx}^{(3)}}{2} + \left(1 - \frac{\alpha}{2}\right)\xi w_{,yy}^{(3)}\right]$$
$$= -H(mn)^2 K(\xi)\left[\frac{5}{4}\left\{\left(f_1^{(1)}w_1^{(2)} + f_2^{(1)}w_2^{(2)}\right)cosny + \left(f_1^{(1)}w_2^{(2)} - f_2^{(1)}w_1^{(2)}\right)sinny\right.\right.$$
$$\left. + \left(f_1^{(1)}w_2^{(2)} + f_2^{(1)}w_1^{(2)}\right)sin3ny + \left(f_1^{(1)}w_1^{(2)} - f_2^{(1)}w_1^{(2)}\right)cos3ny\right\}(1 - cos2mx)$$
$$- \left\{\left(f_2^{(1)}w_2^{(2)} + f_1^{(1)}w_1^{(2)}\right)cosny + \left(f_1^{(1)}w_2^{(2)} - f_2^{(1)}w_1^{(2)}\right)sinny + \left(f_2^{(1)}w_2^{(2)} - f_1^{(1)}w_1^{(2)}\right)cos3ny\right.$$
$$\left. - \left(f_2^{(1)}w_1^{(2)} + f_1^{(1)}w_2^{(2)}\right)sin3ny\right\}(1 + cos2mx)$$
$$+ \frac{5}{4}\left\{\left(w_1^{(1)}f_1^{(2)} + w_2^{(1)}f_2^{(2)}\right)cosny + \left(w_1^{(1)}f_2^{(2)} - w_2^{(1)}f_1^{(2)}\right)sinny\right.$$
$$\left. + \left(w_1^{(1)}f_2^{(2)} + w_2^{(1)}f_1^{(2)}\right)sin3ny + \left(w_1^{(1)}f_1^{(2)} - w_2^{(1)}f_2^{(2)}\right)cos3ny\right\}(1 - cos2mx)$$
$$- \left\{\left(w_2^{(1)}f_2^{(2)} + w_1^{(1)}f_1^{(2)}\right)cosny - \left(w_2^{(1)}f_1^{(2)} - w_1^{(1)}f_2^{(2)}\right)sinny + \left(w_2^{(1)}f_2^{(2)} - w_1^{(1)}f_1^{(2)}\right)cos3ny\right.$$
$$\left. - \left(w_2^{(1)}f_1^{(2)} + w_1^{(1)}f_2^{(2)}\right)sin3ny\right\}(1 + cos2mx)$$
$$+ \frac{5}{4}\left\{\left(\bar{a}f_1^{(2)} + \bar{b}f_2^{(2)}\right)cosny + \left(\bar{a}f_2^{(2)} - \bar{b}f_1^{(2)}\right)sinny + \left(\bar{a}f_2^{(2)} + \bar{b}f_1^{(2)}\right)sin3ny\right.$$
$$\left. + \left(\bar{a}f_1^{(2)} - \bar{b}f_2^{(2)}\right)cos3ny\right\}(1 - cos2mx)$$
$$- \left\{\left(\bar{b}f_2^{(2)} + \bar{a}f_1^{(2)}\right)cosny - \left(\bar{b}f_1^{(2)} - \bar{a}f_2^{(2)}\right)sinny + \left(\bar{b}f_2^{(2)} - \bar{a}f_1^{(2)}\right)cos3ny\right.$$
$$\left.\left. - \left(\bar{b}f_1^{(2)} + \bar{a}f_2^{(2)}\right)sin3ny\right\}(1 + cos2mx)\right] \quad (48)$$

and

$$\bar{\nabla}^4 f^{(3)} - (1+\xi)^2\left(w_{,xx}^{(3)} + \xi r w_{,yy}^{(3)}\right) = -H(1+\xi)(mn)^2 \times \left[\frac{5}{4}\left\{\left(w_1^{(1)}w_1^{(2)} + w_2^{(1)}w_2^{(2)}\right)cosny + \left(w_1^{(1)}w_2^{(2)} - w_2^{(1)}w_1^{(2)}\right)sinny + \left(w_1^{(1)}w_2^{(2)} + w_2^{(1)}w_1^{(2)}\right)sin3ny + \left(w_1^{(1)}w_1^{(2)} - w_2^{(1)}w_2^{(2)}\right)cos3ny\right\}(1 - cos2mx) - \left\{\left(w_2^{(1)}w_2^{(2)} + w_1^{(1)}w_1^{(2)}\right)cosny + \left(w_1^{(1)}w_2^{(2)} - w_2^{(1)}w_1^{(2)}\right)sinny + \left(w_2^{(1)}w_2^{(2)} - w_1^{(1)}w_1^{(2)}\right)cos3ny - \left(w_2^{(1)}w_1^{(2)} + w_1^{(1)}w_2^{(2)}\right)sin3ny\right\}(1 + cos2mx) + \frac{1}{2}\left\{\left\{\frac{5}{4}\left\{\left(\bar{a}w_1^{(2)} + \bar{b}w_2^{(2)}\right)cosny + \left(\bar{a}w_2^{(2)} - \bar{b}w_1^{(2)}\right)sinny + \left(\bar{a}w_2^{(2)} + \bar{b}w_1^{(2)}\right)sin3ny + \left(\bar{a}w_1^{(2)} - \bar{b}w_2^{(2)}\right)cos3ny\right\}(1 - cos2mx) - \left\{\left(\bar{b}w_2^{(2)} + \bar{a}w_1^{(2)}\right)cosny - \left(\bar{b}w_1^{(2)} - \bar{a}w_2^{(2)}\right)sinny + \left(\bar{b}w_2^{(2)} - \bar{a}w_1^{(2)}\right)cos3ny - \left(\bar{b}w_1^{(2)} + \bar{a}w_2^{(2)}\right)sin3ny\right\}(1 + cos2mx)\right\}\right\}\right] \quad (49)$$

A careful inspection reveals that the buckling modes and Airy stress functions of this order of perturbation will generally be in the shapes of $cosnysinmx, sinnysinmx, cos3nysinmx$ and $sinn3nysinmx$. To first determine the Airy stress function of this order, the process is to substitute (30), for $i = 3$, on the right hand side of (49), then multiply through by $cosnysinmx$ and for $p = m, q = n$, the associated Airy stress function is

$$f_{1(m,n)}^{(3)} = -\frac{1}{(m^2 + 4n^2\xi)^2}\left[(1+\xi)^2(n^2 r\xi - m^2)w_{1(m,n)}^{(3)} + \frac{4H(1+\xi)^2 mn^2}{\pi}\left\{w_1^{(1)}w_1^{(2)} + w_2^{(1)}w_2^{(2)} + \frac{1}{2}\left(\bar{a}w_1^{(1)} + \bar{b}w_2^{(2)}\right)\right\}\right] \quad (50)$$

where m is odd. In the same substitution into (49), we next multiply by $sinnysinmx$, and for $p = m$, $q = n$, the Airy stress function and associated buckling mode are related as

$$f_{2(m,n)}^{(3)} = -\frac{1}{(m^2 + n^2\xi)^2}\left[(1+\xi)^2(n^2 r\xi - m^2)w_{2(m,n)}^{(3)} + \frac{4H(1+\xi)^2 mn^2}{\pi}\left\{w_1^{(1)}w_2^{(2)} - w_2^{(1)}w_1^{(2)} + \frac{1}{2}\left(\bar{a}w_2^{(1)} + \bar{b}w_1^{(2)}\right)\right\}\right] \quad (51)$$

valid for m odd. Multiplying through by $cos3nysinmx$ in the same substitution into (49) and for $p = m, q = 3n$, the Airy stress function $f^{(3)}_{1(m,3n)}$ is related to respective buckling $w^{(3)}_{1(m,3n)}$ as

$$f^{(3)}_{1(m,n)} = -\frac{1}{(m^2+9n^2\xi)^2}\left[(1+\xi)^2(9n^2r\xi - m^2)w^{(3)}_{1(m,3n)} + \frac{28H(1+\xi)^2mn^2}{3\pi}\left\{w_1^{(1)}w_1^{(2)} - w_2^{(1)}w_2^{(2)} + \frac{1}{2}\left(\bar{a}w_1^{(2)} - \bar{b}w_2^{(2)}\right)\right\}\right] \quad (52)$$

valid for m odd. Lastly, we next multiply through by $sin3nysinmx$ in the same substitution into (49) and for $p = m, q = 3n$, the Airy stress function $f^{(3)}_{2(m,3n)}$ is related to the associated buckling mode $w^{(3)}_{2(m,3n)}$ as

$$f^{(3)}_{2(m,3n)} = -\frac{1}{(m^2+9n^2\xi)^2}\left[(1+\xi)^2(9n^2r\xi - m^2)w^{(3)}_{2(m,3n)} + \frac{28H(1+\xi)^2mn^2}{3\pi}\left\{w_1^{(1)}w_2^{(2)} + w_2^{(1)}w_1^{(2)} + \frac{1}{2}\left(\bar{a}w_2^{(2)} - \bar{b}w_1^{(2)}\right)\right\}\right] \quad (53)$$

valid for m odd. At this order, the following Airy stress functions and the buckling modes are the fundamental ones from where others are determined and generally take the forms

$$\begin{pmatrix} w^{(3)}_{1(m,n)} \\ f^{(3)}_{1(m,n)} \end{pmatrix} cosnysinmx, \quad \begin{pmatrix} w^{(3)}_{2(m,n)} \\ f^{(3)}_{2(m,n)} \end{pmatrix} sinnysinmx, \quad \begin{pmatrix} w^{(3)}_{1(m,3n)} \\ f^{(3)}_{1(m,3n)} \end{pmatrix} cos3nysinmx,$$
$$and \begin{pmatrix} w^{(3)}_{2(m,3n)} \\ f^{(3)}_{2(m,3n)} \end{pmatrix} sin3nysinmx$$

To determine the buckling modes, substitution is now made into (48) by now multiplying through by $cosnysinmx$ using (50).Thus, for $p = m, q = n$, the simplification for determining the buckling mode $w^{(3)}_{1(m,n)}$ gives (using (33b))

$$w^{(3)}_{1(m,n)} = -\frac{4}{\varphi_1}\left[\frac{H(1+\xi)^2mn^2\left\{\left(\frac{mA}{1+\xi}\right)^2 + n^2\xi\right\}}{\pi(m^2+n^2\xi)^2}\left\{\left(w_1^{(1)}w_1^{(2)} + w_2^{(1)}w_2^{(2)}\right) + \frac{1}{2}\left(\bar{a}w_1^{(2)} + \bar{b}w_2^{(2)}\right)\right\} + \frac{HK(\xi)mn^2}{\pi}\left\{\left(f_1^{(1)}w_1^{(2)} + f_2^{(1)}w_2^{(2)}\right) + \left(\bar{a}f_1^{(2)} + \bar{b}f_2^{(2)}\right)\right\}\right] \quad (54)$$

where m is odd. On still substituting for terms in (48), multiplying through by $sinnysinmx$, using (51) and for $p = m, q = n$, the expression for $w^{(3)}_{2(m,n)}$ eventually gives

$$w^{(3)}_{2(m,n)} = \frac{4}{\varphi_1}\left[\frac{H(1+\xi)^2mn^2\left\{\left(\frac{mA}{1+\xi}\right)^2 + n^2\xi\right\}}{\pi(m^2+n^2\xi)^2}\left\{\left(w_1^{(1)}w_2^{(2)} - w_2^{(1)}w_1^{(2)}\right) + \frac{1}{2}\left(\bar{a}w_2^{(2)} - \bar{b}w_1^{(2)}\right)\right\} - \frac{HK(\xi)mn^2}{\pi}\left\{\left(f_1^{(1)}w_2^{(2)} - f_2^{(1)}w_1^{(2)}\right) + \left(w_1^{(1)}f_2^{(2)} - w_2^{(1)}f_1^{(2)}\right) + \left(\bar{a}f_2^{(2)} - \bar{b}f_1^{(2)}\right)\right\}\right] \quad (55)$$

This is valid for m odd. Again, substituting into (48) and multiplying through by $cos3nysinmx$, using (52) and for $p = m, q = 3n$, the expression for $w^{(3)}_{1(m,3n)}$ gives

$$w_{1(m,3n)}^{(3)} = \frac{28Hmn^2}{3\pi\varphi_5}\left[\frac{\left\{\left(\frac{mA}{1+\xi}\right)^2 + 9n^2\xi\right\}}{(m^2+9n^2\xi)^2}\left\{\left(w_1^{(1)}w_1^{(2)} - w_2^{(1)}w_2^{(2)}\right) + \frac{1}{2}\left(\bar{a}w_2^{(2)} - \bar{b}w_1^{(2)}\right)\right\}\right.$$
$$\left. - K(\xi)\left\{\left(f_1^{(1)}w_1^{(2)} - f_2^{(1)}w_2^{(2)}\right) + \left(w_1^{(1)}f_1^{(2)} - w_2^{(1)}f_2^{(2)}\right) + \left(\bar{a}f_1^{(2)} - \bar{b}f_2^{(2)}\right)\right\}\right] \quad (56)$$

where,

$$\varphi_5 = (m^2+9n^2\xi)^2 + \frac{\left(\left(\frac{mA}{1+\xi}\right)^2 + 9n^2\xi r\right)(1+\xi)^2(9n^2r\xi - m^2)}{(m^2+9n^2\xi)^2} - \lambda\left\{\frac{\alpha m^2}{2} + 9\xi n^2\left(1-\frac{\alpha}{2}\right)\right\} \quad (57)$$

Next, substituting into (48) and multiplying through by $sin3nysinmx$, it is apparent for $p = m, q = 3n$, (using (53)), the expression for $w_{2(m,3n)}^{(3)}$ is

$$w_{2(m,3n)}^{(3)} = \frac{28Hmn^2}{3\pi\varphi_5}\left[\frac{\left\{\left(\frac{mA}{1+\xi}\right)^2 + 9n^2\xi\right\}}{(m^2+9n^2\xi)^2}\left\{\left(w_1^{(1)}w_2^{(2)} + w_2^{(1)}w_1^{(2)}\right) + \frac{1}{2}\left(\bar{a}w_2^{(2)} + \bar{b}w_1^{(2)}\right)\right\}\right.$$
$$\left. - K(\xi)\left\{\left(f_1^{(1)}w_2^{(2)} + f_2^{(1)}w_1^{(2)}\right) + \left(w_1^{(1)}f_2^{(2)} + w_2^{(1)}f_1^{(2)}\right) + \left(\bar{a}f_2^{(2)} + \bar{b}f_1^{(2)}\right)\right\}\right] \quad (58)$$

valid for m odd.

So far, the displacement and associated Airy stress function in the entire deformation can be written as

$$\begin{pmatrix} w(x,y) \\ f(x,y) \end{pmatrix} = \epsilon\left[\begin{pmatrix} w_1^{(1)} \\ f_1^{(1)} \end{pmatrix} cosnysinmx + \begin{pmatrix} w_2^{(1)} \\ f_2^{(1)} \end{pmatrix} sinnysinmx\right]$$
$$+ \epsilon^2\left[\begin{pmatrix} w_1^{(2)} \\ f_1^{(2)} \end{pmatrix} cos2nysinmx + \begin{pmatrix} w_2^{(2)} \\ f_2^{(2)} \end{pmatrix} sin2nysinmx\right]$$
$$+ \epsilon^3\left[\begin{pmatrix} w_{1(m,n)}^{(3)} \\ f_{1(m,n)}^{(3)} \end{pmatrix} cosnysinmx + \begin{pmatrix} w_{2(m,n)}^{(3)} \\ f_{2(m,n)}^{(3)} \end{pmatrix} sinnysinmx + \begin{pmatrix} w_{1(m,3n)}^{(3)} \\ f_{1(m,3n)}^{(3)} \end{pmatrix} cos3nysinmx\right.$$
$$\left. + \begin{pmatrix} w_{2(m,3n)}^{(3)} \\ f_{2(m,3n)}^{(3)} \end{pmatrix} sin3nysinmx\right] + \cdots \quad (59)$$

VI. STATIC BUCKLING LOAD, λ_S

The static buckling load, λ_S will here be determined at the maximum value of the displacement and the necessary conditions for maximum displacement are

$$w_{,x} = w_{,y} = 0 \quad (60)$$

In this case, it is necessary to omit the component of displacement of order ϵ^2 and to take $w(x,y)$ simply as

$$w(x,y) = \epsilon\left(w_1^{(1)}cosny + w_2^{(1)}sinny\right)sinmx + \epsilon^3\left(w_1^{(3)}cosny + w_2^{(3)}sinny\right)sinmx + \cdots \quad (61)$$

This is equivalent to accepting only the buckling modes that are strictly in the shape of imperfection.

Let x_a and y_a be the values of x and y respectively at the maximum displacement and let

$$y_a = y_0 + \epsilon^2 y_2 + \cdots \qquad (62)$$

Substituting (62) into (60) yields

$$\epsilon m\left(w_1^{(1)} cosny_0 + w_2^{(1)} sinny_0\right) cosmx_a + \epsilon^3 m\left(w_1^{(3)} cosny_0 + w_2^{(3)} sinny_0\right) cosmx_a + \cdots = 0 \qquad (63)$$

and

$$\epsilon n\left(-w_1^{(1)} sinny_0 + w_2^{(1)} cosny_0\right) sinmx_a + \epsilon^3 n\left(-w_1^{(3)} sinny_0 + w_2^{(3)} cosny_0\right) sinmx_a + \cdots = 0 \qquad (64)$$

From (63), it is evident that

$$x_a = \left(\frac{2r+1}{2m}\right)\pi, \quad r = 0, 1, 2, 3, \ldots \qquad (65)$$

On substituting (65) into (64), the result is

$$y_0 = \frac{1}{n}\tan^{-1}\left(\frac{w_2^{(1)}}{w_1^{(1)}}\right) = \frac{1}{n}\tan^{-1}\left(\frac{\bar{b}}{\bar{a}}\right) \qquad (66)$$

The maximum displacement w_a, using (61), is

$$w_a = \epsilon\left(w_1^{(1)} cosny_0 + w_2^{(1)} sinny_0\right) + \epsilon^3 m\left(w_{1(m,n)}^{(3)} cosny_0 + w_{2(m,n)}^{(3)} sinny_0\right) + \cdots \qquad (67)$$

where we have taken $r = 0$. The static buckling loadλ_S, will be determined from the maximization [11]

$$\frac{d\lambda}{dw_a} = 0 \qquad (68)$$

for w_a as in (67) and where each of $w_1^{(1)}$, $w_2^{(1)}$, $w_{1(m,n)}^{(3)}$ and $w_{2(m,n)}^{(3)}$ depends on the load parameter λ. For the simplification of (68), it is necessary to rewrite the following simplifications

$$\varphi_6 = \frac{\varphi_3 + \varphi_4}{\varphi_2}, \qquad w_1^{(2)} = \varphi_6\left(2w_1^{(1)}w_2^{(1)} + \bar{b}w_2^{(1)} - \bar{a}w_1^{(1)}\right) \qquad (69)$$

$$w_2^{(2)} = \varphi_6\left(2w_1^{(1)}w_2^{(1)} + \bar{a}w_2^{(1)} + \bar{b}w_1^{(1)}\right) \qquad (70)$$

$$\varphi_7 = -\frac{(1+\xi)^2(4n^2r\xi - m^2)}{(m^2+4n^2\xi)^2}, \qquad \varphi_8 = \frac{2H(1+\xi)^2 mn^2}{\pi(m^2+4n^2\xi)^2} \qquad (71)$$

$$f_1^{(2)} = -\varphi_7 w_1^{(2)} + \varphi_8\left({w_1^{(1)}}^2 - {w_2^{(1)}}^2 + \bar{a}w_1^{(1)} - \bar{b}w_2^{(1)}\right) \qquad (72)$$

$$f_2^{(2)} = -\varphi_7 w_2^{(2)} + \varphi_8\left(w_1^{(1)}w_2^{(1)} - \bar{a}w_2^{(1)} + \bar{b}w_1^{(1)}\right) \qquad (73)$$

Using the above simplifications, and maintaining only the terms that are cubic in the displacement components, it is easy to re-evaluate $w_{1(m,n)}^{(3)}$ and $w_{2(m,n)}^{(3)}$ as :

$$w_{1(m,n)}^{(3)} = -\frac{4}{\varphi_1}\left[\varphi_{11}{w_1^{(1)}}^3 + \varphi_{12}w_1^{(1)}{w_2^{(1)}}^2\right] \qquad (74a)$$

where,

$$\varphi_{11} = (\varphi_0 - \varphi_6 - \varphi_6)\theta_2 - \theta_1\varphi_6 \quad (74b)$$

$$\varphi_{12} = 3\theta_1\varphi_6 + \theta_2(\varphi_6 - \varphi_6\varphi_7 + 2\varphi_8 - 3\varphi_0\varphi_6 - \varphi_6) - \varphi_6 \quad (74c)$$

$$\theta_1 = \frac{Hmn^2(1+\xi)^2\left\{\left(\frac{mA}{1+\xi}\right)^2 + n^2\xi\right\}}{\pi(m^2+4n^2\xi)^2}, \qquad \theta_2 = \frac{HK(\xi)mn^2}{\pi} \quad (74d)$$

and

$$w_{2(m,n)}^{(3)} = \frac{4}{\varphi_1}\left[\varphi_9 {w_2^{(1)}}^3 + \varphi_{10} w_2^{(1)} {w_1^{(1)}}^2\right] \quad (75a)$$

where,

$$\varphi_9 = \varphi_6\varphi_0\theta_2 - \varphi_6\theta_1, \quad \varphi_{10} = 3\theta_1\varphi_6 - 3\theta_2\varphi_0\varphi_6 + 2\theta_2(\varphi_7 - \varphi_8) \quad (75b)$$

We can now rewrite w_a as

$$w_a = \epsilon C_1 + \epsilon^3 C_3 + \cdots \quad (76a)$$

where,

$$C_1 = \left(w_1^{(1)} cosny_0 + w_2^{(1)} sinny_0\right), \qquad C_3 = \left(w_{1(m,n)}^{(3)} cosny_0 + w_{2(m,n)}^{(3)} sinny_0\right) \quad (76b)$$

As in [10], the invocation of (68) is preceded by first reversing the series (76a) in the form

$$\epsilon = d_1 w_a + d_3 w_a^3 + \cdots \quad (77a)$$

The coefficients d_1 and d_3 are determined by substituting for w_a in (77a) and equating the coefficients of powers of ϵ to get

$$d_1 = \frac{1}{C_1}, \qquad d_3 = -\frac{C_3}{C_1^4} \quad (77b)$$

Knowing that each of C_1, C_3 and w_a depends on the load parameter λ, the maximization (68) eventually gives

$$w_{as} = \sqrt{\frac{C_1^3}{3C_3}} \quad (78)$$

where w_{as} is the value of w_a at static buckling. If (77a) is next evaluated at static buckling, the result easily gives

$$\epsilon = \frac{2}{3}\sqrt{\frac{C_1}{3C_3}} = \frac{2}{3}\sqrt{\frac{\left(w_1^{(1)} cosny_0 + w_2^{(1)} sinny_0\right)}{3\left(w_{1(m,n)}^{(3)} cosny_0 + w_{2(m,n)}^{(3)} sinny_0\right)}} \quad (79)$$

where, $w_1^{(1)}$ and $w_2^{(1)}$ are as in (33a) and $w_{1(m,n)}^{(3)}$ and $w_{2(m,n)}^{(3)}$ are as in (74a) and (75a) respectively.

After simplifying and substituting in (79), the final result gives

$$\left[(m^2+n^2\xi)^2+\left\{\left(\frac{mA}{1+\xi}\right)^2+n^2\xi\left(1-\frac{\alpha}{2}\right)\right\}(1+\xi)^2\frac{(n^2r\xi-m^2)}{(m^2+n^2\xi)^2}-\lambda_S\left\{\frac{\alpha m^2}{2}+\xi n^2\left(1-\frac{\alpha}{2}\right)\right\}\right]^{3/2}$$

$$=3\sqrt{3}\,\lambda_S\,\epsilon\left\{\frac{\alpha m^2}{2}+\xi n^2\left(1-\frac{\alpha}{2}\right)\right\}\times\left[\frac{(\varphi_9\bar{b}^3 sinny_0-\varphi_{11}\bar{a}^3 cosny_0)+(\bar{a}^2\bar{b}\varphi_{10}sinny_0-\bar{a}\bar{b}^2\varphi_{12}cosny_0)}{\bar{a}cosny_0+\bar{b}sinny_0}\right]^{1/2} \quad (80)$$

A similar result [5] obtained for the case

$$\bar{w}(x,y)=\bar{a}sinmxsinny \quad (81)$$

is

$$\left[(m^2+n^2\xi)^2+\left\{\left(\frac{mA}{1+\xi}\right)^2+n^2\xi\left(1-\frac{\alpha}{2}\right)\right\}(1+\xi)^2\frac{(n^2r\xi-m^2)}{(m^2+n^2\xi)^2}-\lambda_S\left\{\frac{\alpha m^2}{2}+\xi n^2\left(1-\frac{\alpha}{2}\right)\right\}\right]^{3/2}$$
$$=\frac{3\sqrt{3}}{2}\lambda_S(\bar{a}\epsilon)\left\{\frac{\alpha m^2}{2}+\xi r n^2\left(1-\frac{\alpha}{2}\right)\right\}\sqrt{Q_1Q_7} \quad (82)$$

for Q_1 and Q_7 as there defined.

Analysis of Result

The analysis leading to the result (80) was predicated on the assumption that the imperfection has a two – term Fourier series expansion unlike that of (81). The result (80) thus reflects its dependence on the two Fourier coefficients $\bar{a}$ and $\bar{b}$ which are here assumed small relative to unity. The imperfection amplitude ϵ, satisfies the inequality $0<\epsilon\ll 1$. All along, the nonlinear analysis is substantially simplified by first determining the Airy stress function, which is later substituted in the expression for determining the displacement. All results are valid for m odd.

With the aid of QBasic codes, we can obtain the numerical values for the relationship between the Static Buckling Loads and the Imperfection parameters for some fixed values of r. Here, we take $A=3.5, \xi=0.3, H=0.06, K(\xi)=7, b=1, m=n=1, r=3,5,7,9$. The results are shown in Table 1, Table 2, Figure 1 and Figure 2.

The following are easily derived from the Tables 1 and 2 as well as from the graphical plots:

a) The static buckling load decreases with increased imperfection
b) For the same imperfection, axially stressed toroidal shell (for $\alpha=1$) buckles at lower values of static buckling load compared to hydrostatically stressed loading, where $\alpha=0$
c) The higher the ratio of the radii of the toroidal shell, the greater the static buckling load.

Table 1: Relationship between the Static Buckling Load λ_S and the Imperfection Parameter ϵ for some fixed values of r and for $\alpha=1$.

IMPERFECTION PARAMETER ϵ	STATIC BUCKLING LOAD λ_S FOR r = 3	STATIC BUCKLING LOAD λ_S FOR r = 5	STATIC BUCKLING LOAD λ_S FOR r = 7	STATIC BUCKLING LOAD λ_S FOR r = 9
0.01	1.38699	8.10718	14.83145	21.56799
0.02	1.36799	8.05974	14.75688	21.46943
0.03	1.35478	8.02656	14.70471	21.40049
0.04	1.34432	8.00019	14.66325	21.34572
0.05	1.33553	7.97798	14.62832	21.29952
0.06	1.32788	7.95859	14.59782	21.25919
0.07	1.32106	7.94119	14.57058	21.22319
0.08	1.31489	7.92555	14.54586	21.19049
0.09	1.30923	7.91109	14.52314	21.16047
0.1	1.30399	7.89771	14.50207	21.13261

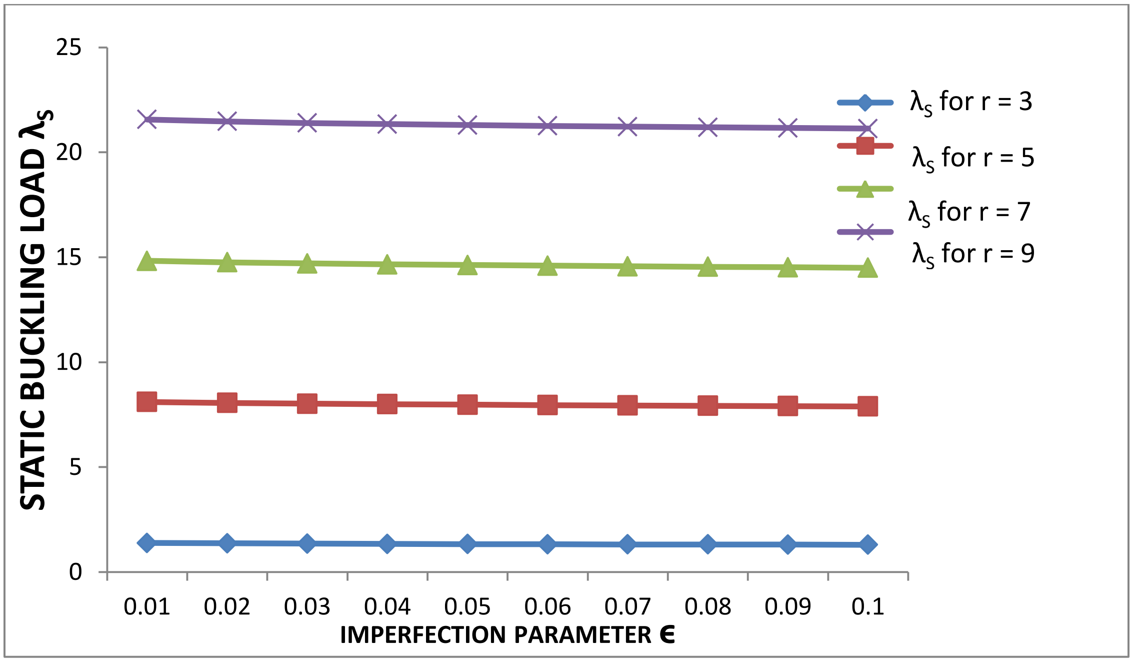

Figure 1: Graphical Plot showing the Relationship Between the Static Buckling Load λ_S and the Imperfection Parameter ϵ for some fixed values of r and for $\alpha = 1$.

Table 2: Relationship between the Static Buckling Load λ_S and the Imperfection Parameter ϵ for some fixed values of r and for $\alpha = 0$

IMPERFECTION PARAMETER ϵ	STATIC BUCKLING LOAD λ_S FOR r = 3	STATIC BUCKLING LOAD λ_S FOR r = 5	STATIC BUCKLING LOAD λ_S FOR r = 7	STATIC BUCKLING LOAD λ_S FOR r = 9
0.01	2.95466	17.92891	32.76923	47.56162
0.02	2.91342	17.85072	32.62473	47.34378
0.03	2.88475	17.79499	32.52268	47.19149
0.04	2.86206	17.75014	32.44109	47.07057
0.05	2.84301	17.71195	32.37195	46.96866
0.06	2.82642	17.67833	32.31136	46.87973
0.07	2.81164	17.64809	32.25706	46.80033
0.08	2.79828	17.62048	32.20763	46.72828
0.09	2.78602	17.59497	32.16209	46.66209
0.1	2.77467	17.57118	32.11976	46.60069

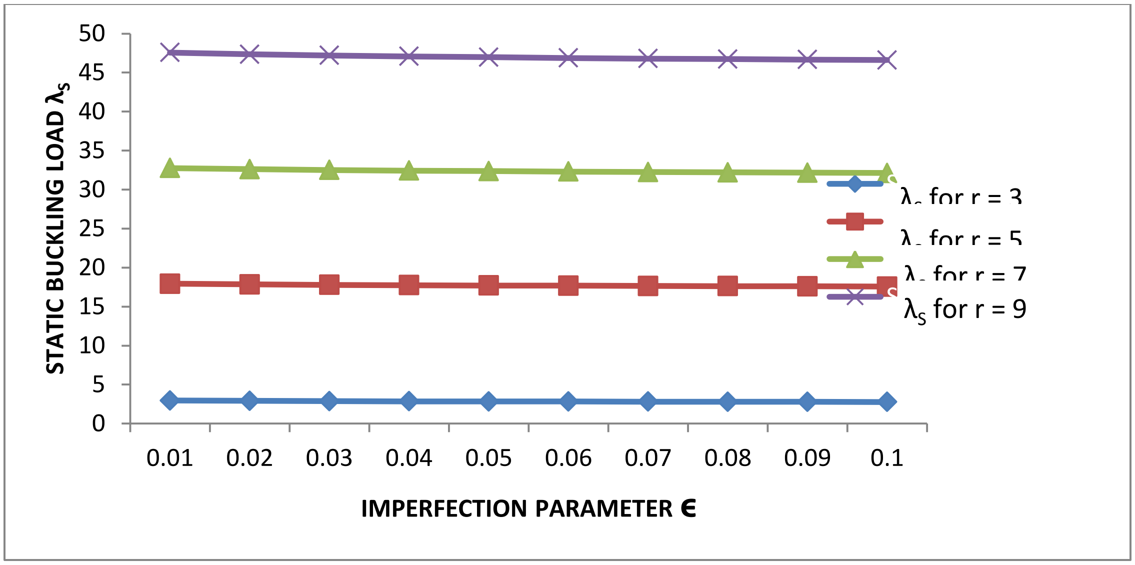

Figure 2: Graphical Plot Showing the Relationship Between the Static Buckling Load λ_S and the Imperfection Parameter ϵ for some fixed values of r and for $\alpha = 0$.

VII. CONCLUSION

We have performed a perturbation procedure to determine the static buckling load of an imperfect toroidal shell that is statically stressed and the results are obtained for both hydrostatically and axially stressed shell. It is clearly observed, among other things, that the axially stressed structure buckles at relatively lower static loads compared to the hydrostatically stressed structure. Besides, the static buckling load increases as the ratio of the radii of the structure increases.

References

[1] Stein, M. and J. A. McElman, J. A. (1965), Buckling of segments of toroidal shells, AIAA Jnl. 3, 1704.
[2] Hutchinson, J. W. (1967), Initial post buckling behavior of toroidal shell segment, Int. J. Solids Struct. 3, 97 -115.
[3] Oyesanya, M. O. (2002), Asymptotic analysis of imperfection sensitivity of toroidal shell segment modal imperfections, J. Nigerian Ass. Maths. Physics 6, 197 – 206.
[4] Oyesanya, M. O. (2005), Influence of extra terms on asymptotic analysis of imperfection sensitivity of toroidal shell segment with random imperfection, Mechanics research Communications 32, 444 – 453.
[5] Ette, A. M., Chukwuchekwa, J. U., Udo – Akpan, I. U. and Ozoigbo, G. E. (2019), On the normal response and buckling of a toroidal shell segment pressurized by a static compressive load, IJMTT, 65(10), 15 – 26.
[6] Amazigo, J. C. (1971), Buckling of stochastically imperfect columns on nonlinear elastic foundation , Quart. Appl. Math. 29, 403 – 409.
[7] Amazigo, J. C. (1974), Asymptotic analysis of the buckling of externally pressurized cylinders with random imperfection, Quart. Appl. Math., 32, 429 – 442.
[8] Ozoigbo, G. E. and Ette, A. M. (2019), On the maximum displacement and static buckling of a circular cylindrical shell, J. of Appl. Math. and Physics, 7, 2868 – 2882.
[9] Bassey, J. E, Ette, A. M. , Chukwuchekwa, J. U._ and C. Atulegwu, Osuji (2019), Dynamic Buckling Of A Clamped Finite Column Resting On A Non – Linear Elastic Foundation, Asian Research Journal Of Mathematics 14(4): 1-47; Article No.Arjom.50223.
[10] `Udo – Akpan, I. U. and Ette, A. M. (2016), On the dynamic buckling of a model structure with quadratic nonlinearities struck by a step load superposed on a quasi – static load, J. of Nigerian Assoc. of Math. Physics, 35, 461 – 472.
[11] Amazigo, J. C. and Ette, A. M. (1987), On a two – small parameter differential equation with application to dynamic buckling, J. of Nigerian Mathematical Soc. 6, 90 – 102.
[12] Kolakowski, Z. and mania, R. (2013), Semi – analytical method versus the FEM for analysing the local post – buckling of thin – walled composite structures , Composite Structures, 97, 99 – 106.
[13] Magnucki, K., Paczos, P. and Kasprzak, T. (2010), Elastic buckling of cold – formed Thin – walled channel beams with drop flanges, J. of Structural Engng., 136 (7), 886 – 896.
[14] Ganaparthi, M. , Gupta, S. S. and patet, B. P (2005), Nonlinear axisymmetric dynamic buckling of laminated angle – ply composite spherical caps, Composite Structures, 59, 89 – 97.
[15] Fan, H. G., Cheng, Z. P., Cheng, J., Huang, S., feng, W. Z. H. and Lin, I. (2016), Analytical research on dynamic buckling of thin cylindricals with thickness variation under axial pressure, Thin – walled Structures, 101, 213 – 221.
[16] Evkin, H. G. and Lykhachova, O. V. (2017), Energy barrier as a criterion for stability estimation of spherical shell under uniform external pressure. Int. J. of Solids and Structures, 14 (23), 118 – 119.
[17] Lockhart, D. and Amazigo, J. C. (1975), Dynamic buckling of externally pressurized imperfect cylindrical shells, J. Appl. Mech. 42 (2), 316 – 320.