HORADAM POLYNOMIAL COEFFICIENT ESTIMATES FOR TWO FAMILIES OF HOLOMORPHIC AND BI-UNIVALENT FUNCTIONS

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ABSTRACT. We aim at introducing two new families of holomorphic and bi-univalent functions in the open unit disc $\mathfrak{D}$ by making use of Horadam polynomials, which are known to generalize some potentially useful polynomials such as the Lucas polynomials, the Pell polynomials and the Chebyshev polynomials of the second kind. For functions belonging to the defined families, the coefficient inequalities and the Fekete-Szegö problem are discussed. Some interesting consequences of the result found here are also presented.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ be the family of normalized functions that have the form

$$g(z) = z + \sum_{j=2}^{\infty} d_j z^j \tag{1}$$

which are holomorphic in $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ be the collection of all members of $\mathcal{A}$ that are univalent in $\mathfrak{D}$. It is well- known that every function $g \in \mathcal{S}$ contains a disc of radius 1/4 (see[8]). According to this, every univalent function g has an inverse g^{-1} satisfying $g^{-1}(g(z)) = z,\ z \in \mathfrak{D}$ and $g(g^{-1}(\omega)) = \omega,\ |\omega| < r_0(g),\ r_0(g) \geq 1/4$, where

$$g^{-1} = s(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2d_3 + d_4)\omega^4 + ... \tag{2}$$

A member g of $\mathcal{A}$ is said to be bi-univalent in $\mathfrak{D}$ if both g and g^{-1} are univalent in $\mathfrak{D}$. We denote the family of bi-univalent functions that have the form (1), by $\sum$. For detailed study and various subfamilies of the family $\sum$, one can refer the works of [4],[5], [6], [7], [14] and [18].

We recall the principle of subordination between two holomorphic functions $g(z)$ and $s(z)$ in $\mathfrak{D}$. We say that $g(z)$ is subordinate to $s(z)$, written as $g(z) \prec s(z),\ z \in \mathfrak{D}$, if there is a $\psi(z)$ holomorphic in $\mathfrak{D}$, with $\psi(0) = 0$ and $|\psi(z)| < 1,\ z \in \mathfrak{D}$, such that $g(z) = s(\psi(z))$. Moreover $g(z) \prec s(z)$ is equivalent to $g(0) = s(0)$ and $g(\mathfrak{D}) \subset s(\mathfrak{D})$, if s is univalent in $\mathfrak{D}$.

Recently, Hörç¸um and Koçer [13] (See also [12]) considered the Horadam polynomials $h_k(x, a, b; p, q)$ (or briefly $h_k(x)$), which are given by the recurrence relation

$$h_k(x) = pxh_{k-1}(x) + qh_{k-2}(x), \quad (k \in \mathcal{N}/\{1, 2\}) \tag{3}$$

with $h_1(x) = a$ and $h_2(x) = bx$, where a, b, p and q are some real constants. It is very clear from (3) that $h_3(x) = pbx^2 + qa$. The generating function of the Horadam polynomials $h_k(x)$ is given

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by (see [13]).

$$\mathcal{G}(x,z) := \sum_{k=1}^{\infty} h_k(x) z^{k-1} = \frac{a+(b-ap)xz}{1-pxz-qz^2}. \tag{4}$$

Remark 1.1. *Here in what follows, the argument $x \in \mathbb{R}$, the set of real number, is independent of the argument $z \in \mathbb{C}$, the set of complex numbers, that is $x \neq R(z)$.*

Few particular cases of Horadam polynomials $h_k(x,a,b;p,q)$ are:
$i)$ $h_k(x,1,1;1,1) = F_k(x)$ the Fibonacci polynomials, $ii)$ $h_k(x,2,1;1,1) = L_k(x)$, the Lucas polynomials, $iii)$ $h_k(x,1,2;2,1) = P_k(x)$, the Pell polynomials, $iv)$ $h_k(x,2,2;2,1) = Q_k(x)$, the Pell-Lucas polynomials, $v)$ $h_k(x,1,1;2,-1) = T_k(x)$, the Chebyshev polynomials of first kind and $vi)$ $h_k(x,1,2;2,-1) = U_k(x)$, the Chebyshev polynomials of the second kind.

In literature, the coefficient estimates and celebrated Fekete- Szegö inequality are found for bi-univalent functions associated with certain polynomials like the Chebyshev polynomials, the Lucas polynomials and the Horadam polynomials. We also note that the above polynomials and other special polynomials are potentially important in the mathematical, physical, statistical and engineering sciences. More details associated with these polynomials can be found in [10], [11], [12], [15], [19] and [21]. Additional informations about Fekete-Szegö problem associated with Haradam polynomials are available with the works of [1] and [20]. Very interesting resources about Fekete-Szegö inequality associated with the q- derivative operator may be found in [3] and [16].

Inspired by recent trends on bi-univalent functions and motivated by the paper [17], we define the following special families of $\sum$ by making use of the Horadam polynomials, which are given by the recurrence relation (3) and the generating function (4).

Definition 1.1. *A function $g(z)$ in $\sum$ of the form (1) is said to be in the family $\mathfrak{S}_{\sum}(x,\gamma,\mu)$, $0 \le \gamma \le 1$, $\mu \ge 0$, $\mu \ge \gamma$, if*

$$\frac{zg'(z)+\mu z^2 g''(z)}{(1-\gamma)z+\gamma z g'(z)} \prec \mathcal{G}(x,z)+1-a \quad \text{and} \quad \frac{\omega s'(\omega)+\mu\omega^2 s''(\omega)}{(1-\gamma)\omega+\gamma\omega s'(\omega)} \prec \mathcal{G}(x,\omega)+1-a,$$

where z, $\omega \in \mathfrak{D}$, $s(\omega)$ is as stated in (2), a, b, p and q are as in (3).

Definition 1.2. *A function $g(z)$ in $\sum$ of the form (1) is said to be in the family $\mathfrak{B}_{\sum}(x,\xi,\tau)$, $\xi \ge 1$, $\tau \ge 1$, if*

$$\frac{(1-\xi)+\xi[(zg'(z))']^{\tau}}{g'(z)} \prec \mathcal{G}(x,z)+1-a \quad \text{and} \quad \frac{(1-\xi)+\xi[(\omega s'(\omega))']^{\tau}}{s'(\omega)} \prec \mathcal{G}(x,\omega)+1-a,$$

where z, $\omega \in \mathfrak{D}$, $s(\omega)$ is as stated in (2), a, b, p and q are as in (3).

For functions of the form (1) belonging to these newly defined families $\mathfrak{S}_{\sum}(x,\gamma,\mu)$ and $\mathfrak{B}_{\sum}(x,\xi,\tau)$, we derive the estimates for the coefficients $|d_2|$ and $|d_3|$ in Section 2 and also, we consider the celebrated Fekete- Szegö problem [9].

2. Coefficient estimates and Fekete-Szegö inequality

Theorem 2.1. *Let $0 \le \gamma \le 1$, $\mu \ge 0$, $\mu \ge \gamma$ and $g(z) \in \mathcal{A}$ be in the family $\mathfrak{S}_{\sum}(x,\gamma,\mu)$. Then*

$$|d_2| \le \frac{|b(x)|\sqrt{|b(x)|}}{\sqrt{|(4\gamma^2-(7+4\mu)\gamma+3(1+2\mu))(bx)^2-4(1-\gamma+\mu)^2(pbx^2+qa)|}}, \tag{5}$$

$$|d_3| \le \frac{b^2x^2}{4(1-\gamma+\mu)^2} + \frac{|b(x)|}{3(1-\gamma+2\mu)} \tag{6}$$

and for $\delta \in \mathbb{R}$

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|b(x)|}{3(1-\gamma+2\mu)} & ; if\ |1-\delta| \leq\ J \\ \frac{|b(x)|^3\,|1-\delta|}{|(4\gamma^2-(7+4\mu)\gamma+3(1+2\mu))(bx)^2-4(1-\gamma+\mu)^2(pbx^2+qa)|} & ; if\ |1-\delta| \geq\ J, \end{cases} \quad (7)$$

where

$$J = \frac{1}{3(1-\gamma+2\mu)}\left|(4\gamma^2-(7+4\mu)\gamma+3(1+2\mu)) - 4(1-\gamma+\mu)^2\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|. \quad (8)$$

Proof. Let $g(z) \in \mathfrak{S}_{\sum}(x,\gamma,\mu)$. Then, for two holomorphic functions $\mathfrak{m}$ and $\mathfrak{n}$ such that $\mathfrak{m}(0) = \mathfrak{n}(0) = 0$, $|\mathfrak{m}(z)| < 1$ and $|\mathfrak{n}(\omega)| < 1,\ z,\omega \in \mathfrak{D}$, and using Definition 1.1, we can write $\frac{zg'(z)+\mu z^2g''(z)}{(1-\gamma)z+\gamma zg'(z)} = \mathcal{G}(x,\mathfrak{m}(z)) + 1 - a$ and $\frac{\omega s'(\omega)+\mu\omega^2s''(\omega)}{(1-\gamma)\omega+\gamma\omega s'(\omega)} = \mathcal{G}(x,\mathfrak{n}(\omega)) + 1 - a.$

Or, equivalently

$$\frac{zg'(z)+\mu z^2g''(z)}{(1-\gamma)z+\gamma zg'(z)} = 1 + h_1(x) - a + h_2(x)\mathfrak{m}(z) + h_3(x)(\mathfrak{m}(z))^2 + ... \quad (9)$$

and

$$\frac{\omega s'(\omega)+\mu\omega^2s''(\omega)}{(1-\gamma)\omega+\gamma\omega s'(\omega)} = 1 + h_1(x) - a + h_2(x)\mathfrak{n}(\omega) + h_3(x)(\mathfrak{n}(\omega))^2 + ... \quad (10)$$

From (9) and (10), in view of (3), we obtain

$$\frac{zg'(z)+\mu z^2g''(z)}{(1-\gamma)z+\gamma zg'(z)} = 1 + h_2(x)\mathfrak{m}_1z + [h_2(x)\mathfrak{m}_2 + h_3(x)\mathfrak{m}_1^2]z^2 + ... \quad (11)$$

and

$$\frac{\omega s'(\omega)+\mu\omega^2s''(\omega)}{(1-\gamma)\omega+\gamma\omega s'(\omega)} = 1 + h_2(x)\mathfrak{n}_1\omega + [h_2(x)\mathfrak{n}_2 + h_3(x){\mathfrak{n}_1}^2]\omega^2 + ... \quad (12)$$

It is well known that if $|\mathfrak{m}(z)| = |\mathfrak{m}_1z + \mathfrak{m}_2z^2 + \mathfrak{m}_3z^3 + ...| < 1,\quad z \in \mathfrak{D}$ and $|\mathfrak{n}(\omega)| = |\mathfrak{n}_1\omega + \mathfrak{n}_2\omega^2 + \mathfrak{n}_3\omega^3 + ...| < 1,\quad \omega \in \mathfrak{D}$, then

$$|\mathfrak{m}_i| \leq 1 \text{ and } |\mathfrak{n}_i| \leq 1\ (i \in \mathcal{N}). \quad (13)$$

Comparing the corresponding coefficients in (11) and (12), we have

$$2(1-\gamma+\mu)d_2 = h_2(x)\mathfrak{m}_1 \quad (14)$$

$$3(1-\gamma+2\mu)d_3 - 4(1-\gamma+\mu)\gamma d_2^2 = h_2(x)\mathfrak{m}_2 + h_3(x)\mathfrak{m}_1^2 \quad (15)$$

$$-2(1-\gamma+\mu)d_2 = h_2(x)\mathfrak{n}_1 \quad (16)$$

$$3(1-\gamma+2\mu)(2d_2^2 - d_3) - 4(1-\gamma+\mu)\gamma d_2^2 = h_2(x)\mathfrak{n}_2 + h_3(x)\mathfrak{n}_1^2. \quad (17)$$

From (14) and (16), we can easily see that

$$\mathfrak{m}_1 = -\mathfrak{n}_1 \quad (18)$$

and also

$$8(1-\gamma+\mu)^2d_2^2 = (\mathfrak{m}_1^2 + \mathfrak{n}_1^2)(h_2(x))^2. \quad (19)$$

If we use (18) in the addition of (15) and (17), then we obtain

$$(4\gamma^2-(7+4\mu)\gamma+3(1+2\mu))2d_2^2 = h_2(x)(\mathfrak{m}_2+\mathfrak{n}_2) + h_3(x)(\mathfrak{m}_1^2+\mathfrak{n}_1^2). \quad (20)$$

Substituting the value of $\mathfrak{m}_1^2 + \mathfrak{n}_1^2$ from (19) in (20), we get

$$2d_2^2 = \frac{(h_2(x))^3(\mathfrak{m}_2+\mathfrak{n}_2)}{[(4\gamma^2-(7+4\mu)\gamma+3(1+2\mu))(h_2(x))^2 - 4(1-\gamma+\mu)^2h_3(x)]}, \quad (21)$$

which yields (5).

Using (18) in the subtraction of (17) from (15), we obtain

$$d_3 = d_2^2 + \frac{h_2(x)(\mathfrak{m}_2 - \mathfrak{n}_2)}{6(1-\gamma+2\mu)}. \tag{22}$$

Then in view of (19), (22) becomes $d_3 = \frac{(h_2(x))^2(\mathfrak{m}_1^2+\mathfrak{n}_1^2)}{8(1-\gamma+\mu)^2} + \frac{h_2(x)(\mathfrak{m}_2-\mathfrak{n}_2)}{6(1-\gamma+2\mu)}$, which yields (6), on using (13).

From (21) and (22), for $\delta \in \mathbb{R}$, we get
$|d_3 - \delta d_2^2| = |h_2(x)| \left| \left(T(\delta,x) + \frac{1}{6(1-\gamma+2\mu)} \right) \mathfrak{m}_2 + \left(T(\delta,x) - \frac{1}{6(1-\gamma+2\mu)} \right) \mathfrak{n}_2 \right|$,
where

$$T(\delta,x) = \frac{(1-\delta)(h_2(x))^2}{2\left[(4\gamma^2 - (7+4\mu)\gamma + 3(1+2\mu))(h_2(x))^2 - 4(1-\gamma+\mu)^2 h_3(x)\right]}.$$

In view of (3), we conclude that

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|h_2(x)|}{3(1-\gamma+2\mu)} & ; if \ 0 \leq |T(\delta,x)| \leq \frac{1}{6(1-\gamma+2\mu)} \\ 2|h_2(x)||T(\delta,x)| & ; if \ |T(\delta,x)| \geq \frac{1}{6(1-\gamma+2\mu)}, \end{cases}$$

which yields (7) with J as in (8). This evidently completes the proof of Theorem 2.1. □

Remark 2.1. *The results obtained in Theorem 2.1 coincide with results found in [[2], Theorem 2.2] for $\mu = 0$ and $\gamma = 0$.*

Corollary 2.3 asserts immediate consequence of Theorem 2.1 for the family $\mathfrak{R}_{\sum}(x,\mu)$ when $\gamma = 0$.

Corollary 2.1. *If $g(z) \in \mathfrak{R}_{\sum}(x,\mu)$, $\mu \geq 0$, a subfamily of $\sum$ satisfying*

$$(g'(z) + \mu z g''(z) - 1) \prec \mathcal{G}(x,z) + 1 - a \quad \text{and} \quad (s'(\omega) + \mu\omega s''(\omega) - 1) \prec \mathcal{G}(x,\omega) + 1 - a,$$

where $z,\ \omega \in \mathfrak{D}$, $s = g^{-1}$ is as in (2) and $a,\ b,\ p$ and q are as in (3), then

$$|d_2| \leq \frac{|b(x)|\sqrt{|b(x)|}}{\sqrt{|3(1+2\mu)(bx)^2 - 4(1+\mu)^2(pbx^2+qa)|}}, \qquad |d_3| \leq \frac{b^2x^2}{4(1+\mu)^2} + \frac{|b(x)|}{3(1+2\mu)}$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|b(x)|}{3(1+2\mu)} & ; if \ \ |1-\delta| \leq J_1 \\ \frac{|b(x)|^3\,|1-\delta|}{|3(1+2\mu)(bx)^2 - 4(1+\mu)^2(pbx^2+qa)|} & ; if \ |1-\delta| \geq J_1. \end{cases}$$

where $J_1 = \frac{1}{3(1+2\mu)} \left| 3(1+2\mu) - 4(1+\mu)^2 \left(\frac{pbx^2+qa}{b^2x^2} \right) \right|$.

Corollary 2.4 asserts an another interesting consequence of Theorem 2.1 for the family $\mathfrak{L}_{\sum}(x,\mu)$ by putting $\gamma = 1$.

Corollary 2.2. *If $g(z) \in \mathfrak{L}_{\sum}(x,\mu)$, $\mu \geq 1$, a subfamily of $\sum$ satisfying*

$$1 + \mu\left(\frac{zg''(z)}{g'(z)}\right) \prec \mathcal{G}(x,z) + 1 - a \quad \text{and} \quad 1 + \mu\left(\frac{\omega s''(\omega)}{s'(\omega)}\right) \prec \mathcal{G}(x,\omega) + 1 - a,$$

where $z,\ \omega \in \mathfrak{D}$, $s = g^{-1}$ is as in (2) and $a,\ b,\ p$ and q are as in (3), then

$$|d_2| \leq \frac{|b(x)|\sqrt{|b(x)|}}{\sqrt{2|\mu(bx)^2 - 2\mu^2(pbx^2+qa)|}}, \quad |d_3| \leq \frac{b^2x^2}{4\mu^2} + \frac{|b(x)|}{6\mu}$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \le \begin{cases} \frac{|b(x)|}{6\mu} & ; if\ |1-\delta| \le \frac{1}{3}\left|1 - 2\mu\left(\frac{pbx^2+qa}{b^2x^2}\right)\right| \\ \frac{|b(x)|^3\,|1-\delta|}{2|\mu(bx)^2 - 2\mu^2(pbx^2+qa)|} & ; if\ |1-\delta| \ge \frac{1}{3}\left|1 - 2\mu\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|. \end{cases}$$

Theorem 2.2. *Let* $\xi \ge 1$, $\tau \ge 1$ *and* $g(z) \in \mathcal{A}$ *be in the family* $\mathfrak{B}_{\sum}(x, \xi, \tau)$. *Then*

$$|d_2| \le \frac{|bx|\sqrt{|bx|}}{\sqrt{|(8\xi\tau^2 - 7\xi\tau + 1)(bx)^2 - 4(2\xi\tau - 1)^2(pbx^2 + qa)|}}, \tag{23}$$

$$|d_3| \le \frac{(bx)^2}{4(2\xi\tau - 1)^2} + \frac{|bx|}{3(3\xi\tau - 1)} \tag{24}$$

and for $\delta \in \mathbb{R}$

$$|d_3 - \delta d_2^2| \le \begin{cases} \frac{|b(x)|}{3(3\xi\tau-1)} & ; if\quad |1-\delta| \le M \\ \frac{|1-\delta||bx|^3}{|(8\xi\tau^2 - 7\xi\tau+1)(bx)^2 - 4(2\xi\tau-1)^2(pbx^2+qa)|} & ; if\quad |1-\delta| \ge M. \end{cases} \tag{25}$$

where $M = \frac{1}{3(3\xi\tau-1)}\left|(8\xi\tau^2 - 7\xi\tau + 1) - 4(2\xi\tau - 1)^2\left(\frac{pbx^2+qa}{b^2x^2}\right)\right|$

Proof. Let $g(z) \in \mathfrak{B}_{\sum}(x, \xi, \tau)$. Then, for some analytic functions $\mathfrak{m}$ and $\mathfrak{n}$ such that $\mathfrak{m}(0) = \mathfrak{n}(0) = 0$, $|\mathfrak{m}(z)| < 1$ and $|\mathfrak{n}(\omega)| < 1$, $z, \omega \in \mathfrak{D}$, and using Definition 1.2, we can write

$$\frac{(1-\xi) + \xi[(zg'(z)')]^\tau}{g'(z)} = \mathcal{G}(x, z) + 1 - a,\ z \in \mathfrak{D} \tag{26}$$

and

$$\frac{(1-\xi) + \xi[(\omega s'(\omega))']^\tau}{s'(\omega)} = \mathcal{G}(x, \omega) + 1 - a,,\ \omega\ \in \mathfrak{D}. \tag{27}$$

Following (9), (10), (11), and (12) in the proof of Theorem 2.1, one gets in view of (26) and (27)

$$(2\xi\tau - 1)2d_2 = h_2(x)\mathfrak{m}_1 \tag{28}$$

$$4(2\xi\tau^2 - 4\xi\tau + 1)d_2^2 + 3(3\xi\tau - 1)d_3 = h_2(x)\mathfrak{m}_2 + h_3(x)\mathfrak{m}_1^2 \tag{29}$$

$$-(2\xi\tau - 1)2d_2 = h_2(x)\mathfrak{n}_1 \tag{30}$$

$$2(4\xi\tau^2 + \xi\tau - 1)d_2^2 - 3(3\xi\tau - 1)d_3 = h_2(x)\mathfrak{n}_2 + h_3(x)\mathfrak{n}_1^2. \tag{31}$$

From (28) and (30), we can easily see that

$$\mathfrak{m}_1 = -\mathfrak{n}_1 \tag{32}$$

and also

$$8(2\xi\tau - 1)^2 d_2^2 = h_2^2(x)(\mathfrak{m}_1^2 + \mathfrak{n}_1^2). \tag{33}$$

If we use (32) in the addition of (29) and (31), then we obtain

$$(8\xi\tau^2 - 7\xi\tau + 1)2d_2^2 = h_2(x)(\mathfrak{m}_2 + \mathfrak{n}_2) + h_3(x)(\mathfrak{m}_1^2 + \mathfrak{n}_1^2). \tag{34}$$

By substituting (33) in (34), we obtain

$$2d_2^2 = \frac{(h_2(x))^3(\mathfrak{m}_2 + \mathfrak{n}_2)}{[(8\xi\tau^2 - 7\xi\tau + 1)(h_2(x))^2 - 4(2\xi\tau - 1)^2(h_3(x))]}, \tag{35}$$

which yields (23). By subtracting (31) from (29) and in light of (32), we deduce that

$$d_3 = d_2^2 + \frac{h_2(x)(\mathfrak{m}_2 - \mathfrak{n}_2)}{6(3\xi\tau - 1)}. \tag{36}$$

Then in view of (33), (36) becomes $d_3 = \frac{(h_2(x))^2(\mathfrak{m}_1^2+\mathfrak{n}_1^2)}{8(2\xi\tau-1)^2} + \frac{h_2(x)(\mathfrak{m}_2-\mathfrak{n}_2)}{6(3\xi\tau-1)}$, which yields (24), on using (13).

From (35) and (36), for $\delta \in \mathbb{R}$, we write

$$|d_3 - \delta d_2^2| = |h_2(x)| \left| \left(L(\delta, x) + \frac{1}{6(3\xi\tau - 1)} \right) \mathfrak{m}_2 + \left(L(\delta, x) - \frac{1}{6(3\xi\tau - 1)} \right) \mathfrak{n}_2 \right|,$$

where

$$L(\delta, x) = \frac{(1-\delta)\,(h_2(x))^2}{[(8\xi\tau^2 - 7\xi\tau + 1)(h_2(x))^2 - 4(2\xi\tau - 1)^2(h_3(x))]}.$$

In view of (3), we conclude that

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|h_2(x)|}{3(3\xi\tau - 1)} & ; if \;\; 0 \leq |L(\delta, x)| \leq \frac{1}{6(3\xi\tau - 1)} \\ 2|h_2(x)||L(\delta, x)| & ; if |L(\delta, x)| \geq \frac{1}{6(3\xi\tau - 1)}, \end{cases}$$

which yields (25). This evidently completes the proof of Theorem 2.2. □

We conclude the below result for the family $\mathscr{M}_{\sum}(x, \xi)$ by putting $\tau = 1$ in Theorem 2.2.

Corollary 2.3. *If $g(z) \in \mathscr{M}_{\sum}(x, \xi)$, a subfamily of $\sum$ satisfying*

$$(1-\xi)\frac{1}{g'(z)} + \xi \left(1 + \frac{zg''(z)}{g'(z)} \right) \prec \mathcal{G}(x, z) + 1 - a, \; z \in \mathfrak{D}$$

and

$$(1-\xi)\frac{1}{s'(\omega)} + \xi \left(1 + \frac{\omega s''(\omega)}{s'(\omega)} \right) \prec \mathcal{G}(x, \omega) + 1 - a, \; \omega \in \mathfrak{D},$$

where $s = g^{-1}$ is as in (2) and a, b, p and q are as in (3), then

$$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|(\xi+1)(bx)^2 - 4(2\xi - 1)^2(pbx^2 + qa)|}}, \quad |d_3| \leq \frac{(bx)^2}{4(2\xi - 1)^2} + \frac{|bx|}{3(3\xi - 1)}$$

and for $\delta \in \mathbb{R}$

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|b(x)|}{3(3\xi - 1)} & ; if \;\; |1-\delta| \leq M_1 \\ \frac{|1-\delta||bx|^3}{|(\xi+1)(bx)^2 - 4(2\xi-1)^2(pbx^2+qa)|} & ; if \; |1-\delta| \geq M_1. \end{cases}$$

where $M_1 = \frac{1}{3(3\xi - 1)} \left| (\xi + 1) - 4(2\xi - 1)^2 \left(\frac{pbx^2 + qa}{b^2x^2} \right) \right|$.

Theorem 2.2 would yield the following corollary for the family $\mathscr{N}_{\sum}(x, \tau)$, when $\xi = 1$.

Corollary 2.4. *If $g(z) \in \mathscr{N}_{\sum}(x, \tau)$, $\tau \geq 1$, a subfamily of $\sum$ satisfying*
$\frac{[(zg'(z))']^{\tau}}{g'(z)} \prec \mathcal{G}(x, z) + 1 - a, \; z \in \mathfrak{D}$ *and* $\frac{[(\omega s'(\omega))']^{\tau}}{s'(\omega)} \prec \mathcal{G}(x, \omega) + 1 - a, , \; \omega \in \mathfrak{D}$,
where $s(\omega)$ is as stated in (2), a, b, p and q are as in (3), then
$|d_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|(8\tau^2 - 7\tau + 1)(bx)^2 - 4(2\tau - 1)^2(pbx^2 + qa)|}}, \quad |d_3| \leq \frac{(bx)^2}{4(2\tau - 1)^2} + \frac{|bx|}{3(3\tau - 1)}$
and for $\delta \in \mathbb{R}$

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|b(x)|}{3(3\tau - 1)} & ; if \;\; |1-\delta| \leq M_2 \\ \frac{|1-\delta||bx|^3}{|(8\tau^2 - 7\tau + 1)(bx)^2 - 4(2\tau-1)^2(pbx^2+qa)|} & ; if \; |1-\delta| \geq M_2. \end{cases}$$

where $M_2 = \frac{1}{3(3\xi - 1)} \left| (\xi + 1) - 4(2\xi - 1)^2 \left(\frac{pbx^2 + qa}{b^2x^2} \right) \right|$.

Remark 2.2. *Taking particular cases of Horadam polynomials as indicated in the introduction, we will have new results similar to Theorem 2.1 and Theorem 2.2, for subfamilies of bi-univalent functions associated with such polynomials.*

3. Conclusion

Using the concept of subordination, we have introduced two families of holomorphic and bi-univalent functions in the open unit disc $\mathfrak{D}$ associated with Horadam polynomials. We have then derived the initial coefficient estimations and also Fekete-Szegö inequalities for functions belonging to these two families. Our main results are obtained in Theorem 2.1 and Theorem 2.2. Further by specializing the parameters, several consequences of these new families are mentioned.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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