The Method of Calculating The Limit

SU Xiao-ya [#1], ZHAI Yan-hui [*2]

[1,2]School of mathematical science, TianGong University, Tianjin 300387, China

***Abstract* —** *The calculation of limit is the most basic operation in mathematical analysis, although the limit calculation method is more, but it is not omnipotent. We should master the essence of each method and solve a specific limit problem in the simplest and proper way. This paper mainly introduces some methods and techniques of limit calculation, including known sequence limit problem and the limit of the recursion relation problem. At the same time, we give appropriate examples and methods to illustrate,which can help us learn mathematical analysis better.*

Keywords—*Sequence limit, Define the method, Forced convergence theorem*

I. INTRODUCTION

Limit plays a very important role in mathematical analysis, and it lays the foundation for many important concepts that we learn later.For example, continuous, definite integrals, derivatives, generalized integrals and partial sums of infinite series are all defined in terms of limits.After that, we studied the double limit and the repeated limit, both of which are very similar to the concept of the unitary limit.So if we want to learn mathematical analysis well, we must firmly grasp the concept of limit and limit calculation methods and skills.Although we have learned a lot of limit calculation methods in mathematical analysis, they do not solve all problems.Therefore, to grasp the essence of each method, we should pursue the easiest and most appropriate method for a specific computational problem of limit.

II. A METHOD FOR CALCULATING THE LIMIT OF A GIVEN SEQUENCE

In general, we have to prove that the limit exists, and then we have to find the right way to calculate the limit.Of course, there are many ways to find the limit, we need to carefully observe different types of limit problems, find the most appropriate, the most simple method.The commonly used methods include the definition of limit, forced convergence theorem, formula, logarithm limit and the definition of definite integral and so on.Below we will grasp the specific examples of the number limit calculation of different methods.

Case 1: By definition

In general, we have to know the guess value a for the limit of the sequence, and for a particular sequence $\{a_n\}$, we can usually solve the inequality $|a_n - a| < \varepsilon$, and get N.

Method: Step 1: Firstly, write $\forall \varepsilon > 0$, solve for n that makes the inequality $|a_n - a| < \varepsilon$ true;

SU Xiao-ya , School of mathematical science, Tiangong University, Tianjin 300387, Tianjin,, PR China

Yan-hui Zhai, School of mathematical science, Tiangong University, Tianjin 300387, Tianjin, PR China

Step 2: You get N and you have to round it off.

Example 1: Prove: $\lim_{n\to\infty}\frac{7^n}{n!}=0$.

Proof. $\forall\varepsilon>0$, to make the following inequality true,

$$|\frac{7^n}{n!}-0|=\frac{7}{1}\cdot\frac{7}{2}\cdots\cdots\frac{7}{7}\cdot\frac{7}{8}\cdots\cdots\frac{7}{n-1}\cdot\frac{7}{n}\leq=\frac{7^7}{6!}\cdot\frac{1}{n}<\varepsilon\quad,\frac{7^7}{7!}\cdot\frac{7}{n}$$

We get $$n>\frac{7^7}{6!}\cdot\frac{1}{\varepsilon},$$

Let $N=[\frac{7^7}{6!}\cdot\frac{1}{\varepsilon}]$,So when $n>N$, we have $|\frac{7^n}{n!}-0|\leq\frac{7^7}{6!}\cdot\frac{1}{n}<\varepsilon$,

Therefore, $\lim_{n\to\infty}\frac{7^n}{n!}=0$ is true.

Case 2:Use the limit algorithm

When we come to the limit of a sequence in the form of summation,difference,product and quotient, we naturally think of using the four rules of the limit.

Four Arithmetic Operations[1]: If $\{a_n\}$ and $\{b_n\}$ are convergent sequences, $\{a_n+b_n\}$, $\{a_n-b_n\}$, $\{a_n\cdot b_n\}$ are also convergent sequences, and

$$\lim_{n\to\infty}(a_n\pm b_n)=\lim_{n\to\infty}a_n\pm\lim_{n\to\infty}b_n\ ,\quad\lim_{n\to\infty}(a_n\cdot b_n)=\lim_{n\to\infty}a_n\cdot\lim_{n\to\infty}b_n\ ,$$

Especially if b_n is a constant c , yes

$$\lim_{n\to\infty}(a_n+c)=\lim_{n\to\infty}a_n+c\ ,\quad\lim_{n\to\infty}c\ a_n=c\lim_{n\to\infty}a_n\ .$$

If $b_n\neq 0$ and $\lim_{n\to\infty}b_n\neq 0$ are satisfied, $\{\frac{a_n}{b_n}\}$is also a convergent sequence, and

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}a_n\Big/\lim_{n\to\infty}b_n\ .$$

Method: For the addition and subtraction of fractions, the general fraction can be decomposed later and then reduced. And then we plug in n approaches infinity.and then n tends to infinite.Some fractions can divide the numerator and denominator by the highest degree of n , some fractions with radical expression can be rationalized by the numerator and denominator, etc.

Example 2: Calculate the limit: $\lim_{n\to\infty} \frac{4n^2 - 5n - 1}{7 + 2n - 8n^2}$.

Solve. We divide the numerator and the denominator $\frac{4n^2 - 5n - 1}{7 + 2n - 8n^2}$ by n^2. According to the algorithm,we get

$$\lim_{n\to\infty} \frac{4n^2 - 5n - 1}{7 + 2n - 8n^2} = \lim_{n\to\infty} \frac{4 - \frac{5}{n} - \frac{1}{n^2}}{\frac{7}{n^2} + \frac{2}{n} - 8} = -\frac{1}{2}.$$

Example 3: Calculate the limit: $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n})$.

Solve. Let numerator rationalization,we get

$$\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty} \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0.$$

Case 3:Use the Sandwich (Forced convergence) theorem

Sandwich (Forced convergence) theorem[2]: Suppose $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ is three sequences, if $\exists N \in N_+, \forall n > N$,let $a_n \le b_n \le c_n$,and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = l$, then $\lim_{n\to\infty} b_n = l$.

Method: We will first ask sequence amplification and the narrowing of two equal sequence limit value, then the convergence theorem tells us that the limit value of the sequence in question exists and is equal to the limit value of the newly obtained sequence. This method is often considered first in cases where multiple fractions are added. If not solved,consider using the definition of definite integrals, which I'll talk about later.

Example 3: Calculate the limit: $\lim_{n\to\infty} (\frac{1}{n^2+n+1} + \frac{2}{n^2+n+2} + \frac{3}{n^2+n+3} + \cdots\cdots + \frac{n}{n^2+n+n})$.

Solve. Let $x_n = \frac{1}{n^2+n+1} + \frac{2}{n^2+n+2} + \frac{3}{n^2+n+3} + \cdots\cdots + \frac{n}{n^2+n+n}$,

Then $$\frac{1+2+\cdots\cdots+n}{n^2+n+1} \ge x_n \ge \frac{1+2+\cdots\cdots+n}{n^2+n+n},$$

So $$\frac{n(n+1)}{2(n^2+n+1)} \ge x_n \ge \frac{n(n+1)}{2(n^2+2n)}$$

Because $$\lim_{n\to\infty}\frac{n(n+1)}{2(n^2+n+1)}=\frac{1}{2}=\lim_{n\to\infty}\frac{n(n+1)}{2(n^2+2n)}.$$

According to Forced convergence theorem,we get

$$\lim_{n\to\infty}\left(\frac{1}{n^2+n+1}+\frac{2}{n^2+n+2}+\frac{3}{n^2+n+3}+\cdots\cdots+\frac{n}{n^2+n+n}\right)=\frac{1}{2}$$

$$\lim_{n\to\infty}\frac{4n^2-5n-1}{7+2n-8n^2}=\lim_{n\to\infty}\frac{4-\frac{5}{n}-\frac{1}{n^2}}{\frac{7}{n^2}+\frac{2}{n}-8}=-\frac{1}{2}.$$

Case 4:Use the necessary condition for several series to converge

For a series whose limit is known to be zero, it can be proved by the necessary condition for several series to converge.

the necessary condition for several series to converge[2]: If the series $\sum_{n=1}^{\infty}u_n$ converges, then $\lim_{n\to\infty}u_n=0$.

Method: A summation sign is added before the sequence u_n, that is $\sum_{n=1}^{\infty}u_n$. Construct a several series,and then the convergence of the series is determined by the criterion of the convergence of the series.If it's a positive series, it can be judged by the method of the partial sum sequence are bounded,comparison test, ratio test and Cauchy test (radical test).If it is a general series, it can be judged by Abel test and Dirichlet test.If it is a staggered series, it can be judged by leibnitz's test.

Example 5: Calculate the limit of the sequence $x_n=\frac{11\cdot 12\cdot 13\cdots\cdots(n+10)}{2\cdot 5\cdot 8\cdots\cdots(3n-1)}(n=1,2,3\cdots\cdots)$.

Solve. Let's think about the positive series $\sum_{n=1}^{\infty}x_n$,

Because $$\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{n+11}{3n+2}=\frac{1}{3}<1,$$

So we know that the series $\sum_{n=1}^{\infty}x_n$ converges,then $\lim_{n\to\infty}x_n=0$.

Case 5:Use the Stolz formula

Stolz formula[6]: Suppose that the sequence $\{y_n\}$ increases monotonically and tends to $+\infty$, while

$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = A$ (finite or infinite), then $\lim_{n\to\infty} \frac{x_n}{y_n} = A$.

Method: The key is to determine whether the denominator of the fraction satisfies the monotone approach to $+\infty$. This method is applicable to the case where the numerator is the sum of the first n terms of a sequence,that is $x_1 + x_2 + \cdots\cdots + x_n$.

Example 6: Set $\lim_{n\to\infty} x_n = a$, $\xi_n = \frac{x_1 + x_2 + \cdots\cdots + x_n}{n}$, prove: $\lim_{n\to\infty} \xi_n = a$.

Proof. Let $z_n = x_1 + x_2 + \cdots\cdots + x_n$, $y_n = n$. Then $\lim_{n\to\infty} y_n = +\infty$

While

$$\lim_{n\to\infty} \frac{z_{n+1} - z_n}{y_{n+1} - y_n}$$

$$= \lim_{n\to\infty} \frac{(x_1 + x_2 + \cdots\cdots + x_{n+1}) - (x_1 + x_2 + \cdots\cdots + x_n)}{(n+1) - n}$$

$$= \lim_{n\to\infty} x_{n+1}$$

$$= a$$

According to the Stolz formula, we get $\lim_{n\to\infty} \xi_n = a$.

Case 6: Using the function limit method, and then using the summation principle (Heine's theorem)

Heine's theorem[1]: If f is defined in $U^0(x_o, \delta)$, $\lim_{x\to x_0} f(x)$ exists if and only if the limit $\lim_{n\to\infty} f(x_n)$ exists and is equal .For any sequence $\{x_n\}$ that involve $U^0(x_o, \delta')$ and with a limit of x_0 .

Note 1: Heine's theorem can also be simplified as:

$$\lim_{x\to x_0} f(x) = A \Leftrightarrow \text{For any } x_n \to x_0 \ (n \to \infty) \text{, we have } \lim_{n\to\infty} f(x_n) = A .$$

Note 2: If you can find a series $\{x_n\}$ with x_0 as the limit, let $\lim_{n\to\infty} f(x_n)$ does not exist, or find two series $\{x_n'\}$ and $\{x_n''\}$ with x_0 as the limit, make $\lim_{n\to\infty} f(x_n')$ and $\lim_{n\to\infty} f(x_n'')$ exist but are not equal, then $\lim_{n\to\infty} f(x_n)$ does not exist.

The function is continuous, differentiable, integrable and so on, while the sequence is a special function, but their domain is different. We can associate the limit of sequence with the limit of function by Heine's theorem, and we can transform the limit of sequence into the limit of function by virtue of these good properties of function, so that the problem can be simplified.

Example 7: Calculate the limit: $\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n, (a \ge 0, b \ge 0)$.

Solve. (i) When either a or b is 0, for example $a=0$, then $\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n = \lim_{n\to\infty}\frac{b}{2^n} = 0 = \sqrt{ab}$.

(ii) When $a>0, b>0$, let $y=\left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}$, then $\ln y = \frac{1}{x}\cdot\ln\frac{a^x+b^x}{2}$.

$$\lim_{x\to0}\ln y = \lim_{x\to0}\frac{1}{x}\cdot\ln\frac{a^x+b^x}{2} = \lim_{x\to0}\frac{2}{a^x+b^x}\cdot\left(\frac{a^x\ln a+b^x\ln b}{2}\right) = \frac{1}{2}(\ln a+\ln b) = \ln\sqrt{ab}.$$

Case 7: Use the monotone bounded theorem

monotone bounded theorem[2]: A monotone bounded sequence must have a limit.

This method is often suitable for calculating the limit of recursive sequence

Method: Step1: From $x_n = f(x_{n-1})$ to judge the sequence $\{x_n\}$ monotony.We can use the difference or quotient method, that is, to judge the positive or negative of $x_n - x_{n-1}$, or to judge whether $\frac{x_n}{x_{n-1}}$ is larger or smaller than 1.You can also judge the magnitude relationship between x_1 and x_2 and then use mathematical induction.

Step2: Judge that the sequence $\{x_n\}$ is bounded.Assume that the limit of $\{x_n\}$ exists and estimate limit a, which is usually the upper or lower bound of $\{x_n\}$.And then judge x_1 is greater than or less than the upper or lower bound .It can be proved by mathematical induction.

Step3: Find the limit a of the sequence $\{x_n\}$.

Example 8: Set $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2a_n}$, $n = 1,2,\cdots\cdots$, prove: $\{a_n\}$ is convergent, and find its limit.

Proof. From known conditions, we get $0 < a_1 < 2$,

If we assume that $0 < a_n < 2$, then $0 < a_{n+1} < \sqrt{2\cdot 2} = 2$,

By mathematical induction, we know that $0 < a_n < 2$,

Now let's prove that $a_{n+1} > a_n$,

In fact, $\frac{a_{n+1}}{a_n} = \frac{\sqrt{2a_n}}{a_n} > \sqrt{\frac{2}{2}} = 1$,therefore $a_{n+1} > a_n$.

So the monotonically increasing sequence $\{a_n\}$ has an upper bound,

So $\lim\limits_{n\to\infty} a_n = l$ exists, then $l = \sqrt{2l}$, we get $l = 2$ or $l = 0$ (rounding),

Therefore $\lim\limits_{n\to\infty} a_n = 2$.

Case 8: Use the definite integral definition

definite integral definition[1]: Let $f(x)$ be a function defined on $[a,b]$, with $n-1$ points inserted in (a,b), $a = x_0 < x_1 < x_2 < \cdots\cdots < x_{n-1} < x_n = b$ .If $\|T\| = \max\limits_{1\le i\le n}(x_i - x_{i-1})$, for $\forall \varepsilon > 0$, there is always $\exists \delta > 0$, so that for any partition T on $[a,b]$,and any $\xi_i \in [x_{i-1}, x_i]\ (i = 1,2,\cdots\cdots,n)$, as long as its fineness is $\|T\| < \delta$, there are real Numbers J , so that $\left|\sum_{i=1}^{n} f(\xi_i)\Delta x_i - J\right| < \varepsilon$ is true.Then the function f is integrable on the interval $[a,b]$ .The number J is called the definite integral of f over $[a,b]$, or the Riemann integral, sign as $J = \int_a^b f(x)dx = \lim\limits_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta x_i$.

It is clear that definite integrals are defined in terms of limits.If the limit of the sequence is the sum of a particular series of integrals of some integrable function, then we can convert the limit of the sequence to a definite integral. What we've seen in general is that the $[0,1]$ interval is divided into n equal parts. The key is to use $\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f(\frac{i}{n}) = \int_0^1 f(x)dx$, but there are special cases.

Method: Step1：Take a sequence of transformations, and transform a_n into a special form of integral sum

$$a_n = \sum_{i=1}^{n} f(\frac{i}{n})\frac{1}{n}.$$

Step2：Find the upper and lower bounds of the integral function f , and let $\frac{i}{n} = x$, then the integral function is $f(\frac{i}{n}) = f(x)$.

Step3：By the definition of the definite integral, we get $a_n = \lim_{n\to\infty}\sum_{i=1}^{n} f(\frac{i}{n})\frac{1}{n}$.

Step4：Calculate the definite integral.

Example 8： Calculate the limit: $\lim_{n\to\infty}(\frac{1}{n+1}+\frac{1}{n+2}+\cdots\cdots+\frac{1}{n+n})$

Solve.

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots\cdots+\frac{1}{n+n}=\frac{1}{n}\left[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\cdots\cdots+\frac{1}{1+\frac{n}{n}}\right]=\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}\cdot\frac{1}{n} \quad (1)$$

Let $f(x) = \frac{1}{1+x}$ ， $0 \le x \le 1$ ， by the definition of a definite integral, we get

$$\int_0^1 \frac{1}{1+x}dx = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}\cdot\frac{1}{n} \quad (2)$$

Because

$$\int_0^1 \frac{1}{1+x}dx = \ln 2 . \quad (3)$$

By (1),(2) ,(3), we get

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots\cdots+\frac{1}{n+n}\right)=\ln 2 .$$

Case 8: Use equivalent substitution

When the limit is very complex, in order to solve it or convert it to a known limit, we can introduce a new variable according to the characteristics of the limit equation, so that the limit obtained can be converted to a new limit.

***Example 9*:** Let a_1 and b_1 are any two positive numbers, and $a_1 \le b_1$, set

$$a_n=\frac{2a_{n-1}b_{n-1}}{a_{n-1}+b_{n-1}}, b_n=\sqrt{a_{n-1}b_{n-1}}, (n=2,3\cdots\cdots) \tag{4}$$

Prove: Sequences $a_1,\ a_2\cdots\cdots$ and $b_1,\ b_2\cdots\cdots$ converge and have the same limit.

Proof. Let $c_n=\frac{1}{a_n},\ d_n=\frac{1}{b_n}$, then

$$c_n=\frac{c_{n-1}+d_{n-1}}{2},\quad d_n=\sqrt{c_{n-1}d_{n-1}},\ (n=2,3\cdots\cdots),\ c_1\ge d_1 .$$

so
$$c_n=\frac{c_{n-1}+d_{n-1}}{2}\ge\sqrt{c_{n-1}d_{n-1}}=d_n . \tag{5}$$

from (5), we have
$$c_n=\frac{c_{n-1}+d_{n-1}}{2}\le\frac{2c_{n-1}}{2}=c_{n-1},$$

then $\{c_n\}$ is monotone decreasing.

While $c_n\ge c_{n-1}\ge\cdots\cdots\ge c_1\ge d_1$,the sequence $\{c_n\}$ has a lower bound, so $\lim_{n\to\aleph} c_n=\frac{1}{l}$ (exist).

then $|c_n|<M$ (bounded), $-M<c_n<M$. $d_n=\sqrt{c_{n-1}d_{n-1}}\ge d_{n-1}$, so $\{d_n\}$ is monotone decreasing.

and $d_n\le c_n<M$,so $\{d_n\}$ is monotonically increasing with an upper bound.

$$\lim_{n\to\aleph} d_n=\frac{1}{s}\text{(exist)},\quad \lim_{n\to\infty} a_n=\lim_{n\to\infty}\frac{1}{c_n}=l,\quad \lim_{n\to\infty} b_n=\lim_{n\to\infty}\frac{1}{d_n}=s .$$

We take the limit of both sides of equation (5),and we have

$$\begin{cases} l=\dfrac{2ls}{l+s}, \\ l=\sqrt{ls} \end{cases}$$

Solve this equation, and we get $s=l$ .So $\lim_{n\to\infty} a_n=\lim_{n\to\infty} b_n$.

Since there are many ways to find the limit of a function, we also need to choose the most appropriate, easiest and fastest method according to different types of problems.The following introduces several typical methods to find the limit of the function, so that the reader can choose the appropriate method when applying the principle of generalization.

III. A METHOD OF CALCULATING THE LIMIT OF A FUNCTION

Case 1:L'Hospital's rule

L'Hospital's rule[1]: If the functions f and g satisfy :

(i) $\lim_{x\to x_0} f(x)=\lim_{x\to x_0} g(x)=0$;

(ii)Both are derivable in some hollow neighborhood $U^{o}(x_0)$ of point x_0, and $g'(x)\neq 0$;

(iii) $\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=A$ (A can be a real number, $+\infty$ or $-\infty$)

then $\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}=A$.

Method: To see if the limit of the function is an infinitive limit.For type $\frac{0}{0}$ and $\frac{\infty}{\infty}$, We can use L 'Hopital's rule directly.For type $0\cdot\infty$,1^{∞} ,0^{0}, ∞^{0}, $\infty-\infty$,we can transform them into type $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Example 10: Calculate the limit: $\lim_{x\to+\infty}\frac{\ln x}{x^{\alpha}}(\alpha>0)$.

Solve. Because

$$\lim_{x\to+\infty}\frac{\ln x}{x^{\alpha}}(\alpha>0)\ \left(\frac{\infty}{\infty}\right)=\lim_{x\to+\infty}\frac{\frac{1}{x}}{\alpha x^{\alpha-1}}=\lim_{x\to+\infty}\frac{1}{\alpha x^{\alpha}}=0\ .$$

According to the L'Hospital's rule,we get $\lim_{n\to\infty}\frac{\ln n}{n^{\alpha}}=0$.

Case 2:Use the substitution of the equivalent infinitesimal (large) to find the limit

When $x\to 0$, the common equivalent infinitesimal is as follows:

$$\sin x\sim x\ ;\quad \tan x\sim x\ ;\quad \arcsin x\sim x\ ;\quad \arctan x\sim x\ ;\quad e^{x}-1\sim x\ ;$$

$$\ln(1+x)\sim x\ ;\ 1-\cos x\sim\frac{x^2}{2}\ ;\quad a^x-1\sim x\ln a\quad;\quad(1+x)^\alpha-1\sim\alpha x\ .$$

Example 11: Calculate the limit: $\lim\limits_{x\to 0}\dfrac{\arctan x}{\ln(1+\sin x)}$

Solve. By equivalent infinitesimal, $\arctan x\sim x$, $\ln(1+\sin x)\sim\sin x$ $(x\to 0)$,

then
$$\lim_{x\to 0}\frac{\arctan x}{\ln(1+\sin x)}=\lim_{x\to 0}\frac{x}{\sin x}=1\ .$$

Case 3:Using Taylor's formula

The commonly used Taylor formula is as follows:

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}+o(x^5)\ ,$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}+o(x^6)\ ,$$

$$\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+o(x^6)\ ,$$

$$e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+o(x^n)\ ,$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots+(-1)^{n-1}\frac{x^n}{n}+o(x^n)\ .$$

Example 12: Calculate the limit: $\lim\limits_{x\to 0}\dfrac{x\ln(1+x)}{x-\ln(1+x)}$.

Solve. By $\ln(1+x)=x-\dfrac{x^2}{2}+o(x^2)$, $\ln(1+x)\sim x$ $(x\to 0)$, we have $$\lim_{x\to 0}\frac{x\ln(1+x)}{x-\ln(1+x)}=\lim_{x\to 0}\frac{x^2}{x-(x-\frac{x^2}{2}+o(x^2))}=2.$$

Case 4: Using two important limit formulas

two important limit formulas[1]: $\lim_{x\to 0}\frac{\sin x}{x}=1,\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$.

Example 13: Calculate the limit: $\lim_{x\to\infty}(\frac{x^2+1}{x^2-1})^{x^2}$.

Solve. $$\lim_{x\to\infty}(\frac{x^2+1}{x^2-1})^{x^2}=\lim_{x\to\infty}(\frac{1+\frac{1}{x^2}}{1-\frac{1}{x^2}})^{x^2}=\lim_{x\to\infty}\frac{(1+\frac{1}{x^2})^{x^2}}{(1-\frac{1}{x^2})^{x^2}}=\lim_{x\to\infty}(1+\frac{1}{x^2})^{x^2}(1-\frac{1}{x^2})^{-x^2}$$

$$=\lim_{x\to\infty}(1+\frac{1}{x^2})^{x^2}\cdot\lim_{n\to\infty}(1+\frac{1}{-x^2})^{-x^2}=e\cdot e=e^2.$$

Case 5: Using logarithmic

The limit of infinitive can be calculated by logarithm,such as type $1^\infty,\infty^0$.

Example 14: Calculate the limit: $\lim_{x\to+\infty}x^{\frac{1}{x}}$.

Solve. $$\lim_{x\to+\infty}x^{\frac{1}{x}}=\lim_{x\to+\infty}e^{\ln x^{\frac{1}{x}}}=e^{\lim_{n\to\infty}\frac{1}{x}\ln x}=e^{\lim_{n\to\infty}\frac{\frac{1}{x}}{1}}=e^0=1.$$

IV. THREE METHODS OF LIMITING RECURSION OF KNOWN SEQUENCE

Case 1: We first determine the existence of this limit, and then calculate the limit

Method: First use monotonic definite theorem to prove the existence of the sequence limit , and then solve it.

Example 15: Set $a>0$, $0<x_1<a$, $x_{n+1}=x_n(2-\frac{x_n}{a}),\ n\in N$, prove: $\{x_n\}$ is convergent, and find its limit.

Proof. Let's prove by mathematical induction that

$$0 < x_n < a, n \in N \tag{6}$$

When $n = 1$, the conclusion is true.Assuming that the conclusion holds for n, then we prove $n+1$,

Because $$x_{n+1} = x_n(2 - \frac{x_n}{a}) = -\frac{1}{a}(x_n - a)^2 + a,$$

so $$0 < x_{n+1} < a.$$

That is to say, formula (6) holds.

then $$\frac{x_{n+1}}{x_n} = 2 - \frac{x_n}{a} > 2 - \frac{a}{a} = 1.$$

So the monotonically increasing sequence $\{x_n\}$ has an upper bound, $\lim\limits_{n\to\infty} x_n = b$ exists,

We take the limit of both sides of equation $x_{n+1} = x_n(2 - \frac{x_n}{a})$,and we have

$$b = b(2 - \frac{b}{a}) \tag{7}$$

From formula (6) and $\{x_n\}$ monotonically increasing, it is obvious that $b \neq 0$, and $b = a$ can be obtained .

from (7).Therefore $\lim\limits_{n\to\infty} x_n = a$.

Case 2:Use compression mapping

compression mapping[6]: If the sequence $\{x_n\}$ satisfies the condition $|x_n - x_{n-1}| \leq r|x_{n-1} - x_{n-2}|$ $n = 3,4,\cdots(0 < r < 1)$,then it is called the compressed variation sequence (abbreviated as the compressed sequence).Any compressed sequence must converge.

Method: Step1：According to $x_n = f(x_{n-1})$, first judge whether sequence $\{x_n\}$ is a compressed sequence.That is to say whether $|f'(x)|$ is less than 1.

Step2：Suppose the limit of $\{x_n\}$ is a.

Step3：Find the limit a of the sequence $\{x_n\}$.

Example 16: Set $f(x)=\dfrac{x+2}{x+1}$,the sequence $\{x_n\}$ is defined by the following recursive formula: $x_0=1$, $x_{n+1}=f(x_n)(n=0,1,2,\cdots)$ .Prove: $\lim\limits_{n\to\infty} x_n=\sqrt{2}$.

Proof. By $x_0=1$,

$$x_{n+1}=\frac{x_n+2}{x_n+1}=1+\frac{1}{x_n+1}\geq 1.\quad (n=0,1,2,\cdots) \tag{8}$$

$$|f'(x)|=|-\frac{1}{(x+1)^2}|\leq\frac{1}{2}\text{(when } x\geq 1\text{)}.$$

so $$|x_{n+1}-x_n|=|f(x_n)-f(x_{n-1})|=|f'(\xi)|\cdot|x_n-x_{n-1}|\leq\frac{1}{2}|x_n-x_{n-1}|.$$

Then $\{x_n\}$ is the compressed sequence.

So $\lim\limits_{n\to\infty} x_n=l$ exists,from (8), we have $l=\dfrac{l+2}{l+1}$,that is $l^2=2$.

Then $l=\sqrt{2}$ or $l=-\sqrt{2}$ (rounding),

Therefore $\lim\limits_{n\to\infty} x_n=\sqrt{2}$.

Case 3: Writing down the general term and then calculate the limit

Method:Obtain the general term expression of the sequence based on the known recursion relation, and then find its limit.

Example 17: Let $x_0=0, x_1=1, x_{n+1}=\dfrac{x_n+x_{n-1}}{2}$, then calculate the limit: $\lim\limits_{n\to\infty} x_n$.

Solve. $$x_{n+1}-x_n=\frac{x_n+x_{n-1}}{2}-x_n=-\frac{x_n-x_{n-1}}{2}.$$

Apply this result repeatedly,

$$x_{n+1}-x_n=(-\frac{1}{2})^n(x_1-x_0)=\frac{(-1)^n}{2^n}(n=1,2,\cdots).$$

Therefore

$$x_{n+1}=(x_{n+1}-x_n)+(x_n-x_{n-1})+\cdots+(x_1-x_0)+x_0=(-\frac{1}{2})^n+(-\frac{1}{2})^n+\cdots+1=\frac{1-(-\frac{1}{2})^{n+1}}{1-(-\frac{1}{2})},$$

So $$\lim_{n\to\infty} x_n=\lim_{n\to\infty}\frac{1-(-\frac{1}{2})^n}{1-(-\frac{1}{2})}=\frac{2}{3}.$$

V. CONCLUSIONS

Limit is a good practice of leap thinking in dialectical thinking logic. The method of using limit is a remarkable sign that mathematical analysis is different from elementary mathematics.Almost all concepts in mathematical analysis cannot be separated from limits, and a lot of knowledge learned later is defined in terms of limits. Therefore, the concept of limits is a very important and fundamental concept in mathematical analysis.In general, the limit problems that we encounter are tricky to find. If we have a problem proving that the limit is equal to some constant, we need to consider the definition;In order to solve the limit problem of the sum of multiple factors, we need to consider two methods: the pinch theorem and the definition of definite integral;When we encounter the limit problem of recursive relations, we must first consider the monotonically defined principle...Therefore, we have to choose the most appropriate solution according to the different types of problems.

In this paper, the basic concepts and operation properties of the limit are given, various representative methods of solving the limit are introduced in detail, and some appropriate examples are listed.This can concentrate on the essence of various methods, but also has a comparative, convenient for readers to understand each method and flexible use.

REFERENCES

[1] Mathematical Analysis, East China Normal University.Mathematical analysis (3rd edition) [M]. Shanghai.Higher Education Press.2001.

[2] Liu Yulian, Fu Peiren, Lin Ding, Yuan Dexin, Liu Ning.Handout on Mathematical Analysis (5th edition) [M]. Beijing.Higher education Publisher, 2008.

[3] Wang Yan, Zhou Wenli, Dong Minghui.Several methods to calculate the limit [J]. Journal of Xi 'an Eurasia University,2005.

[4] Pei Liwen.Typical problems and methods in Mathematical analysis (2nd edition) [M].Beijing.Higher Education Press.2006

[5] Cao Xuefeng, Sun Xinrong.Application of infinitesimal quantities in calculating limits [J].Mathematics Learning and Research (Teaching and Research edition), 2008, (01).

[6] Qian Jilin.The Essence of Mathematical Analysis [M]. Wuhan.Chongwen Bookstore. 2011.

[7] Chen Chuanzhang, Zhu Xue-yan et al. Mathematical Analysis. Department of Mathematics, Fudan University [M].Higher Education Press.2006.