An equivalence between the rank 1 convexity and polyconvexity of energy functions on $SL^{+}(3)$

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Abstract:
In this research paper, we have proposed as a work to demonstrate the equivalence between the rank 1 convexity and the polyconvexity of deformation energy functions. The study is done in the three-dimensional deformation with a case of a cylindrical hyperelastic incompressible isotropic tube. To achieve our objective, the kinematics with an isotropic and incompressible strain energy function was used. We have state and demonstrate three propositions which allowed us to show that there is an equivalence between the rank 1 convexity and the polyconvexity of the energy potential which is a function of the gradient tensor. We also obtain from this study, a new theorem on the convexity and polyconvexity of energy functions in three dimensions.

Keywords:
Incompressible deformation, isotropic elementary invariants, eigenvalues, Frobenius norm, energy function, Rank 1 convexity, polyconvexity.

1 Introduction

In the theoretical research of solutions of nonlinear mechanical systems, a particular attention has been paid to the study of elliptic, parabolic and hyperbolic equations in recent years. In structural mechanics, this attention has been focused on the ellipticity of the equations resulting from the study of compressible or incompressible material systems [1]. Necessary and sufficient conditions have been studied for the strong ellipticity in the isotropic case for any dimension [2]. For an isotropic and compressible material, the loss of ellipticity criterion depends on the determinant of the Hessian matrix of the energy potential and therefore of the invariants of the Cauchy Green tensor [3]. Thus, the absence of a stability criterion may lead to comparing the ellipticity condition considered as a local stability criterion to the local form of classical stability analysis by the Lyapunov theorem [4].
Others criterias such as rank 1 convexity and polyconvexity are properties that play an important role in the theory of nonlinear hyperelasticity. The notion of polyconvexity was introduced into the context of nonlinear elasticity theory by John Ball [5].
A study shows that the rank 1 convexity implies the polyconvexity of energy function of in case of a planar deformation [6].

In this paper, we first study an isotropic and incompressible three-dimensional deformation. Then we study the conditions of rank 1 convexity and polyconvexity of the energy functions in this same dimension.
From those conditions, we will abtain three propositions and one theorem what will allow us to show the equivalence between rank 1 convexity and polyconvexity of energy functions of deformation in three dimensions.

2 Preliminaries

Let's consider here a continuous material body in which a material particule is described by $\mathbf{X}(R, \Theta, Z)$ in the undeformed reference configuration and by $\mathbf{x}(r, \theta, z)$ in the deformed configuration, what yields in a cylindrical coordinates the general following kinematic:

$$r = r(R, \Theta, Z), \quad \theta = \theta(R, \Theta, Z), \quad z = z(R, \Theta, Z). \tag{1}$$

To describe the local deformation, we use the deformation gradient tensor $\mathbf{F}$ which is the tangent linear application. This tensor allow to have the volume change from his determinant which is always positive, so that

$$\mathbf{F} = Grad\,\mathbf{x}, \quad det(\mathbf{F}) \geq 0; \tag{2}$$

In our study case which is a three dimensional deformation, the gradient tensor can be written as a function of eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$:

$$\mathbf{F} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \tag{3}$$

In incompressible deformation, there is no volume change, so that:

$$det(\mathbf{F}) = \lambda_1 \lambda_2 \lambda_3 = 1. \tag{4}$$

It is important to specify that from the previous deformation gradient tensor $\mathbf{F}$, we can have the familiar symetric tensors that are: right Cauchy-Green tensor $\mathbf{C}$ and left Cauchy-Green $\mathbf{B}$, with

$$\mathbf{C} = \mathbf{F^T F}; \quad \mathbf{B} = \mathbf{F F^T}. \tag{5}$$

In isotropic deformation, we can calculate the three first elementary invariants which are:

$$\begin{aligned} I_1 &= tr(\mathbf{C}) = tr(\mathbf{B}); \\ I_2 &= tr(\mathbf{C}^*) = tr(\mathbf{B}^*); \\ I_3 &= det(\mathbf{C}) = det(\mathbf{B}). \end{aligned} \tag{6}$$

where $\mathbf{C}^*$ and $\mathbf{B}^*$ are respectively the adjoint tensors of $\mathbf{C}$ and $\mathbf{B}$, tr is the *trace* operator and det is the *determinant* operator.
The behavior of a material is described by a thermodynamic potential called deformation energy function which is a function of $\mathbf{F}$, so a function of eigenvalues λ_1, λ_2 and λ_3, or a function of elementary invariants I_1, I_2 and I_3.

$$W = W(\mathbf{F}) = W(\lambda_1, \lambda_2, \lambda_3) = W(I_1, I_2, I_3). \tag{7}$$

Let's consider here the group $GL^+(3)$, which represents the set of matrices defined by:

$$GL^+(3) = \left\{\mathbf{X} \in \mathbb{R}^3 \times \mathbb{R}^3 / det(\mathbf{X}) > 0\right\}, \tag{8}$$

and his subset $SL^+(3)$ defined by:

$$SL^+(3) = \left\{\mathbf{X} \in \mathbb{R}^3 \times \mathbb{R}^3 / det(\mathbf{X}) = 1\right\}. \tag{9}$$

There is also the orthogonal group $O(3)$ which represents the set of matrices defined by:

$$O(3) = \left\{\mathbf{X} \in \mathbb{R}^3 \times \mathbb{R}^3 / \mathbf{X}^T\mathbf{X} = \mathbf{1}\right\}, \tag{10}$$

where $\mathbf{1}$ is the identity tensor.
The energy function of relation (7) will be called objective if for all two elements Q_1 and Q_2 of $O(3)$, we have:

$$W(\mathbf{Q_1 F Q_2}) = W(\mathbf{F}), \forall \mathbf{F} \in GL^+(3). \tag{11}$$

Convexity is a criterion that must be verified by deformation energy functions. Following a definition of Ball [5], W is rank-one convex on $GL^+(3)$ if for all $\mathbf{F} \in GL^+(3)$, $\theta \in [0,1]$ and all $\xi, \eta \in \mathbb{R}^3$ with $\mathbf{F} + t.\xi \bigotimes \eta \in GL^+(3)$ for all $t \in [0,1]$, and as

$$W(\mathbf{F} + (1-\theta)\,\xi \otimes \eta) = W\left(\theta \mathbf{F} + (1-\theta)\left(\mathbf{F} + \xi \otimes \eta\right)\right),$$

then:

$$W(\mathbf{F} + (1-\theta)\,\xi \otimes \eta) \leq \theta W(\mathbf{F}) + (1-\theta)\, W\left(\mathbf{F} + \xi \otimes \eta\right). \tag{12}$$

where $\xi \otimes \eta$ denotes the dyadic product.
Let's now suppose a deformation energy function $W : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R} \cup \{+\infty\}$.
W is polyconvex if for all $\mathbf{F} \in \mathbb{R}^3 \times \mathbb{R}^3$, it exist a convex function
$P : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ with $1 \leq n \leq 3$ and $M(\mathbf{F})$ a minor of the gradient tensor $\mathbf{F}$, so that:

$$W(\mathbf{F}) = P(M(\mathbf{F})). \tag{13}$$

The Frobenius norm [7] of the deformation gradient tensor is defined by:

$$\parallel \mathbf{F} \parallel = \sqrt{\sum_{i,j=1}^{3} \mid F_{ij} \mid}. \tag{14}$$

3 Formulation of the problem

Let's consider a continuous cylindrical hyperelastic tube where a material point occupies the position (R, Θ, Z) before the deformation and the position (r, θ, z) after deformation and which is represented by the following kinematic

$$r = R, \quad \theta = \Theta, \quad z = Z + \gamma R. \tag{15}$$

where γ represents a positive constant.
From (15), the gradient tensor of deformation gives :

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma & 0 & 1 \end{pmatrix}, \tag{16}$$

It's follows the symmetrical right and left Cauchy-Green tensors which are

$$\mathbf{C} = \begin{pmatrix} 1+\gamma^2 & 0 & \gamma \\ 0 & 1 & 0 \\ \gamma & 0 & 1 \end{pmatrix}, \tag{17}$$

and

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & \gamma \\ 0 & 1 & 0 \\ \gamma & 0 & \gamma^2+1 \end{pmatrix}. \tag{18}$$

From (17) or (18), we can then calculate the first three isotropic elementary invariants of deformation

$$\begin{aligned} I_1 &= 3 + \gamma^2; \\ I_2 &= 3 + \gamma^2; \\ I_3 &= 1. \end{aligned} \tag{19}$$

Remark:
relations (19) yield us an interesting result of the volume change given by $I_3 = 1$. This means that for our kinematics defined in (15), the deformation will always be incompressible.
The Frobenius norm of the deformation gradient tensor becomes:

$$\| \mathbf{F} \| = \sqrt{3 + \gamma^2}. \tag{20}$$

4 Equivalence between rank 1 Convexity and Polyconvexity

in this section, we give three new propositions and a new theorem on the energy deformation function that we will prove in three dimension.
As it has been recently shown that the rank 1 convexity implies the polyconvexity of energy functions for a planar incopressible deformation, we have set ourselves to broaden this result in three dimension. The objective is to show the equivalence between rank 1 convexity and polyconvexity in three-dimensional incompressible and isotropic deformation.

4.1 Proposition 1

Let's $W : SL^{+}(3) \longrightarrow \mathbb{R}$ be an isotropic and twice differentiable objective function such that there are two unique functions, pairs $\varphi : [3, +\infty] \longrightarrow \mathbb{R}$, $\phi : [0, +\infty] \longrightarrow \mathbb{R}$ with

$$W(\mathbf{F}) = \varphi(I) = \phi(\gamma), I = \parallel \mathbf{F} \parallel, \forall \mathbf{F} \in SL^{+}(3), \tag{21}$$

then the following hypotheses are equivalent:
$i)$ W is rank 1 convex.
$ii)$ ϕ is non decreasing and convex on $[0, +\infty]$.
$iii)$ $\frac{d\varphi}{dI} \geq 0$ and $2(I-2)\frac{d^2\varphi}{dI^2} + \frac{d\varphi}{dI} \geq 0$ for all $I \in [3, +\infty]$.

Proof:
$i) \Longrightarrow ii)$
Let's consider our deformation gradient tensor

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma & 0 & 1 \end{pmatrix},$$

we can see that $det(\mathbf{F}) = 1, \forall \gamma \in [0, +\infty]$; so the rank 1 convexity of W implies that

$$\gamma \longrightarrow W \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma & 0 & 1 \end{pmatrix},$$

is convex, what's gives the convexity of ϕ.
Let's show that ϕ is not decreasing on $[0, +\infty]$.
Let's consider $0 \leq t_1 \leq t_2$, so t_1 is lies in the convex hull of $-t_2$ and t_2, what's means that for all value of $s \in [0, 1]$, we can pose:

$$t_1 = s(-t_2) + (1-s)t_2;$$

so,

$$\phi(t_1) = \phi(s(-t_2) + (1-s)t_2) \leq s\phi(-t_2) + (1-s)\phi(t_2) = \phi(t_2),$$
$$\phi(t_1) \leq \phi(t_2),$$

then ϕ is not decreasing on $[0, +\infty]$.
Hence $i) \Longrightarrow ii.$
$ii) \Longrightarrow iii$
In our three-dimensional incompressible objective energy function case, we have

$$I = \| \mathbf{F} \| = \sqrt{3+\gamma^2},$$

what yields

$$\gamma = \sqrt{I-3}.$$

With the existence of two functions pairs that are twice differentiable
$\varphi : [3, +\infty] \longrightarrow \mathbb{R}$, $\phi : [0, +\infty] \longrightarrow \mathbb{R}$, as

$$W(\mathbf{F}) = \varphi(I) = \phi(\gamma) = \phi(\sqrt{I-3}), \forall \mathbf{F} \in SL^+(3),$$

it's easy to see that for $\gamma = \sqrt{I-3}$, we are getting:

$$\frac{d\varphi}{dI} = \frac{d\phi}{d\gamma}\frac{1}{2\sqrt{I-3}} = \frac{1}{2\gamma}\frac{d\phi}{d\gamma} \Longrightarrow \frac{d\phi}{d\gamma} = 2\gamma\frac{d\varphi}{dI}$$

and

$$\frac{d^2\varphi}{dI^2} = \frac{1}{4\gamma^2}\frac{d^2\phi}{d\gamma^2} - \frac{1}{4\gamma^3}\frac{d\phi}{d\gamma} \Longrightarrow \frac{d^2\phi}{d\gamma^2} = 2(I-3)\frac{d^2\varphi}{dI^2} + \frac{d\varphi}{dI}.$$

The monotonicity of ϕ is equivalent to:

$$\frac{d\varphi}{dI}(I) \geq 0, \forall I \in [3, +\infty].$$

The convexity of ϕ is equivalent to:

$$2(I-3)\frac{d^2\varphi}{dI^2}(I) + \frac{d\varphi}{dI}(I) \geq 0, \forall I \in [3, +\infty].$$

So we have $ii) \Longrightarrow iii)$
$iii) \Longrightarrow i)$
When we consider the inequalities

$$\frac{d\varphi}{dI}(I) \geq 0 \text{ and } 2(I-3)\frac{d^2\varphi}{dI^2}(I) + \frac{d\varphi}{dI}(I) \geq 0, \forall I \in [3, +\infty];$$

with the condition $W(\mathbf{F}) = \varphi(I) = \phi(\gamma), \phi(\sqrt{I-3}), \forall \mathbf{F} \in SL^+(3)$, we obtain:

$$\frac{d\varphi}{dI} = \frac{d\phi}{d\gamma}\frac{1}{2\sqrt{I-3}} = \frac{1}{2\gamma}\frac{d\phi}{d\gamma}$$

and

$$\frac{d^2\varphi}{dI^2} = \frac{1}{4\gamma^2}\frac{d^2\phi}{d\gamma^2} - \frac{1}{4\gamma^3}\frac{d\phi}{d\gamma}.$$

The condition $iii)$ and the two previous relations gives us the convexity of ϕ.
According to $W(\mathbf{F}) = \varphi(I) = \phi(\gamma), \phi(\sqrt{I-3}), \forall \mathbf{F} \in SL^+(3)$, the convexity of ϕ implies the rank 1 convexity of W.
Hence $iii) \Longrightarrow i)$.

4.2 Proposition 2

Let's $W : SL^+(3) \longrightarrow \mathbb{R}$ be an isotropic and twice differentiable objective function such that there is a unique function, pair $\phi : [0, +\infty] \longrightarrow \mathbb{R}$ with

$$W(\mathbf{F}) = \phi(\gamma), I = \parallel \mathbf{F} \parallel, \forall \mathbf{F} \in SL^+(3), \tag{22}$$

then the following hypotheses are equivalent:
$i)$ ϕ is non decreasing and convex on $[0, +\infty]$.
$ii)$ W is polyconvex.

Proof:
$i) \Longrightarrow ii)$
ϕ is non decreasing and convex on $[0, +\infty]$, what implies according to the proposition 1 that W is rank 1 convex.
With

$$\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma & 0 & 1 \end{pmatrix},$$

we obtain

$$W(\mathbf{F}) = W\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma & 0 & 1 \end{pmatrix} = \phi(\gamma),$$

$$W(\mathbf{F}) = \phi(\gamma).$$

And as γ is a minor of $\mathbf{F}$ by deleting the 2 first lines and the 2 last column, we can conclude that W is polyconvex.
So $i) \Longrightarrow ii)$
$ii) \Longrightarrow i)$
W is polyconvex; according to the definition of polyconvexity, there will be a convex function ϕ with

$$W(\mathbf{F}) = W\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma & 0 & 1 \end{pmatrix} = \phi(\gamma),$$

hence the confirmation that ϕ is convex.
And when we consider the convexity of ϕ, we showed previously that for $0 \leq t_1 \leq t_2$ and $s \in [0,1]$, as $t_1 = s(-t_2) + (1-s)t_2$, we have:

$$\phi(t_1) \leq \phi(t_2),$$

then ϕ is not decreasing on $[0, +\infty]$.
Hence $ii) \Longrightarrow i$.

4.3 Proposition 3

Let's $W : SL^+(3) \longrightarrow \mathbb{R}$ be an isotropic and twice differentiable objective function such that there are two unique functions, pairs $\varphi : [3, +\infty] \longrightarrow \mathbb{R}$, $\phi : [0, +\infty] \longrightarrow \mathbb{R}$ with

$$W(\mathbf{F}) = \varphi(I) = \phi(\gamma), I = \parallel \mathbf{F} \parallel, \forall \mathbf{F} \in SL^+(3), \tag{23}$$

then the following hypotheses are equivalent:
$i)$ W is rank 1 convex.
$ii)$ ϕ is non decreasing and convex on $[0, +\infty]$.
$iii)$ W is polyconvex.

Proof:
the proof of this previous proposition is given by the proposition 1 and the proposition 2.

4.4 Theorem

Let's consider $\mathbf{F}$, the deformaion gradient tensor of the following kinematic in cylindrical coordinates:

$$r = R, \quad \theta = \Theta, \quad z = Z + \gamma R. \tag{24}$$

Let's $W : SL^+(3) \longrightarrow \mathbb{R}$ be an isotropic, twice differentiable, objective function which is rank 1 convex; let's the existence of two unique functions, pairs: $\varphi : [3, +\infty] \longrightarrow \mathbb{R}$, $\phi : [0, +\infty] \longrightarrow \mathbb{R}$, with

$$W(\mathbf{F}) = \varphi(I) = \phi(\gamma), I = \parallel \mathbf{F} \parallel, \forall \mathbf{F} \in SL^+(3), \tag{25}$$

Then:
$i)$ ϕ is non decreasing and convex on $[0, +\infty]$.
$ii)$ $\frac{d\varphi}{dI} \geq 0, 2(I-2)\frac{d^2\varphi}{dI^2} + \frac{d\varphi}{dI} \geq 0$ for all $I \in [3, +\infty]$.
$iii)$ W is polyconvex.

Proof:
the proof of this theorem is also obtained from the three previous propositions.

5 Conclusion

In this research work, we have proposed to study the equivalence between the rank 1 convexité and the polyconvexity of deformation energy functions on $SL^{+}(3)$. A kinematic of deformation was given and from it, the deformation gradient tensor, its norm of Frobenius, its Cauchy-Green tensors and its isoptropic elementary invariants was calculated.
From the calculated expressions, we have obtained three propositions which allowed us to show that there is an equivalence between the rank 1 convexity and the polyconvexity of the energy potential which is a function of the gradient tensor. finaly, this allowed us to obtain a theorem on this convexity and polyconvexity in three dimensions.

References

[1] Aubert : Necessary and sufficient conditions for isotropic rank one functions in dimension 2; Journal of Elasticity, 39, 31-46 (1995).

[2] B. Dacorogna., "Necessary and sufficient conditions for strong ellipticity of isotropic functions in any dimension". Discrete and continuous dynamical systems-series B, Vol 1, Number 2, pp. 257-263 (2001).

[3] E. Diouf. " Sur l'ellipticité des équations non linéaires pour une certaine classe de matériau isotrope et compressible". International Journal of Innovation and Applied Studies, ISSN 2028-9324 Vol. 13 No. 2 Oct., pp. 327-334 (2015).

[4] A. Lyapunov. "The general problem of the stability of motion". CRC Press, (ISBN 0748400621), (1992).

[5] J. M. Ball. ?Convexity conditions and existence theorems in nonlinear elasticity". Arch. Rat. Mech. Anal. 63, pp. 337?403 (1977).

[6] I. D. Ghiba, R. G. martin, P. Neff. "Rank-one convexity implies polyconvexity in isotropic planar incompressible elasticity". arXiv:1609.01339v1 [math.CA], 5 Sep 2016.

[7] Hjørungnes. "Complex-Valued Matrix Derivatives: With Applications in Signal Processing and Communications". CUP, p. 121- , (2011).