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SEMI Δ-OPEN SETS IN TOPOLOGICAL SPACES

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ABSTRACT

In this paper, we introduce the class of semi Δ-open sets in Topology. It is obtained by generalizing Δ-open sets in the same way that semi-open sets were generalized open sets. We study some properties of semi Δ-open sets. We also define the semi Δ-interior and the semi Δ-closure of a set A in a space (X, τ).

Keywords: *Δ-open sets, Δ-closed sets, semi Δ-open sets, semi Δ-closed sets, semi Δ-interior and semi Δ-closure of a set in X.*

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INTRODUCTION

Let (X, τ) be a topological space and let $A \subseteq X$. Then the closure of A and the interior of A will be denoted by cl(A) and int(A), respectively. A subset $A \subseteq X$ is said to be semi-open [3] if there exists $O \in \tau$ such that $O \subseteq A \subseteq cl(A)$. It is evident that A is semi-open if and only if $A \subseteq cl(int(A))$. The complement of a semi-open set is said to be semi-closed [1]. Δ-open sets are defined and studied by Veera [5]. A subset A of a space X is said to be Δ-open if $A = (B - C) \cup (C - B)$, where B and C are open subsets of X. The complement of a Δ-open set is said to be Δ-closed [5]. The notions of Δ-interior and Δ-closure of a set A in X [5] are defined analouge to interior and closure of a set A in X.

<u>Definition 1.</u> A subset S of a space (X, τ) is said to be semi Δ-open if $S = (A - B) \cup (B - A)$, where A and B are semi-open sets in X.

It is evident that every Δ-open set as also every semi-open set is semi Δ-open. But the converse implications are not true in general. Following is an example:

Example 1. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}\}$ is a topology on X. Then {b} is semi Δ-open. But is neigher Δ-open nor semi-open.

Theorem 1. If S is a semi Δ-open subset of a space (X, τ), then there exists a Δ-open set O such that $O \subseteq S \subseteq cl(O)$.

Proof. S is semi Δ-open, there exist semi-open sets A and B such that $S = (A - B) \cup (B - A)$. Now, A and B are semi-open sets, it follows that there exist open sets U and V in X such that $U \subseteq A \subseteq cl(U)$ and $V \subseteq B \subseteq cl(V)$. This implies that

$$\begin{aligned}(U - V) \cup (V - U) &\subseteq (A - B) \cup (B - A) \\ &\subseteq [cl(U) - cl(V)] \cup [cl(V) - cl(U)] \\ &\subseteq cl(U - V) \cup cl(V - U) \\ &= cl[(U - V) \cup (V - U)].\end{aligned}$$

Hence the result.

Theorem 2. If O is open and S is semi Δ-open in a space (X, τ), then $O \cap S$ is semi Δ-open in X.

Proof. $O \cap S = O \cap [(A - B) \cup (B - A)]$, where A and B are semi-open sets in X.

$$= [O \cap (A - B)] \cup [O \cap (B - A)]$$

$$= [(O \cap A) - (O \cap B)] \cup [(O \cap B) - (O \cap A)]$$

which is semi Δ-open since $O \cap A$ and $O \cap B$ are semi open sets [2].

Theorem 3. A subset A of a space (X, τ) is semi Δ-open if and only if $A \subseteq cl(\Delta\, int(A))$.

Proof. Let A be a semi Δ-open subset of X. Then there exists a Δ-open set U such that $U \subseteq A \subseteq cl(U)$. But $U = \Delta\, int(U) \subseteq \Delta\, int(A)$ and so $cl(U) \subseteq cl(\Delta\, int(A))$. Thus $A \subseteq cl(U) \subseteq cl(\Delta\, int(A))$. Now, suppose $A \subseteq cl(\Delta\, int(A))$. Put $U = \Delta\, int(A)$. Then U is Δ-open with $U \subseteq A \subseteq cl(\Delta\, int(A))$. Hence $U \subseteq A \subseteq cl(U)$ and A is semi Δ-open.

Theorem 4. If $\{A_\alpha : \alpha \in I\}$ is a family of semi Δ-open subsets of (X, τ), then $\bigcup_{\alpha \in I} A_\alpha$ is semi Δ-open.

Proof. For each $\alpha \in I$, there exists a semi Δ-open set U_α such that $U_\alpha \subseteq A_\alpha \subseteq cl(U_\alpha)$. Now $\bigcup_{\alpha \in A} A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha \subseteq \bigcup_{\alpha \in I} cl(U_\alpha) \subseteq cl\left(\bigcup_{\alpha \in I} U_\alpha\right)$. Then $\bigcup_{\alpha \in I} A_\alpha$ is semi Δ-open.

Definition 2. Let (X, τ) be a topological space and let $A \subseteq X$. Then the union of all semi Δ-open sets contained in A, denoted by $s\Delta\,int(A)$, is called the semi Δ-interior of A. It is clear that $int(A) \subseteq sin\,t(A) \subseteq s\Delta\,int(A)$, for any subset A of X. Recall that sint(A) is the semi-interior of $A \subseteq X$ [2].

Theorem 5. Let A be a subset of a space (X, τ). Then $s\Delta\,int(A) = A \cap cl(\Delta\,int(A))$.

Proof. $A \cap cl(\Delta\,int(A)) \subseteq cl[\Delta\,int(A \cap int(A))]$

$$\subseteq cl[\Delta\,int(A \cap cl(\Delta\,int(A))].$$

Thus $A \cap cl(\Delta\,int(A))$ is semi Δ-open set contained in A. Hence $A \cap cl(\Delta\,int(A)) \subseteq s\Delta\,int(A)$. On the other hand, since $s\Delta\,int(A)$ is a semi Δ-open set, we have $s\Delta\,int(A) \subseteq cl(\Delta\,int(s\Delta\,int(A))) \subseteq cl(s\Delta\,int(A))$. Hence, $s\Delta\,int(A) = A \cap cl(\Delta\,int(A))$.

Theorem 6. Let (Y, τ_Y) be a subspace of a space (X, τ) and let $A \subseteq Y$. If A is semi Δ-open in X, then A is semi Δ-open in Y.

Proof. A is semi Δ-open in X, there exists a Δ-open set U in X such that $U \subseteq A \subseteq cl(U)$. Then $U = U \cap Y \subseteq A \subseteq cl(U) \cap Y = cl_Y(U)$. Thus A is semi Δ-open in Y.

Theorem 7. Let Y be a Δ-open set in a space (X, τ). If $A \subseteq Y$ and A is semi Δ-open in Y, then A is semi Δ-open in X.

Proof. Let A be semi Δ-open in Y. Then there exists a Δ-open subset U of (Y, τ_Y) such that $U \subseteq A \subseteq cl_Y(U)$. Since Y is Δ-open in X, therefore, U is Δ-open in X and $U \subseteq A \subseteq cl_Y(U) \subseteq cl(U)$. Hence A is semi Δ-open in X.

Definition 3. A subset A of a space (X, τ) is said to be semi Δ-closed iff X□A is semi Δ-open.

Remark 1. Since all open sets are semi Δ-open, it follows that all closed sets are semi Δ-closed.

Definition 4. Let A be a subset of a space (X, τ), then the semi Δ-closure of A, denoted by $s\Delta cl(A)$, is defined as the intersection of all semi Δ-closed subsets of X containing A.

Remark 2. $s\Delta cl(A) \subseteq scl(A)$ for any $A \subseteq X$.

Theorem 8. A subset F of a space (X, τ) is semi Δ-closed if and only if $int(\Delta cl(F)) \subseteq F$.

Proof. Obvious.

Theorem 9. If A is a subset of a space (X, τ), then $s\Delta cl(A) = A \cup int(\Delta cl(A))$.

Proof. $int[\Delta cl(A \cup int(\Delta clA))] \subseteq int[\Delta cl(A \cup cl(A))]$

$$= int[\Delta cl(A)] \subseteq A \cup int[\Delta cl(A)].$$

Thus by Theorem 8, $A \cup int(\Delta cl(A))$ is a semi Δ-closed set containing A and so $s\Delta cl(A) \subseteq A \cup int(\Delta cl(A))$.

On the other hand, since $s\Delta cl(A)$ is semi Δ-closed, therefore, $int[\Delta cl(s\Delta clA)] \subseteq s\Delta cl(A)$. Hence $int[\Delta cl(A)] \subseteq int[\Delta cl(s\Delta cl(A))] \subseteq s\Delta cl(A)$ and consequently $A \cup int(\Delta clA) \subseteq s\Delta cl(A)$. Thus $s\Delta cl(A) = A \cup int(\Delta clA)$.

Theorem 10. Let A is a subset of (X, τ). Then $s\Delta cl(A) \subseteq scl(A) \cap \Delta cl(A)$.

Proof. $s\Delta cl(A) = A \cup int(\Delta cl(A))$

$$\subseteq int(cl(A)) = scl(A) \quad \text{(Theorem 1.5 of [2])}.$$

Also, $s\Delta cl(A) \subseteq A \cup \Delta cl(A) = \Delta cl(A)$. Therefore, $s\Delta cl(A) \subseteq scl(A) \cap \Delta cl(A)$.

Remark 3. The equality in Theorem 10 does not hold. Following is an example:

Example 2. Let $X = \{a, b, c\}$ and let $\tau = \{X, \phi, \{a,c\}, \{c\}\}$ be a topology on X. Then $scl(\{a, c\}) = X = \Delta cl(\{a, c\})$ but $s\Delta cl(\{a, c\}) = \{a, c\}$.

Theorem 11. If F is closed and S is semi Δ-closed in a space (X, τ), then $F \cup S$ is semi Δ-closed.

Proof. (X □ F) is open and (X □ S) semi Δ-open. Then by Theorem 2, $(X - F) \cap (X - S)$ is semi Δ-open. That is $X - (F \cup S)$ is semi Δ-open. Hence $F \cup S$ is semi Δ-closed.

Theorem 12. Let (X, τ) be a topological space and $A \subseteq X$. Then:

(a) A is semi Δ-open iff $A = s\Delta\, int(A)$.

(b) A is semi Δ-closed iff $A = s\Delta cl(A)$.

(c) $s\Delta\, int(X - A) = X - s\Delta cl(A)$.

(d) $s\Delta cl(X - A) = X - s\Delta\, int(A)$.

Theorem 13. If A is a subset of a space (X, τ), then the following are equivalent:

(a) A is a dense subset of X.

(b) $s\Delta cl(A) = X$.

(c) If F is a semi Δ-closed subset of X and $U \subseteq F$, then $F = X$.

(d) For any non-empty Δ-open subset S of (X, τ), $S \cap A \neq \phi$.

(e) $s\Delta\, int(X - A) = \phi$.

Proof. Same proof as Theorem [4].

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