GENERALIZED HYERS-ULAM TYPE STABILITY OF THE QUADRATIC FUNCTIONAL EQUATION INEQUALITIES WITH $2n$-VARIABLES ON AN APPROXIMATE GROUP AND RING HOMOMORPHISM

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Abstract. *In this paper we study to solve quadratic functional inequality with $2n-$ variables and their Hyers-Ulam type stability. First are investigated results with a direction method of group homomorphism and last are investigated in ring homomorphism. Then I will show that the solutions of inequality are quadratic mapping. These are the main results of this paper*

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1. Introduction

The study of the functional equation stability originated from a question of S.M. Ulam [34], concerning the stability of group homomorphisms. Let $\Big(\mathbb{G},*\Big)$ be a group and let $\Big(\mathbb{G}',\circ,d\Big)$ be a metric group with metric $d\Big(\cdot,\cdot\Big)$. Given $\epsilon>0$, does there exist a $\delta>0$ such that if $f:\mathbb{G}\to\mathbb{G}'$ satisfies

$$d\Bigg(f\Big(x*y\Big),f\Big(x\Big)\circ f\Big(y\Big)\Bigg)<\delta$$

for all $x,y\in\mathbb{G}$ then there is a homomorphism $h:\mathbb{G}\to\mathbb{G}'$ with

$$d\Bigg(f\Big(x\Big),h\Big(x\Big)\Bigg)<\epsilon$$

for all $x\in\mathbb{G}$?, if the answer, is affirmative, we would say that equation of homomorphism $h\Big(x*y\Big)=h\Big(y\Big)\circ h\Big(y\Big)$ is stable. The concept of stability for a functional equation arises when we replace functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given function equation? Hyers[18] gave a first affirmative answes the question of Ulam as follows:

Let $\epsilon\geq0$ and let f be a function defined on an Abelian group $\Big(\mathbb{G},+\Big)$ with values in Banach spaces $\Big(\mathbb{Y},+\Big)$ satisfying

$$\Big\|f\Big(x+y\Big)-f\Big(x\Big)-f\Big(y\Big)\Big\|\leq\epsilon,$$

for all $x,y\in\mathbb{G}$ and some $\epsilon\geq0$. Then there exists a unique additive mapping $T:\mathbb{G}\to\mathbb{Y}$, such that

$$\Big\|f\Big(x\Big)-T\Big(x\Big)\Big\|\leq\epsilon,\forall x\in\mathbb{G}.$$

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Next Th. M. Rassias [29] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded: (Th. M. Rassias.) Consider $\mathbb{E}, \mathbb{E}'$ to be two Banach spaces, and let $f : \mathbb{E} \to \mathbb{E}'$ be a mapping such that $f\left(tx\right)$ is continous in t for each fixed x. Assume that there exist $\theta > 0$ and $p \in [0, 1]$ such that

$$\left\|f\left(x+y\right)-f\left(x\right)-f\left(y\right)\right\| \leq \epsilon\left(\left\|x\right\|^p+\left\|y\right\|^p\right), \forall x, y \in \mathbb{E}.$$

then there exists a unique linear $L : \mathbb{E} \to \mathbb{E}'$ satifies

$$\left\|f\left(x\right)-L\left(x\right)\right\| \leq \frac{2\theta}{2-2^p}\|x\|^p, x \in \mathbb{E}.$$

Next Badora in [8] provided the following result concering the stability of a ring homomorphism:

Let $\Re$ be a ring and $\mathbb{Y}$ be Banach algebra and $\epsilon, \delta \geq 0$. Assume that $f : \Re \to \mathbb{Y}$ satisfies

$$\left\|f\left(x+y\right)-f\left(x\right)-f\left(y\right)\right\| \leq \epsilon$$

and

$$\left\|f\left(x \cdot y\right)-f\left(x\right)f\left(y\right)\right\| \leq \delta,$$

$\forall x, y \in \Re$. Then there exists a unique ring homomorphisms $T : \Re \to \mathbb{Y}$ such that

$$\left\|f(x)-T(x)\right\| \leq \epsilon, \forall x \in \Re.$$

In 1990, Th. M. Rassias [31] during the 27th International Symposium on Functional Equation asked the question whether such a theorem can also be proved for $p \geq 1$.
In 1991, Z. Gajda [15] following the same approach as in Th. M. Rassias [31], gave an affirmative solution to this question for $p > 1$.
It was proved by Gajda [15], as well as by Th. M. Rassias and P. Semrl [32] that one can not prove a Th. M. Rassias type therem when $p = 1$.
In 1994, P. Găvruta [17] provided a further generalization of Th. M. Rassias theorem in which he replaced the bouned $\epsilon\left(\left\|x\right\|^p+\left\|y\right\|^p\right)$ by a general control function $\psi\left(x, y\right)$ for the existence of a unique linear mapping. In [12], Czerwik proved the generalizaed Hyers-Ulam stability of the quadratic functional equation. Borelli and Forti [10] generalizaed stability the result as follows [19]:
Let $\mathbf{G}$ be an *Abelian* group, and $\mathbf{X}$ a Banach space. Assume that a mapping $f : \mathbf{G} \to \mathbf{X}$ satisfies the functional inequality

$$\left\|f\left(x+y\right)+f\left(x-y\right)-2\left(x\right)-2f\left(y\right)\right\| \leq \varphi\left(x, y\right), \forall x, y \in \mathbb{G}$$

and $\varphi : \mathbb{G} \times \mathbb{G} \to \left[0, \infty\right)$ is function such that

$$\psi\left(x, y\right) = \sum_{i=0}^{\infty} \varphi\left(2^i x, 2^i y\right) < \infty$$

$\forall x, y \in \mathbb{G}$. Then there exists a unique quadratic mapping $Q : \mathbb{G} \to \mathbb{X}$ with the properties

$$\left\|f\left(x\right)-Q\left(x\right)\right\| \leq \psi\left(x, x\right), \forall x, y \in \mathbb{G}.$$

Here, we cannot fail to notice that S-M. Jung [19] dealt with stability problem for the quadratic function equation of pexider type

$$f_1\Big(x+y\Big)+f_2\Big(x-y\Big)=f_3\Big(x\Big)+f_4\Big(y\Big).$$

In addition, the conditional stability of quadratic equation and stability of the quadratic mappings in Banach modules were stdied by M. S. Mosilehian [22] and C. Park [27].

Next in 2007 Mohammad Sal Moslehian, Themistocles M. Rassias [21] proved the generalized Hyers-Ulam stability of Cauchy additive functional equation and quadratic functional equation.
Recently, in [3-6, 21] the authors studied the Hyers-Ulam stability for the following inequalities functional in group and ring homomorphisms. So that we solve and proved the Hyers-Ulam type stability for functional inequalities

$$\left\|f\left(\sum_{j=1}^{n}x_j+\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)+f\left(\sum_{j=1}^{n}x_j-\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)-2\sum_{j=1}^{n}f\Big(x_j\Big)-2\sum_{j=1}^{n}f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}}\le\epsilon,\forall\epsilon\ge0 \tag{1.1}$$

and

$$\left\|f\left(\prod_{j=1}^{n}x_j+\frac{1}{n}\prod_{j=1}^{n}x_{n+j}\right)+f\left(\prod_{j=1}^{n}x_j-\frac{1}{n}\prod_{j=1}^{n}x_{n+j}\right)-2\prod_{j=1}^{n}f\Big(x_j\Big)-2\prod_{j=1}^{n}f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}}\le\delta,\forall\delta\ge0 \tag{1.2}$$

in group and ring homomorphisms. Ie the functional inequalities with $2n$-variables. Under suitable assumptions on spaces $\mathbb{X}$ and $\mathbb{Y}$, we will prove that the mappings satisfying the functional inequalities (1.1) and (1.2). Thus, the results in this paper are generalization of those in [3-6, 21, 35] for functional inequalities with $2n$-variables.
The paper is organized as followns:
In section preliminaries we remind some basic notations in [3-6, 20] such as solutions of the inequalities.

Section 3 is devoted to prove the Hyers-Ulam type stability of the quadratic functional inequalities (1.1) when $\mathbb{X}$ be an Abelian group and $\mathbb{Y}$ be a Banach space.

Section 4 is devoted to prove the Hyers-Ulam type stability of the quadratic functional inequalitie (1.1) and (1.2) when $\mathbb{X}$ be a ring and $\mathbb{Y}$ be a Banach algebra, $\mathbb{X}$ be an Abelian group and $\mathbb{Y}$ be a Banach space

2. PRELIMINARIES

2.1. Banach spaces.

Definition 2.1. Let $\Big\{x_n\Big\}$ be a sequence in a normed space $\mathbb{X}$.

(1) A sequence $\left\{x_n\right\}_{n=1}^{\infty}$ in a space $\mathbb{X}$ is a Cauchy sequence iff the sequence $\left\{x_{n+1} - x_n\right\}_{n=1}^{\infty}$ converges to zero;

(2) The sequence $\left\{x_n\right\}_{n=1}^{\infty}$ is said to be convergent if, there exists $x \in \mathbb{X}$ such that, for any $\epsilon > 0$, there is a positive integer N such that

$$\left\|x_n - x\right\| \leq \epsilon . \forall n \geq N.$$

Then the point x$\in X$ is called the limit of sequence x_n and denoted by $\lim_{n\to\infty} x_n = x$;

(3) If every sequence Cauchy in $\mathbb{X}$ converger, then the normed space X is called a Banach space.

2.2. **Solutions of the inequalities.** The functional equation

$$f\Big(x+y\Big) + f\Big(x-y\Big) = 2f\Big(x\Big) + 2f\Big(y\Big)$$

is called the quadratic equation. In particular, every solution of the qudratic equation is said to be a *quadratic mapping*.

3. Stability of Approximate Gourp Homomorphisms

Now, we first study the solutions of (1.1). Note that for this inequalitie, $\mathbb{X}$ be an Abelian gourp and $\mathbb{Y}$ is a Banach spaces. Under this setting, we can show that the mapping satisfying (1.1) is quadratic. These results are give in the following.

Theorem 3.1. *Let $\mathbb{X}$ be an Abelian group and $\mathbb{Y}$ be Banach space. If $\epsilon \geq 0$, $n \in \mathbb{N}$, $n \geq 2$, $f\Big(0\Big) = 0$ and $f : \mathbb{X} \to \mathbb{Y}$ such that*

$$\left\| f\left(\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right) + f\left(\sum_{j=1}^{n} x_j - \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right) - 2\sum_{j=1}^{n} f\Big(x_j\Big) - 2\sum_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}} \leq \epsilon \quad (3.1)$$

for all $x_1, x_2, ..., x_{2n} \in X$, then there exists a unique quadratic mapping $H : \mathbb{X} \to \mathbb{Y}$ such that

$$\left\|f(x) - H(x)\right\|_{\mathbb{Y}} \leq \frac{1}{2\cdot 2n - 2}\epsilon, \forall x \in \mathbb{X}. \quad (3.2)$$

Proof. We will show that

$$\left\|\frac{f\Big((2n)^k x\Big)}{2\cdot(2n)^k} - f\Big(x\Big)\right\|_{\mathbb{Y}} \leq \frac{\epsilon}{2}\sum_{m=1}^{k}(2n)^{-m}, \forall x \in \mathbb{X}. \quad (3.3)$$

for any positive integer k and for all $x \in \mathbb{X}$. The proof of (3.3) follows by induction on k. With $k = 1$ and letting $x_j = x, x_{n+j} = nx$ for all $j = 1, 2, ..., n$ by the hypothesis (3.1), we have

$$\left\|\frac{f\Big(2nx\Big)}{2\cdot 2n} - f\Big(x\Big)\right\|_{Y} = \frac{1}{2}\cdot\frac{1}{2n}\left\|f\Big(2nx\Big) - 2\cdot 2nf\Big(x\Big)\right\|_{\mathbb{Y}} \leq \frac{\epsilon}{2}\cdot\frac{1}{2n}, \forall x \in \mathbb{X}.$$

Assume now that (3.3) holds for k and we want to prove it for the case $k+1$. Replacing x by $4nx$ in (3.3) we obtain

$$\left\|\frac{f\Big(2\cdot(2n)^k\cdot 2nx\Big)}{2\cdot(2n)^k}-f\Big(2\cdot 2nx\Big)\right\|_{\mathbb{Y}}\leq\frac{\epsilon}{2}\sum_{m=1}^{k}(2n)^{-m},\forall x\in\mathbb{X}.$$

therefore

$$\left\|\frac{f\Big(2\cdot(2n)^{k+1}x\Big)}{2\cdot(2n)^{k+1}}-\frac{1}{2n}f\Big(2\cdot 2nx\Big)\right\|_{\mathbb{Y}}\leq\frac{\epsilon}{2}\sum_{m=2}^{k+1}(2n)^{-m},\forall x\in\mathbb{X}.$$

Now, using the triangle inequality we deduce

$$\begin{aligned}\left\|\frac{f\Big(2\cdot(2n)^{k+1}x\Big)}{2\cdot(2n)^{k+1}}-f\Big(x\Big)\right\|_{\mathbb{Y}}&\leq\left\|\frac{f\Big(2\cdot(2n)^{k+1}x\Big)}{2\cdot(2n)^{k+1}}-\frac{1}{2n}f\Big(2\cdot 2nx\Big)\right\|_{\mathbb{Y}}\\&+\left\|\frac{1}{2n}f\Big(2\cdot 2nx\Big)-f\Big(x\Big)\right\|_{\mathbb{Y}}\\&\leq\frac{\epsilon}{2\cdot 2n}+\frac{\epsilon}{2}\sum_{m=2}^{k+1}(2n)^{-m}\\&\leq\frac{\epsilon}{2}\sum_{m=1}^{k+1}(2n)^{-m}.\end{aligned}$$

Thus, (3.3) is valid for all $k\in\mathbb{N}$. Since $\sum_{m=1}^{k}(2n)^{-m}$ is increasingly convergent to $\frac{1}{2n-1}$, we get from (3.3) that

$$\left\|\frac{f\Big(2\cdot(2n)^{k+1}x\Big)}{2\cdot(2n)^{k+1}}-f\Big(x\Big)\right\|_{Y}\leq\frac{1}{2\cdot 2n-2}\epsilon,\forall x\in X.\tag{3.4}$$

Fixing an $x\in\mathbb{X}$, for all $h,k\in\mathbb{N}$ with $h>k$, we have, from (3.4) that

$$\begin{aligned}\left\|\frac{f\Big(2\cdot(2n)^{h}x\Big)}{2\cdot(2n)^{h}}-\frac{1}{2\cdot(2n)^k}f\Big(2\cdot(2n)^kx\Big)\right\|_{\mathbb{Y}}&=\frac{1}{(2n)^h}\left\|\frac{1}{(2n)^{h-k}}f\Big(2\cdot(2n)^hx\Big)-f\Big(2\cdot(2n)^kx\Big)\right\|_{\mathbb{Y}}\\&\leq\frac{1}{(2n)^k}\cdot\frac{1}{2\cdot 2n-2}\epsilon.\end{aligned}$$

Therefore

$$\lim_{h,k\to\infty}\left\|\frac{f\Big(2\cdot(2n)^{h}x\Big)}{2\cdot(2n)^{h}}-\frac{1}{2\cdot(2n)^k}f\Big(2\cdot(2n)^kx\Big)\right\|_{\mathbb{Y}}=0.$$

Since Y is Banach space, the sequence $\left\{\frac{f\left(2\cdot(2n)^kx\right)}{2\cdot(2n)^k}\right\}$ converges. Set

$$H\Big(x\Big)=\lim_{k\to\infty}\frac{f\Big(2\cdot(2n)^kx\Big)}{2\cdot(2n)^k},\forall x\in\mathbb{X}.\tag{3.5}$$

Then we obtain a mapping $H : \mathbb{X} \to \mathbb{Y}$. From (3.1), for all $x_1, x_2, ..., x_{2n} \in X$ and for all $k \in \mathbb{N}$, We compute that

$$\left\| f\left(2\cdot(2n)^k\left(\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right) + f\left(2\cdot(2n)^k\left(\sum_{j=1}^{n} x_j - \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right)\right.\right.\right.$$
$$\left.\left. - 2\sum_{j=1}^{n} f\left(2\cdot(2n)^k x_j\right) - 2\sum_{j=1}^{n} f\left(2\cdot(2n)^k \frac{x_{n+j}}{n}\right)\right\|_Y \le \epsilon,$$

and so

$$\frac{1}{2\cdot(2n)^k}\left\| f\left(2\cdot(2n)^k\left(\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right)\right) + f\left(2\cdot(2n)^k\left(\sum_{j=1}^{n} x_j - \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right)\right)\right.$$
$$\left. - 2\sum_{j=1}^{n} f\left(2\cdot(2n)^k x_j\right) - 2\sum_{j=1}^{n} f\left(2\cdot(2n)^k \frac{x_{n+j}}{n}\right)\right\|_Y \le \frac{\epsilon}{2}\cdot\frac{1}{(2n)^k}.$$

We will prove that H is quadratic.
Consequently,

$$\left\| H\left(\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right) + H\left(\sum_{j=1}^{n} x_j - \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right) - 2\sum_{j=1}^{n} H\left(x_j\right) - 2\sum_{j=1}^{n} H\left(\frac{x_{n+j}}{n}\right)\right\|_Y$$
$$= \lim_{k\to\infty}\left\| \frac{1}{2\cdot(2n)^k} f\left(2\cdot(2n)^k\left(\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right)\right) + \frac{1}{2\cdot(2n)^k} f\left(2\cdot(2n)^k\left(\sum_{j=1}^{n} x_j - \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right)\right)\right.$$
$$\left. - 2\sum_{j=1}^{n} \frac{1}{2\cdot(2n)^k} f\left(2\cdot(2n)^k x_j\right) - 2\sum_{j=1}^{n} \frac{1}{2\cdot(2n)^k} f\left(2\cdot(2n)^k \frac{x_{n+j}}{n}\right)\right\|_Y$$
$$\le \lim_{k\to\infty}\left\| \frac{\epsilon}{2}\frac{1}{(2n)^k}\right\| = 0.$$

It follows from (3.5) that

$$\left\| H\left(\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right) + H\left(\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right) - 2\sum_{j=1}^{n} H\left(x_j\right)\right.$$
$$\left. - 2\sum_{j=1}^{n} H\left(\frac{x_{n+j}}{n}\right)\right\|_Y = 0.$$

Hence

$$H\left(\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right) + H\left(\sum_{j=1}^{n} x_j - \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right) = 2\sum_{j=1}^{n} H\left(x_j\right) + 2\sum_{j=1}^{n} H\left(\frac{x_{n+j}}{n}\right)$$

for all $x_1, x_2, ..., x_{2n} \in \mathbb{X}$.

Clearly, $H\left(0\right) = 0$ and so H is a quadratic mapping. From (3.4) and (3.5) we see that (3.2) is valid. Now we prove the uniqueness of H. Assume that $H_1 : \mathbb{X} \to \mathbb{Y}$ is a quadratic mapping such that

$$\left\| f\left(x\right) - H_1\left(x\right)\right\| \le \frac{1}{2\cdot 2n - 2}\epsilon, \forall x \in \mathbb{X}.$$

Since both H and H_1 are quadratic, we deduce that, for each $\forall x \in \mathbb{X}$ and for all $n \in \mathbb{N}$,

$$
\begin{aligned}
2\cdot 2n\left\|H(x)-H_1\left(x\right)\right\|_{\mathbb{Y}} &= \left\|H(2\cdot 2nx)+H_1(2\cdot 2nx)\right\|_{\mathbb{Y}}\\
&\le \left\|H(2\cdot 2nx)-f\left(2\cdot 2nx\right)\right\|_{\mathbb{Y}}+\left\|f(2\cdot 2nx)+H_1(2\cdot 2nx)\right\|_{\mathbb{Y}}\\
&\le \frac{2\epsilon}{2\cdot 2n-2},
\end{aligned}
$$

so that

$$
\left\|H(x)-H_1\left(x\right)\right\|_{\mathbb{Y}} \le \frac{\epsilon}{2\cdot 2n\left(2n-1\right)}
$$

for all $x\in\mathbb{X}$ and hence $H\left(x\right)=H_1\left(x\right)$ for all $x\in\mathbb{X}$. This completes the proof. □

Corollary 3.2. *Let $\mathbb{X}$ be an Abelian group and $\mathbb{Y}$ be Banach space. If $\epsilon\ge 0$, $n\in\mathbb{N}$, $n\ge 2$, $f\left(0\right)=0$ and $f:\mathbb{X}\to\mathbb{Y}$ such that*

$$
\left\|f\left(\sum_{j=1}^{n}x_j+\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)+f\left(\sum_{j=1}^{n}x_j-\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)-2\sum_{j=1}^{n}f\left(x_j\right)-2\sum_{j=1}^{n}f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}}\le\epsilon, \tag{3.6}
$$

for all $x_1,x_2,...,x_{2n}\in X$, then there exists a unique quadratic group homomorphism $H:\mathbb{X}\to\mathbb{Y}$ such that

$$
\left\|f(x)-H(x)\right\|_{\mathbb{Y}}\le\frac{1}{2\cdot 2n-2}\epsilon, \forall x\in\mathbb{X}. \tag{3.7}
$$

4. Stability of a ring Homomorphism

Now, we first study the solutions of (1.2). Note that for this inequalitie, $\mathbb{X}$ be a ring and $\mathbb{Y}$ is a Banach algebra and $\mathbb{X}$ be an Abelian gourp and $\mathbb{Y}$ is a Banach spaces. Under this setting, we can show that the mapping satisfying (1.2) is quadratic. These results are give in the following.

Theorem 4.1. *Let $\Re$ be a ring and $\mathbb{Y}$ be Banach algebra and $\epsilon,\delta\ge 0$ and $n\in\mathbb{N}$, $n\ge 2$, $f\left(x\right)=0$. If a mapping $f:\Re\to\mathbb{Y}$ satisfies*

$$
\begin{aligned}
\Bigg\|f\left(\sum_{j=1}^{n}x_j+\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)+f\left(\sum_{j=1}^{n}x_j-\frac{1}{n}\sum_{j=1}^{n}x_{n+j}\right)-2\sum_{j=1}^{n}f\left(x_j\right)&\\
-2\sum_{j=1}^{n}f\left(\frac{x_{n+j}}{n}\right)\Bigg\|_{\mathbb{Y}}&\le\epsilon
\end{aligned}\tag{4.1}
$$

and

$$
\begin{aligned}
\Bigg\|f\left(\prod_{j=1}^{n}x_j+\frac{1}{n}\prod_{j=1}^{n}x_{n+j}\right)+f\left(\prod_{j=1}^{n}x_j-\frac{1}{n}\prod_{j=1}^{n}x_{n+j}\right)-2\prod_{j=1}^{n}f\left(x_j\right)&\\
-2\prod_{j=1}^{n}f\left(\frac{x_{n+j}}{n}\right)\Bigg\|_{\mathbb{Y}}&\le\delta
\end{aligned}\tag{4.2}
$$

for all $x_1, x_2, ..., x_{2n} \in \Re$, then there exists a unique quadratic mapping $H : \Re \to \mathbb{Y}$ such that

$$H\left(\prod_{j=1}^{n} x_j + \frac{1}{n}\prod_{j=1}^{n} x_{n+j}\right) + H\left(\prod_{j=1}^{n} x_j - \frac{1}{n}\prod_{j=1}^{n} x_{n+j}\right) = 2\prod_{j=1}^{n} H\left(x_j\right) + 2\prod_{j=1}^{n} H\left(\frac{x_{n+j}}{n}\right) \tag{4.3}$$

for all $x_1, x_2, ..., x_{2n} \in \Re$ and

$$\left\|f\left(x\right) - H\left(x\right)\right\|_{\mathbb{Y}} \leq \frac{1}{2 \cdot 2n - 2}\epsilon, \forall x \in \Re. \tag{4.4}$$

Proof. Theorem 3.1 show that there exists a unique additive mapping $H : \Re \to \mathbb{Y}$ satisfies (4.4). By the proof of Theorem 3.1, we see that the mapping H is give by

$$H\left(x\right) = \lim_{k\to\infty} \frac{1}{2 \cdot \left(2n\right)^k} f\left(2 \cdot \left(2n\right)^k x\right), \forall x \in \Re \tag{4.5}$$

for all $x_1, x_2, ..., x_{2n} \in \Re$, let

$$h\left(x_1, x_2, ..., x_{2n}\right) = f\left(\prod_{j=1}^{n} x_j + \frac{1}{n}\prod_{j=1}^{n} x_{n+j}\right) + f\left(\prod_{j=1}^{n} x_j - \frac{1}{n}\prod_{j=1}^{n} x_{n+j}\right) - 2\prod_{j=1}^{n} f\left(x_j\right) - 2\prod_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right).$$

The using inequality (4.1), we get

$$\lim_{k\to\infty} \frac{1}{2 \cdot \left(2n\right)^k} h\left(2 \cdot \left(2n\right)^k x_1, x_2, ..., x_{2n}\right) = 0.$$

Therefore

$$
\begin{aligned}
&H\left(\prod_{j=1}^{n} x_j + \frac{1}{n}\prod_{j=1}^{n} x_{n+j}\right) + H\left(\prod_{j=1}^{n} x_j - \frac{1}{n}\prod_{j=1}^{n} x_{n+j}\right)\\
&= H\Big(x_1x_2\cdots x_n + \frac{1}{n}x_{n+1}x_{n+2}\cdots x_{2n}\Big) + H\Big(x_1x_2\cdots x_n - \frac{1}{n}x_{n+1}x_{n+2}\cdots x_{2n}\Big)\\
&= \lim_{k\to\infty}\frac{1}{2\cdot\big(2n\big)^k} f\Big(2\cdot\big(2n\big)^k\Big(x_1x_2\cdots x_n + \frac{1}{n}x_{n+1}x_{+2}\cdots x_{2n}\Big)\\
&+ \lim_{k\to\infty}\frac{1}{2\cdot\big(2n\big)^k} f\Big(2\cdot\big(2n\big)^k\Big(x_1x_2\cdots x_n - \frac{1}{n}x_{n+1}x_{+2}\cdots x_{2n}\Big)\\
&= \lim_{k\to\infty}\frac{1}{2\cdot\big(2n\big)^k} h\Big(\Big(2\cdot\big(2n\big)^k x_1\Big)x_2\cdots x_n + \frac{1}{n}\Big(2\cdot\big(2n\big)^k x_{n+1}\Big)x_{n+2}\cdots x_{2n}\Big)\Big)\\
&+ \lim_{k\to\infty}\frac{1}{2\cdot\big(2n\big)^k} h\Big(2\cdot\Big(\big(2n\big)^k x_1\Big)x_2\cdots x_n - \frac{1}{n}\Big(2\cdot\big(2n\big)^k x_{n+1}\Big)x_{n+2}\cdots x_{2n}\Big)\Big)\\
&= \lim_{k\to\infty}\frac{1}{2\cdot\big(2n\big)^k} h\Big(\Big(2\cdot\big(2n\big)^k x_1\Big)x_2\cdots x_n + \frac{1}{n}\Big(2\cdot\big(2n\big)^k x_{n+1}\Big)x_{n+2}\cdots x_{2n}\Big)\Big)\\
&+ \lim_{k\to\infty}\frac{1}{2\cdot\big(2n\big)^k} h\Big(\Big(2\cdot\big(2n\big)^k x_1\Big)x_2\cdots x_n - \frac{1}{n}\Big(2\cdot\big(2n\big)^k x_{n+1}\Big)x_{n+2}\cdots x_{2n}\Big)\Big)\\
&= \lim_{k\to\infty}\frac{1}{2\cdot\big(2n\big)^k}\Bigg[h\Big(2\cdot(2n)^k x_1, x_2, ..., x_{2n}\Big) + 2f\Big(2\cdot(2n)^k x_1\Big)f\Big(x_2\Big)\cdots f\Big(x_n\Big)\\
&+ 2f\left(2\cdot\big(2n\big)^k\cdot\frac{x_{n+1}}{n}\right)f\left(\frac{x_{n+2}}{n}\right)\cdots f\left(\frac{x_{2n}}{n}\right)\Bigg]\\
&= 2\prod_{j=1}^{n} H\Big(x_j\Big) + 2\prod_{j=1}^{n} H\left(\frac{x_{n+j}}{n}\right),
\end{aligned}
$$

$\forall x_1, x_2, ..., x_{2n} \in \Re$.
From the last equation and the additivity of H we see that, for all $k \in \mathbb{N}$

$$2H\left(x_1\right)f\left(2\cdot(2n)^kx_2\right)\cdots f\left(x_n\right)+2H\left(\frac{x_{n+1}}{n}\right)f\left(2\cdot\left(2n\right)^k\cdot\frac{x_{n+2}}{n}\right)\cdots f\left(\frac{x_{2n}}{n}\right)$$
$$=H\left(x_12\cdot(2n)^kx_2\cdots x_n+\frac{1}{n}x_{n+1}2\cdot(2n)^kx_{n+2}\cdots x_{2n}\right)$$
$$+H\left(x_12\cdot(2n)^kx_2\cdots x_n-\frac{1}{n}x_{n+1}2\cdot(2n)^kx_{n+2}\cdots x_{2n}\right)$$
$$=H\left(2\cdot(2n)^k\cdot x_1\cdot x_2\cdots x_n+\frac{1}{n}2\cdot(2k)^kx_{n+1}\cdot x_{n+2}\cdots x_{2n}\right)$$
$$+H\left(2\cdot(2n)^k\cdot x_1\cdot x_2\cdots x_n-\frac{1}{n}2\cdot(2k)^kx_{n+1}\cdot x_{n+2}\cdots x_{2n}\right)$$
$$=2\cdot 2\dot{(}2n)^kH\left(x_1\right)f\left(x_2\right)\cdots f\left(x_n\right)$$
$$+2\cdot 2\cdot(2n)^kH\left(\frac{x_{n+1}}{n}\right)f\left(\frac{x_{n+2}}{n}\right)\cdots f\left(\frac{x_{2n}}{n}\right)$$

and so

$$2H\left(x_1\right)\frac{f\left(2\cdot(2n)^kx_2\right)}{2\cdot(2n)^k}\cdots f\left(x_n\right)+2H\left(\frac{x_{n+1}}{n}\right)\frac{f\left(2\cdot\left(2n\right)^k\cdot\frac{x_{n+2}}{n}\right)}{2\cdot(2n)^k}\cdots f\left(\frac{x_{2n}}{n}\right)$$
$$=2H\left(x_1\right)f\left(x_2\right)\cdots f\left(x_n\right)+2H\left(\frac{x_{n+1}}{n}\right)f\left(\frac{x_{n+2}}{n}\right)\cdots f\left(\frac{x_{2n}}{n}\right).$$

Sending k to to infinty, we see that

$$2H\left(x_1\right)H\left(x_2\right)H\left(x_3\right)\cdots f\left(x_n\right)+2H\left(\frac{x_{n+1}}{n}\right)H\left(\frac{x_{n+2}}{n}\right)H\left(\frac{x_{n+3}}{n}\right)\cdots f\left(\frac{x_{2n}}{n}\right)$$
$$=H\left(x_1x_2\cdots x_n+\frac{1}{n}x_{n+1}x_{n+2}\cdots x_{2n}\right)+H\left(x_1x_2\cdots x_n-\frac{1}{n}x_{n+1}x_{n+2}\cdots x_{2n}\right),\tag{4.6}$$

$\forall x_1, x_2, ..., x_{2n}\in\Re$.
Suppose that

$$2H\left(x_1\right)H\left(x_2\right)H\left(x_3\right)\cdots H\left(x_{n-1}\right)f\left(x_n\right)+2H\left(\frac{x_{n+1}}{n}\right)H\left(\frac{x_{n+2}}{n}\right)H\left(\frac{x_{n+3}}{n}\right)$$
$$\cdots H\left(\frac{x_{2n-1}}{n}\right)f\left(\frac{x_{2n}}{n}\right)$$
$$=H\left(x_1x_2\cdots x_n+\frac{1}{n}x_{n+1}x_{n+2}\cdots x_{2n}\right)+H\left(x_1x_2\cdots x_n-\frac{1}{n}x_{n+1}x_{n+2}\cdots x_{2n}\right),\tag{4.7}$$

$\forall x_1, x_1, ..., x_{2n} \in \Re$. Then, from (4.7), we get that, for all $k \in \mathbb{N}$.

$$\begin{aligned}
2 \cdot & \frac{1}{2\cdot(2n)^k} H\Big(x_1\Big)H\Big(x_2\Big)H\Big(x_3\Big)\cdots H\Big(x_{n-1}\Big)f\Big(2\cdot(2n)^k x_n\Big) \\
&+ 2\cdot\frac{1}{2\cdot(2n)^k} H\left(\frac{x_{n+1}}{n}\right)H\left(\frac{x_{n+2}}{n}\right)H\left(\frac{x_{n+3}}{n}\right)\cdots H\left(\frac{x_{2n-1}}{n}\right)f\left(2\cdot(2n)^k\cdot\frac{x_{2n}}{n}\right) \\
&= \frac{1}{2\cdot(2n)^k} H\left(2\cdot(2n)^k\left(x_1x_2\cdots x_n + \frac{1}{n}x_{n+1}x_{n+2}\cdots x_{2n}\right)\right) \\
&+ \frac{1}{2\cdot(2n)^k} H\left(2\cdot(2n)^k\left(x_1x_2\cdots x_n + \frac{1}{n}x_{n+1}x_{n+2}\cdots x_{2n}\right)\right) \\
&= H\left(x_1x_2\cdots x_n + \frac{1}{n}x_{n+1}x_{n+2}\cdots x_{2n}\right) + H\left(x_1x_2\cdots x_n - \frac{1}{n}x_{n+1}x_{n+2}\cdots x_{2n}\right).
\end{aligned} \tag{4.8}$$

By letting $k \to \infty$ we see that

$$\begin{aligned}
&2H\Big(x_1\Big)H\Big(x_2\Big)H\Big(x_3\Big)\cdots H\Big(x_n\Big) + 2H\left(\frac{x_{n+1}}{n}\right)H\left(\frac{x_{n+2}}{n}\right)H\left(\frac{x_{n+3}}{n}\right)\cdots H\left(\frac{x_{2n}}{n}\right) \\
&= H\left(x_1x_2\cdots x_n + \frac{1}{n}x_{n+1}x_{n+2}\cdots x_{2n}\right) + H\left(x_1x_2\cdots x_n - \frac{1}{n}x_{n+1}x_{n+2}\cdots x_{2n}\right).
\end{aligned} \tag{4.9}$$

$\forall x_1, x_2, ..., x_{2n} \in \Re$ which is the desired identity (4.3) □

From proving the theorems we have corollarys:

Corollary 4.2. *Let $\Re$ be a ring with a unit 1 and $\mathbb{Y}$ be Banach algebra with a unit e and $\epsilon, \delta \geq 0$ and $n \in \mathbb{N}$, $n \geq 2$. If a mapping $f : \Re \to \mathbb{Y}$ satisfies*

$$\left\| f\left(\sum_{j=1}^{n} x_j + \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right) + f\left(\sum_{j=1}^{n} x_j - \frac{1}{n}\sum_{j=1}^{n} x_{n+j}\right) - 2\sum_{j=1}^{n} f\Big(x_j\Big) - 2\sum_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}} \leq \epsilon \tag{4.10}$$

and

$$\left\| f\left(\prod_{j=1}^{n} x_j + \frac{1}{n}\prod_{j=1}^{n} x_{n+j}\right) + f\left(\prod_{j=1}^{n} x_j - \frac{1}{n}\prod_{j=1}^{n} x_{n+j}\right) - 2\prod_{j=1}^{n} f\Big(x_j\Big) - 2\prod_{j=1}^{n} f\left(\frac{x_{n+j}}{n}\right)\right\|_{\mathbb{Y}} \leq \delta \tag{4.11}$$

for all $x_1, x_2, ..., x_{2n} \in \Re$ and $f\Big(1\Big) = e$, then there exists a unique ring homomorphism $H : \Re \to \mathbb{Y}$ such that

$$\Big\| f(x) - H(x) \Big\|_{\mathbb{Y}} \leq \frac{1}{2\cdot 2n - 2}\epsilon, \forall x \in \Re. \tag{4.12}$$

5. Conclusion

In this paper, I have shown that the solutions of the functional inequalities (1.1) and (1.2) are quadratic mappings. The Hyers-Ulam type stability for these given from theorems. These are the main results of the paper, which are the general of the results [3-6, 21, 35] .

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