An analytical solution of a hollow cylindrical tube subjected to inflation

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Abstract:

In this paper, we have proposed as work, the study of a hollow tubular cylindrical structure which has as deformation an inflation in order to obtain an analytical solution of displacement and also to see solution components behaviors when the radius varies. A deformation kinematics governing this behavior has been given in order to be able to solve the problem analytically. From this resolution, we obtained from the kinematics principal components of the stress tensor, equilibrium equations and boundary conditions, the analytical solution of the displacement, a solution that has a complex component. The simulation of the real and imaginary parts of this component allowed us to see a different behavior between these two parts when the radius varies. This study showed us that the geometrical shapes that does not only modify the volume contributions, the porosity, the directions, the therml coating as well as the variation of the scalar products of the material vectors but also creates certain behaviors at the solution level when the radius varies.

Keywords: *Tubular cylindrical material, Inflation, Navier equation, Analytical solution.*

I. Introduction

On the human scale, which can be said to be macroscopic, the mechanical behavior of a material can be described from empirical parameters, directly measured during a test of the fabricated part, and without a detailed knowledge of its origins.

At the microscopic scale, the mechanical properties can be related to physical mechanisms whose scale varies from the dimension of the atom to dimensions typical of the microstructure of the material. The relationships that characterize the physical properties of materials are commonly referred to as constitutive equations. Due to the variety of materials and loads, the study and development of these constitutive equations are certainly one of the most interesting and developed fields of mechanics. Although the theory of continuum mechanics has established some principles for the development of behavioral laws many studies based on empirical relationships confirm and verify this theory [1]. The equations governing this class of continuous environments result from the relationships between conservation laws and behavioral relations of elastic domains. The concept of continuous medium is a macroscopic physical modeling resulting from current experience, the relevance of which is proven according to the problems addressed and according to the scale of phenomena involved.

In the classical mathematical formulation of this concept, a mechanical system is represented at the differential level by a volume consisting of particles. The geometrical state of these particles is characterized by the knowledge of their position. The continuity of the medium is expressed by the spatial and temporal continuity of the correspondence between the initial and current position of the particle. All physical quantities are then defined in this way with imposed conditions such as continuous differentiability.The study of materials in the theory of continuous media has therefore grown rapidly. Several types of models have been developed. This is the case, in a very felt past of functional quality materials (FGM). Among the first mentions and applications of the FGMs were the thermal problems in some materials such as ceramics. In Japan, for example, at the beginning of the nineteen eighties, the peculiarity of these materials consisted of their surface, a coating or thermal protection. This resulted in a fairly high temperature gradient, especially in terms of electronic structures [2,3]. Functional quality materials (FGM) are composite materials in which the shape, the direction of the elemental differences and the value of their concentration are taken into account. These different geometrical shapes influence the volume contributions, the porosity, the directions, the thermal coating as well as the variation of the scalar products of the material vectors (variation of angle and length) [4]. The analysis of FGMs structures involving thermal effects often led researchers to take into account the influence of temperature on the properties of each element constituting the composite material. Thus, if the effect of a nonuniform temperature on the properties is taken into account, the heat equations or the differential equations of the boundary problem involve temperature-dependent coefficients [5,6].

The structure of functional quality materials is attracting more and more attention from scientists in the field of analytical or applied physics. FGMs play a vital role in most integrated systems. Authors such as He et al. [7]

have studied the FGMs plates composed of piezoelectric sensors. These studies may lead to buckling of piezoelectric plates subjected to thermo-electro-mechanical loading. Yang et al. [8] studied the dynamic behavior of stresses in FGMs elastics cylinders and derived the inter-diffusion effect at the constituent elements of the material. Almajid and Taya [9] and Almajid et al. [10] have studied the displacement and stress fields in microstructure piezocomposite plates in application to piezoelectricity.
It should also be noted that an exact three-dimensional solution was obtained from studies on anisotropic elastic [11] and piezoelectric plates [12] made of functional graded materials.

II. Formulation of the problem

In linear elasticity, it is established that there is a relationship between fear and deformation at each point in the body. This observation is well verified by experimental evidence, provided that the transformations remain sufficiently small.
We will refer to this theoretical framework as linear elasticity and we will refer to the relation between constraints and constraints like Hooke's law.
We assume the isotropic continuous medium. What is expressed in mathematical form by:
If the strain $\boldsymbol{\varepsilon}$ corresponds to the stress $\boldsymbol{\sigma}$ as

$$\sigma_{ij} = a_{ijkh}\varepsilon_{kh} \tag{2.1}$$

then, at the deformation $\boldsymbol{\varepsilon}' = \mathbf{Q}\boldsymbol{\varepsilon}\mathbf{Q}^T$, where $\mathbf{Q}$ is an orthogonal matrix, it will correspond to the constraint $\boldsymbol{\sigma}' = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T$,
where a_{ijkh} can be referred to as elastic stiffnesses, elastic constants, or elastic moduli.
More generally, the elastic behavior is characterized by a linear relation between stresses and deformations. In the context of three-dimensional elasticity, this relation is written:

$$\begin{cases} \sigma_{ij} = a_{ijkh}\varepsilon_{kh} \\ \sigma = a[\varepsilon] \end{cases} \tag{2.2}$$

or

$$\begin{cases} \varepsilon_{ij} = \Lambda_{ijkh}\sigma_{kh} \\ \varepsilon = \Lambda[\sigma] \end{cases} \tag{2.3}$$

where a_{ijkh} and Λ_{ijkh} are the components of two applications $\mathbf{a}$ and $\boldsymbol{\Lambda}$ which are inverse of each other, of the space of the symmetric tensors in itself.
To solve a problem of elasticity, it is necessary to find a displacement field $\mathbf{u}$ and a field of the stress $\boldsymbol{\sigma}$ verifying the problems of elasticity. Equations of movement or equilibrium depending on whether one is interested in the dynamic or quasi-static problem:

$$\sigma_{ij,j} + f_i = \rho\frac{\partial^2 u_i}{\partial t^2}, \text{or } 0, \tag{2.4}$$

and the law of behavior defined in (2),
with

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.5}$$

We get a system of equations and the problem will be good posed, regular, and will admit a unique solution provided that it adds boundary conditions and possibly adequate initial conditions.
Classically displacements-type conditions are defined at the border:

$$u_{i/\partial\Omega} = u_i^d, \tag{2.6}$$

or type of effort on the border:

$$\sigma_{ij}n_{j/\partial\Omega} = T_i^d, \tag{2.7}$$

To solve analytically a problem of elasticity, one postulates a priori a particular form for the solution then one tries to verify all the equations. If we succeed, then according to the uniqueness theorem for a regular problem, it is the solution of the problem.
This results in two methods, depending on whether one tries a field of displacement or a field of stress.
If we start from the displacement field $\boldsymbol{u}$ we can calculate the tensor of the deformations by (2.5) and the stress tensor by the constitutive law (2.2). It remains only to verify that the equations of motion (2.4), the boundary conditions of type displacement and effort type and possibly the initial conditions. Reporter (2.2) and (2.5) in the equation of motion (2.4) makes it possible to write the equation which must be verified by the displacement field in dynamics or in static [13]:

$$a_{ijkh}\frac{\partial^2 u_k}{\partial x_j \partial x_h} + f_i = \rho\frac{\partial^2 u_i}{\partial t^2}, or\ 0 \quad (2.8)$$

where the symmetry of a_{ijkh} is used and assuming the homogeneous material (the operator $\boldsymbol{a}$ constant). In elasticity, time does not intervene in the law of behavior. It intervenes only in the equation of motion and disappears in quasi-static with the exception of friction problems, where it remains in the boundary condition (equations 2.6 and 2.7). In elasticity, one never speaks of quasi-static problems but only of static problems. To solve a quasi-static problem, it suffices to solve at each moment the corresponding static problem.
In the context of the isotropic linear elasticity-classical elasticity-equation (2.8) becomes:

$$(\lambda+\mu)u_{i,ik} + \mu u_{i,kh} + f_i = 0, \quad (2.9)$$

either, by introducing the operators of vector analysis:

$$(\lambda+\mu)grad(div(u)) + \mu\Delta(u) + f = 0, \quad (2.10)$$

or equivalent

$$(\lambda+2\mu)\mathrm{grad}(\mathrm{div}(u)) - \mu\,\mathrm{rot}(\mathrm{rot}(u)) + f = 0. \quad (2.11)$$

This equation of Navier is, in particular, very useful if one knows that the displacement field u sought is such that its rotational is zero. This comes from a symmetry property of the regular or well-posed problem.

In this case, we obtain the reduced form

$$(\lambda+2\mu)\mathrm{grad}(\mathrm{div}(\boldsymbol{u})) + \boldsymbol{f} = 0. \quad (2.12)$$

Which necessarily implies that
$\overrightarrow{\mathrm{rot}}(\boldsymbol{u}) = \mathbf{0}$,
and so that f derives from a potential

$$f = -\underline{grad}(\phi).$$

The reduced form equation above, then admits a first integral

$$(\lambda+2\mu)(\mathrm{div}(u)) + \phi = C. \quad (2.13)$$

where C is an integration constant, which simplifies the search for $\boldsymbol{u}$.

In this study, we consider a circular section hollow tube composed of two coaxial cylinders R_i and R_o. The material is subjected to inflation through its wall, thus translating the loading into pressure p. The characteristic of this material domain that we study is that it is thick-walled in materials of functional quality and whose volume contribution of the cylinder R_i is given by the function [14]:

$$c(r) = c_0[1 - k(r/r_i)^m] \quad (2.14)$$

where r is the radius and c_o, k and m are the material parameters.
With the form of the analytic function defined in (2.14), we can obtain several models to describe the nonlinear and continuous behavior of the functional quality material through the contribution of the volume ratio c(r).

We propose to study the boundary problem through a cylindrical tube. Classically, for small transformations the relations between radial and circumferential deformations on the one hand and displacement are given by:

$$\begin{cases} \varepsilon_r = \frac{du}{dr} \\ \varepsilon_\theta = \frac{u}{r} \end{cases} \quad (2.15)$$

The structure of the material of which we make the analytical study, is of the type FGM. Using the law of Lamé in (2.2) and the deformation in (2.5), we can define the principal stresses by:

$$\begin{cases} \sigma_r = \bar{\lambda}\frac{u}{r} + (\bar{\lambda} + 2\bar{\mu})\frac{du}{dr} \\ \sigma_\theta = (\bar{\lambda} + 2\bar{\mu})\frac{u}{r} + \bar{\lambda}\frac{du}{dr}, \\ \sigma_z = \bar{\lambda}\left(\frac{u}{r} + \frac{du}{dr}\right) \end{cases} \quad (2.16)$$

where the coefficients of Lamé $\bar{\lambda}$ and $\bar{\mu}$ [14]in this current material configuration are defined as functions of the parameters in the reference configuration by the function defined in equation (2.14):

$$\begin{cases} \bar{\lambda} = c(r)\lambda_0 + [1 - c(r)]\lambda_1 \\ \bar{\mu} = c(r)\mu_0 + [1 - c(r)]\mu_1 \end{cases}. \quad (2.17)$$

The equation of equilibrium translating the problem to the limits, in a cartesian system is given by:

$$\frac{d}{dr}(r\sigma_r) - \sigma_\theta = 0. \quad (2.18)$$

The function defining the volume contribution $c(r)$ is continuous, differentiable. By assuming that the cylindrical tube of basic components FGM, undergoes small disturbances, we can then linearize $c(r)$. By means of a development limited to order 1 around the inner radius R_i, we can then write, with Landeau's notation:

$$c(r) = c(R_i) + c'(R_i)(r - R_i) + o(r - R_i), \quad (2.19_1)$$

that we can still write in the form:

$$c(r) = \alpha_0 r + \beta_0, \quad (2.19_2)$$

with $(.)' = \frac{d}{dr}(.)$.

By referring the expression of (2.19_2) in the system (2.17) and considering the equations defined in (2.16), the equation of equilibrium (2.18) is transformed into:

$$r^2\left(\alpha_1 + \frac{\alpha_2}{r}\right)\frac{d^2u}{dr^2} + (\alpha_2 + 2\alpha_1 r)\frac{du}{dr} + \left(\alpha_3 - \frac{\alpha_2}{r}\right)u = 0, \quad (2.20)$$

where the expressions of the different constants are:

$$\begin{cases} \alpha_0 = c'(R_i) \\ \beta_0 = c(R_i) - \alpha_0 R_i + o(r - R_i) \\ \alpha_1 = \alpha_0(\lambda_0 - \lambda_1) + 2\alpha_0(\mu_0 - \mu_1) \\ \alpha_2 = \beta_0(\lambda_0 - \lambda_1) + \lambda_1 + 2(\beta_0(\mu_0 - \mu_1) + \mu_1) \\ \alpha_3 = -2\alpha_0(\mu_0 - \mu_1) \end{cases}.$$

III. Results and discussion

Very few ordinary differential equations have exact analytical solutions. This is not easy, not because ingenuity has failed, but because the repertoire of standard functions for the expression of solutions is very limited. Even if a solution can be found, the formula is often too complicated to clearly display the main features of the solution.
This is particularly the case for implicit solutions and solutions that come in the form of integrals or infinite series. This is the case, in general, in nonlinear mechanics, where the boundary problem is in the form of one or more Bessel equations or Liouville equations.
The analytical expressions obtained are often series of functions to which the asymptotic behavior must be studied.

A. Exact analytical solution

The recurrent movements fall into two types, one of them being explicitly representable by means of continuous and periodic functions of the variables they contain. This first type is called continuous type and seems to include all the recurrent movements whose existence has been established. In this paper, we choose the form of displacement as follows:

$$u(r) = f(r)\cos(wr), \tag{3.1}$$

Equation (2.20) then becomes:

$$\left[r^2\left(\alpha_1 + \frac{\alpha_2}{r}\right)\left(f''(r) - w^2 f(r)\right) + (\alpha_2 + 2\alpha_1 r)f'(r) + \left(\alpha_3 - \frac{\alpha_2}{r}\right)f(r)\right]\cos(wr)$$
$$-w\left[r^2\left(\alpha_1 + \frac{\alpha_2}{r}\right)f'(r) + (\alpha_2 + 2\alpha_1 r)f(r)\right]\sin(wr) = 0.$$

We get two decoupled equations, which gives the system:

$$\begin{cases} r^2\left(\alpha_1 + \frac{\alpha_2}{r}\right)\left(f''(r) - w^2 f(r)\right) + (\alpha_2 + 2\alpha_1 r)f'(r) + \left(\alpha_3 - \frac{\alpha_2}{r}\right)f(r) = 0 \\ r^2\left(\alpha_1 + \frac{\alpha_2}{r}\right)f'(r) + (\alpha_2 + 2\alpha_1 r)f(r) = 0 \end{cases}, \tag{3.2}$$

The differential equation (3.1a) admits an exact analytical solution:

$$\mathrm{f(r)} = \mathrm{C_0 exp}\left(\mathrm{F(r)}\right) + \mathrm{C_1}, \tag{3.3$_1$}$$

with

$$\mathrm{F(r)} = \left(\frac{w^2\alpha_2^2}{\alpha_1^2} - \frac{\alpha_3}{\alpha_1}\right)\log(\alpha_2 + 2\alpha_1 r) + \log\left(\frac{2\alpha_1}{\alpha_2 r^2} - \frac{2}{r^3}\right) + 2\log\left(\frac{4\alpha_1 r}{\alpha_2(\alpha_1 r + \alpha_2)} + \frac{2}{\alpha_1 r + \alpha_2}\right) - \frac{w^2}{2} r\left(r + \frac{\alpha_2}{\alpha_1}\right), \tag{3.3$_2$}$$

In the theory of differential equations, an important theoretical question is to know the number of solutions. The answer to this question is provided by the existence and uniqueness theorem.

An axial symmetry state is thus considered in the problem of a thick-walled tube subjected to an internal pressure p. The boundary conditions are [14]:

$$\begin{cases} \sigma_r(R_i) = -p \\ \sigma_r(R_o) = 0 \end{cases}, \tag{3.4}$$

Taking into account the boundary conditions (3.4), the nature of the function (3.3), we obtain the general solution of the elasticity problem (2.18) or (3.2) by:

$$u(r) = \left[\mathrm{C_0 exp}\left(\mathrm{F(r)}\right) + \mathrm{C_1}\right]\cos(wr), \tag{3.5}$$

with the integration constants whose expressions are given by:

$$\begin{cases} \mathrm{C_0} = -p\frac{B'}{AB' - A'B} \\ \mathrm{C_1} = p\frac{A'}{AB' - A'B} \end{cases}.$$

where:

$$\begin{cases} A = \left[\frac{\sigma_{i1}}{R_i}\cos(wR_i) + \sigma_{i2}F'(R_i)\cos(wR_i) - w\sigma_{i2}\sin(wR_i)\right]exp(F(R_i)) \\ B = \frac{\sigma_{i1}}{R_i}\cos(wR_i) - w\sigma_{i2}\sin(wR_i) \\ A' = \left[\frac{\sigma_{o1}}{R_o}\cos(wR_o) + \sigma_{o2}F'(R_o)\cos(wR_o) - w\sigma_{o2}\sin(wR_o)\right]exp(F(R_o)) \\ B' = \frac{\sigma_{o1}}{R_o}\cos(wR_o) - w\sigma_{o2}\sin(wR_o) \end{cases},$$

with:

$$\begin{cases} \sigma_{i1} = \alpha_0(\lambda_0 - \lambda_1)R_i + \lambda_0\beta_0 + \lambda_1 - \beta_0\lambda_1 \\ \sigma_{o1} = \alpha_0(\lambda_0 - \lambda_1)R_o + \lambda_0\beta_0 + \lambda_1 - \beta_0\lambda_1 \end{cases},$$

$$\begin{cases} \sigma_{i2} = \sigma_{i1} + 2\alpha_0(\mu_0 - \mu_1)R_i + 2(\beta_0\mu_0 + \mu_1 - \beta_0\mu_1) \\ \sigma_{o2} = \sigma_{o1} + 2\alpha_0(\mu_0 - \mu_1)R_o + 2(\beta_0\mu_0 + \mu_1 - \beta_0\mu_1) \end{cases},$$

Remark

We can also examine the case of a particular solution to the boundary problems in (2.18).
In the limited development in equation (2.19_1), passing to the limit:($o(r - R_i) \to 0$), we can deduce that ($\alpha_2 \to 0$).
Under these conditions, the equilibrium equation (2.20) becomes:

$$u"(r) + \frac{2}{r}u'(r) + \frac{\alpha_3}{\alpha_1}\frac{u(r)}{r^2} = 0. \qquad (3.6)$$

Following the sign of $\frac{\alpha_3}{\alpha_1}$, this equation can be put in the form:

$$u"(r) + \frac{2}{r}u'(r) \pm k^2\frac{u(r)}{r^2} = 0, \qquad (3.7)$$

The characteristic equation is written:

$$r^2 + r \pm k^2 = 0, \qquad (3.8)$$

where we pose: $\frac{\alpha_3}{\alpha_1} = \pm k^2$

We thus obtain the solution of equation (3.7) according to the shape of the roots of the characteristic equation.

i. $u(r) = c_0 r^{r_1} + c_1 r^{r_2}$,
if (3.8) admits two real roots: r_1, r_2,

ii. $u(r) = r^a\left[c_0\cos\left(b.ln(r)\right) + c_1\sin\left(b.ln(r)\right)\right]$, (3.9)
if (3.8) admits two conjugated complex roots: $a \pm ib$

iii. $u(r) = \frac{1}{\sqrt{r}}[c_0 + c_1\ln(r)]$
if (3.8) admits a double root.

The integration constants are determined under the same conditions as the problem at the limits of the preceding general case.

La fonction $F(r)$ définie en (3.3_2) est une fonction complexe :
$F(r) = real(F) + i(imag(F))$.
Les figures (1), (2) montrent les allures des parties réelles et imaginaires en fonction du rayon.
Nous remarquons que plus le rayon est grand, plus la partie réelle tend vers 0 et la partie imaginaire de l'ordre de $+10^3$.

Compte tenu de la forme des fonctions F et f, des équations (3.1), (3.2) et (3.3), nous en déduisons la forme du déplacement $u(r)$ solution de (2.20) :

$$u(r) = \left[C_0\cos(imag(F))e^{real(F)} + C_1\right]\cos(wR)$$

B. Simulation and interpretation

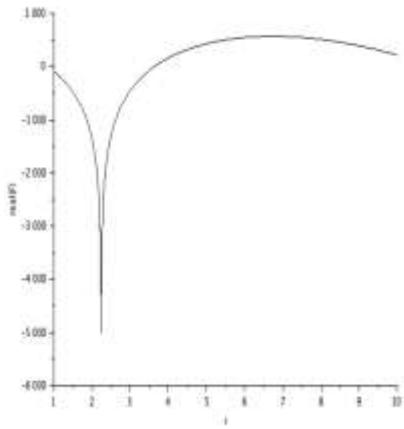

Figure1

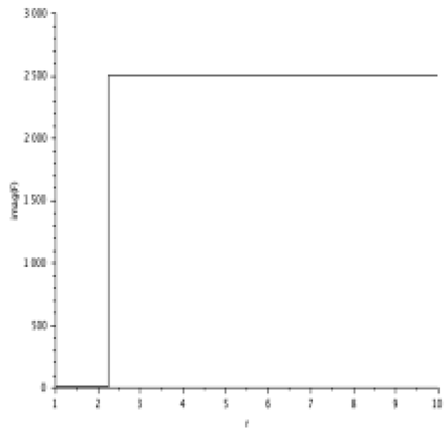

Figure2

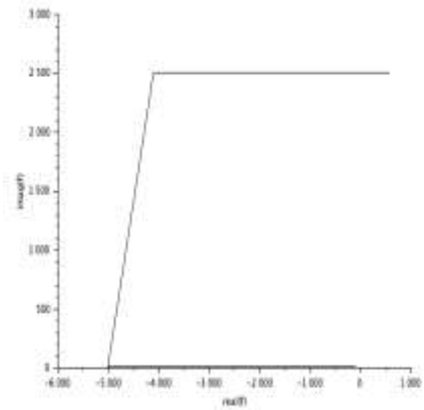

Figure 3

We find in **Figure 1** that when the radius increases, the real part of F decreases exponentially between 0 and -5000 for values of r between 1 and 2.25 before changing the variation follow a logarithmic growth after $r = 2.25$ in tend to 500, a value that is reached for a radius equal to 7, before approaching asymptotically the value zero as the ray continues to grow.

Figure 2 shows us that as the radius increases we have the imaginary part of F that is zero for values of r between 1 and 2.25. But for $r = 2.25$ this imaginary part leaves zero to reach the value 2500 and becomes constant in stabilizing itself in this value for r greater than 2.25.

The graphical representation of the imaginary part as a function of the real part of F (**Figure 3**) shows that for values of the real part less than -5000 we have the imaginary part which is zero. For a real part between -5000 and -4250, we find a rectilinear and very fast growth of the imaginary part. When the real part reaches the value -4250, the imaginary part reaches the value 2500 by stabilizing itself in this value for larger imaginary values.

Summarized, these figures obtained showed us that when the radius varies between 0 and 7, the real part of F follows the pace of a thin cone reverse with a spade obtained for $r = 2.25$ while its imaginary part follows a pace shaped staircase.

Conclusion:

In our study, we considered a hollow circular tubular section which is composed of two coaxial cylinders, The material is subjected to swelling across its wall with a boundary problem. After we gave the kinematics of deformation describing the behavior of the material subjected to inflation, many mathematical objects have been calculated for the resolution of the problem.

The main components of the stress tensor and the equilibrium equations allowed us to have a Navier equation, so then a differential equation and from this equation we obtained an exact solution of the displacement. This solution obtained contains a component which is a complex function.
We are interested in the behavior of the real and imaginary parts of the complex component as a function of the radius. This allowed us to see that the imaginary part which follows the shape of a thin taper reverse with a spike obtained for $r = 2.25$ behaves differently from the imaginary part which follows a stair-like shape.
Finally, this study showed us that the geometrical material shapes does not only modify criterias as the volume contributions, the porosity, the directions, the thermal coating as well as the variation of the scalar products of the material vectors but also creates certain behaviors of the material when it is subjected to inflation as the solution shown us at the level of the complex component when the radius varies.

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