Stability and Hopf Branch of Partial Profit Model with Three Time Delays

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Abstract — *This paper mainly investigated a partial profit model with three discrete time delays. With two combinations of three time delays* τ_1, τ_2, τ *as bifurcation parameters, we investigated the local stability of equilibrium point of the system and the sufficient condition for the existence of Hopf branch in two cases. Furthermore, we obtained the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions by using the center manifold theorem and normal form theory.At last, some numerical simulation results are confirmed that the feasibility of the theoretical analysis.*

Keywords—*partial profit model, stability,time delays ,center manifold theorem,Hopf bifurcation*

I. INTRODUCTION

Population dynamics is a very important branch of biomathematics. It has been widely used in ecology, especially in the fields of animal and plant protection and ecological environment management and development. Cooperative systems are very important to describe population interaction in the study of population dynamics. In recent years, there have been a large number of excellent research achievements on population cooperation system at home and abroad[1-9].Given the cooperation of the two species, there is a special relationship:interspecific interactions benefits only one, and the other cannot survive independently. Such a two-species cooperative system (partial profit system) is also an important aspect of cooperative system. Jiang Q Y et al.[10] took into account that plants can survive independently, and the pollination of insects can increase the growth rate of plants. Noting that insects feeding on pollen cannot live alone without plants, they established the following model:

$$\begin{cases} \dot{x}(t) = \gamma_1 x(t)[1 - \dfrac{x(t)}{N_1} + \dfrac{\sigma_1 y(t)}{N_2}], \\ \dot{y}(t) = \gamma_2 y(t)[-1 + \dfrac{\sigma_2 x(t)}{N_1} - \dfrac{y(t)}{N_2}]. \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ denote the population density of the first and the second population at time t ,respectively. $\gamma_i, i = 1,2$ is the intrinsic growth rate of the i-th population; $N_i, i = 1,2$ represents the

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environmental capacity of the i-th population;and $\sigma_i, i = 1,2$ describes the cooperation efficiency of the two populations.

It is well known that the interactions between species in nature are delayed, and the delay effect complicates the dynamical behavior of the system.. Jiang Y S et al. considered the partial profit cooperative system with the following time-delay effects:

$$\begin{cases} \dot{x}(t) = \gamma_1 x(t)[1 - \dfrac{x(t)}{N_1} + \dfrac{\sigma_1 y(t-\tau_1)}{N_2}], \\ \dot{y}(t) = \gamma_2 y(t)[-1 + \dfrac{\sigma_2 x(t-\tau_2)}{N_1} - \dfrac{y(t)}{N_2}]. \end{cases} \tag{2}$$

where τ_1 and τ_2 represent growth time delay of the two populations respectively.

Since biological populations all have gestation period, this paper, based on models (2),this paper introduces time delay τ as the growth incubation time of the two populations (the gestation period), and obtains a partial profit model with three time delays. The specific model is as follows:

$$\begin{cases} \dot{x}(t) = \gamma_1 x(t)[1 - \dfrac{x(t-\tau)}{N_1} + \dfrac{\sigma_1 y(t-\tau_1)}{N_2}], \\ \dot{y}(t) = \gamma_2 y(t)[-1 + \dfrac{\sigma_2 x(t-\tau_2)}{N_1} - \dfrac{y(t-\tau)}{N_2}]. \end{cases} \tag{3}$$

where $x(t)$, $y(t)$ $\gamma_i, i = 1,2$, , $N_i, i = 1,2$, $\sigma_i, i = 1,2$, τ_1 and τ_2 have the same meaning as model (1),the time delay τ represents the incubation time of the two populations.

First, let the equilibrium point of model (3) be $E(x_0, y_0)$, so it satisfies the following equation:

$$1 - \frac{x_0}{N_1} + \frac{\sigma_1 y_0}{N_2} = 0 \quad ; \quad -1 + \frac{\sigma_2 x_0}{N_1} - \frac{y_0}{N_2} = 0$$

That is

$$x_0 = \frac{N_1(1-\sigma_1)}{1-\sigma_1\sigma_2}, \qquad v_0 = \frac{N_2(\sigma_2 - 1)}{1-\sigma_1\sigma_2}.$$

Theorem 1. For system (3), if (H_1) $\sigma_1 < 1, \sigma_2 > 1, \sigma_1\sigma_2 < 1$ so, the system has a unique positive equilibrium point $E(\dfrac{N_1(1-\sigma_1)}{1-\sigma_1\sigma_2}, \dfrac{N_2(\sigma_2-1)}{1-\sigma_1\sigma_2})$.

II. STABILITY AND LOCAL HOPF BIFURCATION ANALYSIS

In this section, we focus on the problems of the Hopf bifurcation and stability for the system(3).

Let $\bar{x}(t) = x(t) - x_0, \bar{y}(t) = y(t) - y_0$,then the linearized approximation equation corresponding to model (3) at the equilibrium point $E(x_0, y_0)$ is:

$$\begin{cases} \dfrac{d\bar{x}(t)}{dt} = -\dfrac{\gamma_1(1-\sigma_1)}{1-\sigma_1\sigma_2}\bar{x}(t-\tau) + \dfrac{N_1\gamma_1\sigma_1(1-\sigma_1)}{N_2(1-\sigma_1\sigma_2)}\bar{y}(t-\tau_1), \\ \dfrac{d\bar{y}(t)}{dt} = -\dfrac{\gamma_2(\sigma_2-1)}{1-\sigma_1\sigma_2}\bar{y}(t-\tau) + \dfrac{N_2\gamma_2\sigma_2(\sigma_2-1)}{N_1(1-\sigma_1\sigma_2)}\bar{x}(t-\tau_2). \end{cases} \tag{4}$$

The correspoding characteristic equation of system (3) is as follows.

$$\lambda^2 + P\lambda e^{-\lambda\tau} + Qe^{-2\lambda\tau} - Re^{-\lambda(\tau_1+\tau_2)} = 0. \tag{5}$$

Where

$$P = \frac{\gamma_1(1-\sigma_1)}{1-\sigma_1\sigma_2} + \frac{\gamma_2(\sigma_2-1)}{1-\sigma_1\sigma_2} > 0,$$
$$Q = \frac{\gamma_1\gamma_2(1-\sigma_1)(\sigma_2-1)}{(1-\sigma_1\sigma_2)^2} > 0,$$
$$R = \frac{\gamma_1\gamma_2\sigma_1\sigma_2(1-\sigma_1)(1-\sigma_2)}{(1-\sigma_1\sigma_2)^2} < 0.$$

In order to study the stability and branching of the equilibrium point E of the system, we only need to discuss the distribution of the roots of the characteristic equation (5).If all the roots of equation (5) have negative real parts, the equilibrium point E is asymptotically stable. If one root of the equation contains positive real parts, the equilibrium point E is unstable. Since the dynamic properties of differential equations with multiple delays are very complex, we discuss system (3) with two combinations of three time .

Case 1: $\tau_1 \neq \tau_2$ and $\tau = \tau_1 + \tau_2 > 0$

Lemma 1. For system (3),when $\tau = 0$,the equilibrium E is stable.

Proof. When $\tau = 0$,the characteristic equation of system (5) becomes

$$\lambda^2 + P\lambda + Q - R = 0. \tag{6}$$

It is known that the following formula is true when $\sigma_1 < 1, \sigma_2 > 1, \sigma_1\sigma_2 < 1$ is satisfied,

$$P = \frac{\gamma_1(1-\sigma_1)}{1-\sigma_1\sigma_2} + \frac{\gamma_2(\sigma_2-1)}{1-\sigma_1\sigma_2} > 0$$

$$Q - R = \frac{\gamma_1\gamma_2(1-\sigma_1)(\sigma_2-1)}{(1-\sigma_1\sigma_2)^2} - \frac{\gamma_1\gamma_2\sigma_1\sigma_2(1-\sigma_1)(1-\sigma_2)}{(1-\sigma_1\sigma_2)^2} > 0\ ,$$

the two roots of equation (6) always have negative real parts. So when $\tau = 0$, the equilibrium point of system (3) is asymptotically stable.

Next, we discuss the distribution of the characteristic roots of equation (5) when $\tau \neq 0$.

Lemma 2. If (H_2) $Q^4 - Q^2R^2 < 0$ is satisfied, the equation(7) has two pairs of purely imaginary roots

$\pm i\omega_0$ when $\tau = \tau^j = \tau_{k_0}^j$, where $\tau_k^j = \frac{1}{\omega_k}[\arccos(\frac{R}{Q-\omega_k^2}) + 2j\pi]$, $k = 1,2,3,4$; $j = 0,1,2,\cdots$;

$\tau^j = \tau_{k_0}^j = \min\{\tau_k^j : k = 1,2,3,4\}(j = 0,1,2,\cdots)$; $\omega_0 = \omega_{k_0}$.

Proof. When $\tau \neq 0$, the characteristic equation of system (3) becomes

$$\lambda^2 + (P\lambda - R)e^{-\lambda\tau} + Qe^{-2\lambda\tau} = 0. \tag{7}$$

First, we assume that $\lambda = i\omega$ ($\omega > 0$) is a root of the characteristic equation (7), then it satisfies the following equation

$$-\omega^2 + i\omega e^{-i\omega\tau}P + Qe^{-2i\omega\tau} - Re^{-i\omega\tau} = 0. \tag{8}$$

That is $$-\omega^2(\cos\omega\tau + i\sin\omega\tau) + i\omega P + Q(\cos\omega\tau - i\sin\omega\tau) - R = 0. \tag{9}$$

The separation of the real and imaginary parts, it follows

$$\begin{cases}(\omega^2 - Q)\cos\omega\tau = -R, \\ (\omega^2 + Q)\sin\omega\tau = P\omega.\end{cases} \tag{10}$$

From (10) we obtain

$$R^2(\omega^2 + Q)^2 + P^2\omega^2(\omega^2 - Q)^2 = (\omega^2 + Q)^2(\omega^2 - Q)^2. \tag{11}$$

Simplify,we get $$\omega^8 + m_3\omega^6 + m_2\omega^4 + m_1\omega^2 + m_0 = 0 \tag{12}$$

where $m_3 = -P^2$; $m_2 = 2P^2Q - R^2 - 2Q^2$; $m_1 = -2QR^2 - P^2Q^2$; $m_0 = Q^4 - Q^2R^2$.

Let $u = \omega^2$, then (12) becomes

$$f(u) = u^4 + m_3u^3 + m_2u^2 + m_1u + m_0 = 0. \tag{13}$$

Since $\lim_{u\to 0} f(u) = m_0$, $\lim_{u\to 0} f(u) = +\infty$, as long as $m_0 = Q^4 - Q^2R^2 < 0$, that is, condition (H_2) is true.

Then equation (13) has at least one positive real root.

The following assumes that the condition (H_2) is always satisfied, without loss of generality, suppose that fourth-order equation (13) has 4 positive roots, denoted as $u_k (k = 1,2,3,4)$ respectively, then equation (12) has 4 positive roots $\omega_k = \sqrt{u_k} (k = 1,2,3,4)$.

From(10), we can get

$$\cos \omega_k \tau = \frac{R}{Q - \omega_k^2} \tag{14}$$

So

$$\tau_k^j = \frac{1}{\omega_k}[\arccos(\frac{R}{Q - \omega_k^2}) + 2j\pi], k = 1,2,3,4; j = 0,1,2,\cdots \tag{15}$$

So when $\tau = \tau_k^j$, equation (7) has a pair of pure virtual roots $\pm i\omega_0$.

Let $\tau^j = \tau_{k_0}^j = \min\{\tau_k^j : k = 1,2,3,4\}(j = 0,1,2,\cdots); \omega_0 = \omega_{k_0}$.

Then, the minimum value of τ in equation (7) with a pure virtual root is τ^0 . This completes the proof.

Lemma 3. Let $\lambda(\tau^j) = \alpha(\tau^j) + i\omega(\tau^j)$ be a root of the characteristic equation (7),which satisfies $\alpha(\tau^j) = 0$ and $\omega(\tau^j) = \omega_0$,then we have the following transversality condition $\mathrm{Re}(\frac{d\lambda}{d\tau})\Big|_{\tau=\tau^j} > 0$ is satisfied.

Proof. By differentiating both sides of equation (7) with regard to τ and applying the implicit function theorem, we have :

$$\frac{d\lambda}{d\tau} = \frac{Q\lambda e^{-\lambda\tau} - \lambda^3 e^{\lambda\tau}}{(2\lambda + \lambda^2\tau)e^{\lambda\tau} + P - Q\tau e^{-\lambda\tau}},$$

Therefore,

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{2\lambda e^{\lambda\tau} + P}{Q\lambda e^{-\lambda\tau} - \lambda^3 e^{\lambda\tau}} - \frac{\tau}{\lambda},$$

So

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Big|_{\tau=\tau^j}$$

$$=\frac{(P-2\omega_0\sin\omega_0\tau^j)(-P\omega_0^2+2Q\omega_0\sin\omega_0\tau^j)+2\omega_0\cos\omega_0\tau^j(-R\omega_0+2Q\omega_0\cos\omega_0\tau^j)}{(-P\omega_0^2+2Q\omega_0\sin\omega_0\tau^j)^2+(-R\omega_0+2Q\omega_0\cos\omega_0\tau^j)^2}$$

$$=\frac{P^2\omega_0^2(Q-\omega_0^2)^4+2R^2\omega_0^2(3Q+\omega_0^2)(\omega_0^2+Q)^2}{(\omega_0^2+Q)^2(\omega_0^2-Q)^2[(-P\omega_0^2+2Q\omega_0\sin\omega_0\tau^j)^2+(-R\omega_0+2Q\omega_0\cos\omega_0\tau^j)^2]}$$

$$>0,$$

Since $\mathrm{sign}\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)\right]_{\tau=\tau^j}=\mathrm{sign}\left[\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right]_{\tau=\tau^j}$,the proof is completed.

Theorem 2. For system (3), suppose that (H_1) and (H_2) are true. When $\tau_1\neq\tau_2$ and $\tau=\tau_1+\tau_2>0$,the following conclusions are true:

(a)If $\tau\in[0,\tau^0)$, the equilibrium point E is asymptotically uniformly stable.;

(b)If $\tau>\tau^0$,model (3) is unstable at the equilibrium point E .

(c)If $\tau=\tau^j(j=0,1,2,\cdots)$,model (3) generates Hopf branch at the equilibrium point E .

Case 2: $\tau_1=\tau_2=\tau>0$

Suppose that (H_4) is true, then the characteristic equation of system (3) becomes

$$\lambda^2+2\lambda Ae^{-\lambda\tau}+Be^{-2\lambda\tau}=0. \tag{16}$$

Where

$$A=\frac{1}{2}\left[\frac{\gamma_1(1-\sigma_1)}{1-\sigma_1\sigma_2}+\frac{\gamma_2(\sigma_2-1)}{1-\sigma_1\sigma_2}\right]>0,$$

$$B=\frac{\gamma_1\gamma_2(1-\sigma_1)(\sigma_2-1)}{(1-\sigma_1\sigma_2)^2}-\frac{\gamma_1\gamma_2\sigma_1\sigma_2(1-\sigma_1)(1-\sigma_2)}{(1-\sigma_1\sigma_2)^2}>0.$$

Lemma 4. When $\tau=0$,both of the characteristic roots of equation (16) have negative real parts.

Proof. When $\tau=0$,the characteristic equation (16) becomes

$$\lambda^2+2\lambda A+B=0. \tag{17}$$

Therefore, the two characteristic roots of equation (17) are

$$\lambda_1 = -A - \sqrt{A^2 - B} \le \lambda_2 = -A + \sqrt{A^2 - B} < 0. \tag{18}$$

Obviously, the two roots of equation (17) always have negative real parts. So when $\tau = 0$, the equilibrium point is asymptotically stable. The proof is completed.

Next, we discuss the distribution of the characteristic roots of equation (16) when $\tau \ne 0$.

Lemma 5. For the system (3), assume that (H_1) is satisfied.

(i) When $A^2 - B \ge 0$, then equation(16) has two pairs of purely imaginary roots $\pm i\lambda_k$ when $\tau = \tau_{(k)}^j, (k = 1,2;\ j = 0,1,2,\cdots)$, where

$$\tau_{(k)}^j = -\frac{1}{\lambda_k}\left(2j\pi + \frac{\pi}{2}\right), (k = 1,2;\ j = 0,1,2,\cdots) \tag{19}$$

(ii) When $A^2 - B < 0$, then equation(16) has two pairs of purely imaginary roots $\pm i\sqrt{B}$ when $\tau = \tau_{(k)}^j, (k = 3,4;\ j = 0,1,2,\cdots)$, where

$$\tau = \tau_{(3)}^j = \frac{1}{\sqrt{B}}\left[\arccos\left(\sqrt{1 - \frac{A^2}{B}}\right) + 2j\pi\right], (j = 0,1,2,\cdots), \tag{20}$$

$$\tau = \tau_{(4)}^j = \frac{1}{\sqrt{B}}\left[-\arccos\left(\sqrt{1 - \frac{A^2}{B}}\right) + (2j+1)\pi\right], (j = 0,1,2,\cdots). \tag{21}$$

Proof. From equation (18), equation (16) can be transformed into

$$(\lambda - \lambda_1 e^{-\lambda\tau})(\lambda - \lambda_2 e^{-\lambda\tau}) = 0. \tag{22}$$

First, we assume that $\lambda = i\omega$ ($\omega > 0$)is a root of the equation (16), then it satisfies the following equation

$$(i\omega - \lambda_1 e^{-i\omega\tau})(i\omega - \lambda_2 e^{-i\omega\tau}) = 0. \tag{23}$$

So
$$i\omega - \lambda_1 e^{-i\omega\tau} = 0 \text{ or } i\omega - \lambda_2 e^{-i\omega\tau} = 0 \tag{24}$$

(i) When $A^2 - B \ge 0$, then $\lambda_2 \le \lambda_2 < 0$,equation(24) becomes

$$i\omega - \lambda_1(\cos \omega\tau - i\sin \omega\tau) = 0 \tag{25}$$

or
$$i\omega - \lambda_2(\cos \omega\tau - i\sin \omega\tau) = 0 \tag{26}$$

The separation of the real and imaginary parts, it follows

$$\begin{cases} \lambda_1 \cos \omega\tau = 0, \\ \omega + \lambda_1 \sin \omega\tau = 0. \end{cases} \text{or} \begin{cases} \lambda_1 \cos \omega\tau = 0, \\ \omega + \lambda_1 \sin \omega\tau = 0. \end{cases} \tag{27}$$

From (27) we obtain

$$\omega_1 = -\lambda_1, \tau = \tau_{(1)}^j = -\frac{1}{\lambda_1}\left(2j\pi + \frac{\pi}{2}\right), (j = 0,1,2,\cdots), \tag{28}$$

$$\omega_2 = -\lambda_2, \tau = \tau_{(2)}^j = -\frac{1}{\lambda_2}\left(2j\pi + \frac{\pi}{2}\right), (j = 0,1,2,\cdots), \tag{29}$$

Because $\lambda_2 \le \lambda_2 < 0$, so $\tau_{(2)}^j \le \tau_{(1)}^j, (j = 0,1,2,\cdots)$.

Therefore, we have the following conclusions:if $\tau \in [0, \tau_{(2)}^0)$, all characteristic roots of equation (16) have negative real parts;If $\tau = \tau_{(2)}^0$,equation (16) has a pair of pure virtual roots $\pm i\lambda_2$, while the other characteristic roots have negative real parts.If $\tau > \tau_{(2)}^0$,equation (16) has at least one characteristic root with a positive real part.

（ii）When $A^2 - B < 0$,so λ_1 and λ_2 are a pair of conjugate complex roots, equation(24) becomes

$$i\omega + (A - i\sqrt{B - A^2})(\cos \omega\tau - i\sin \omega\tau) = 0 \tag{30}$$

or
$$i\omega + (A + i\sqrt{B - A^2})(\cos \omega\tau - i\sin \omega\tau) = 0 \tag{31}$$

The separation of the real and imaginary parts, it follows

$$\begin{cases} A\cos \omega\tau - \sqrt{B - A^2} \sin \omega\tau = 0, \\ A\sin \omega\tau + \sqrt{B - A^2} \cos \omega\tau = \omega. \end{cases} \text{or} \begin{cases} A\cos \omega\tau + \sqrt{B - A^2} \sin \omega\tau = 0, \\ A\sin \omega\tau - \sqrt{B - A^2} \cos \omega\tau = \omega. \end{cases} \tag{32}$$

From (32) we obtain $\omega = \sqrt{B}$

$$\tau = \tau^{j}_{(3)} = \frac{1}{\sqrt{B}}\left[\arccos\left(\sqrt{1-\frac{A^2}{B}}\right) + 2j\pi\right], (j = 0,1,2,\cdots), \tag{33}$$

$$\tau = \tau^{j}_{(4)} = \frac{1}{\sqrt{B}}\left[-\arccos\left(\sqrt{1-\frac{A^2}{B}}\right) + (2j+1)\pi\right], (j = 0,1,2,\cdots). \tag{34}$$

So $\tau^{j}_{(3)} < \tau^{j}_{(4)}, (j = 0,1,2,\cdots)$.

Therefore, we have the following conclusions:If $\tau \in [0, \tau^{0}_{(3)})$, all characteristic roots of equation (16) have negative real parts;If $\tau = \tau^{0}_{(3)}$,equation (16) has a pair of pure virtual roots $\pm i\sqrt{B}$, while the other characteristic roots have negative real parts. If $\tau > \tau^{0}_{(3)}$,equation (16) has at least one characteristic root with a positive real part.

Lemma 6. Assume that (H_1) is satisfied, we have the following transversality condition $\mathrm{Re}(\frac{d\lambda}{d\tau})\Big|_{\tau=\tau^{j}_{(k)}} > 0$ is satisfied.

Proof.Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be a root of the characteristic equation (16) near $\tau = \tau^{j}_{(k)} (k = 1,2,3,4; j = 0,1,2,\cdots)$,which satisfies $\alpha(\tau^{j}_{(k)}) = 0$, $\omega(\tau^{j}_{(k)}) = -\lambda_k \ (k = 1,2)$, $\omega(\tau^{j}_{(k)}) = \sqrt{B} \ (k = 3,4)$,then we have the following transversality condition $\mathrm{Re}(\frac{d\lambda}{d\tau})\Big|_{\tau=\tau^{j}_{(k)}} > 0$ is satisfied.

Theorem 3. For system (3), suppose that (H_1) is true. When $\tau_1 = \tau_2 = \tau > 0$,the following conclusions are true:

(a)When $A^2 - B \geq 0$ .If $\tau \in [0, \tau^{0}_{(2)})$, the equilibrium point E is asymptotically uniformly stable; If $\tau \in \{\tau^{j}_{(1)}, \tau^{j}_{(2)}\}$,model (3) generates Hopf branch at the equilibrium point E ; If $\tau > \tau^{0}_{(2)}$,model (3) is unstable at the equilibrium point E .

(b)When $A^2 - B < 0$ .If $\tau \in [0, \tau^{0}_{(3)})$, the equilibrium point E is asymptotically uniformly stable; If $\tau \in \{\tau^{j}_{(3)}, \tau^{j}_{(4)}\}$,model (3) generates Hopf branch at the equilibrium point E ; If $\tau > \tau^{0}_{(3)}$,model (3) is unstable at the equilibrium point E .

III. DIRECTION AND STABILITY OF THE HOPF BIFURCATION

In the analysis in the above section, we have obtained the conditions for the system to generate Hopf branch. In this section we focus on case 2,we shall study the direction and stability of the bifurcating periodic solutions by applying the normal form theory and center manifold theorem introduced by Hassard et al.[11] .According to Theorem 3, the system (3) generates a Hopf branch at the branch point $\tau = \tau_{(k)}^{j} (k = 1,2,3,4; j = 0,1,2,\cdots)$. For the convenience of discussion,replace $\tau_{(k)}^{j}$ with $\bar{\tau}$, and mark the characteristic root of the characteristic equation (16) as $\pm i\omega_0$.

Let $u_1(t) = x(\tau t) - x_0$, $u_2(t) = y(\tau t) - y_0$,we consider the Taylor expansion of model (3) at the equilibrium point E ,

$$\begin{cases} \dot{u}_1(t) = \tau\gamma_1(u_1(t) + x_0)[-\dfrac{u_1(t-1)}{N_1} + \dfrac{\sigma_1 u_2(t-1)}{N_2}], \\ \dot{u}_2(t) = \tau\gamma_2(u_2(t) + y_0)[\dfrac{\sigma_2 u_1(t-1)}{N_1} - \dfrac{u_2(t-1)}{N_2}]. \end{cases} \tag{35}$$

Let $\tau = \bar{\tau} + \mu$, $u(t) = (u_1(t), u_2(t))^T$,then $\mu = 0$ represents the Hopf branch parameter of system (3).Then the model (3) is equivalent to the following Functional Differential Equation (FDE) system

$$\dot{u}(t) = L_\mu u_t + F(u_t, \mu). \tag{36}$$

$$L_\mu(\phi) = \tau B\varphi(-1), \tag{37}$$

and

$$F(\phi,\mu) = \tau\begin{pmatrix} -a_{11}\varphi_1(0)\varphi_1(-1) + a_{12}\varphi_1(0)\varphi_2(-1) \\ a_{21}\varphi_2(0)\varphi_1(-1) - a_{22}\varphi_2(0)\varphi_2(-1) \end{pmatrix}. \tag{38}$$

Where

$$B = \begin{pmatrix} -a_{11}x_0 & a_{12}x_0 \\ a_{21}y_0 & -a_{22}y_0 \end{pmatrix},$$

$$\frac{\gamma_1}{N_1} = a_{11}, \frac{\gamma_1\sigma_1}{N_2} = a_{12}, \frac{\gamma_2\sigma_2}{N_1} = a_{21}, \frac{\gamma_2}{N_2} = a_{22}.$$

By the Riesz representation theorem, there exists a bounded variation function $\eta(\theta,\mu)$, $\theta \in [-1,0]$, such that

$$L_{\mu}\phi = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), \quad \phi \in C. \tag{39}$$

In fact, we can choose

$$\eta(\theta,\mu) = \begin{cases} 0, & \theta \in (-1,0], \\ -\tau B, & \theta = 0. \end{cases} \tag{40}$$

For $\phi \in C^{'}([-1,0], R^{2})$, the operators A and R are defined as follow

$$A(\mu)\phi(\theta) = \begin{cases} \dfrac{d(\phi(\theta))}{d\theta}, & \theta \in [-1,0), \\ \int_{-1}^{0} d(\eta(t,\mu)\phi(t)), & \theta = 0. \end{cases} \tag{41}$$

$$R(\mu)\phi(\theta) = \begin{cases} 0, & \theta \in [-1,0), \\ F(\mu,\theta), & \theta = 0. \end{cases} \tag{42}$$

Hence the system (3) can be written as the following form:

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t \tag{43}$$

Where $u_t = u(t+\theta), \theta \in [-1,0)$.

For $\psi \in C^{'}[0,1]$, we define the adjoint operator $A^{*}(0)$ of $A(0)$ as

$$A^{*}\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds}, & s \in (0,1], \\ \int_{-1}^{0} d(\eta^{T}(s,0)\psi(-s)), & s = 0. \end{cases} \tag{44}$$

For $\phi \in C^{'}([-1,0], R^{2})$ and $\psi \in C^{'}[0,1]$, we define a Bilinear form

$$<\psi,\phi> = \overline{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\varepsilon=0}^{\theta} \overline{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi. \tag{45}$$

where $\eta(\theta) = \eta(\theta,0)$

Lemma 7. The eigenvectors $q(\theta) = (1,\rho_1)^{T} e^{i\omega_0\bar{\tau}\theta}$ and $q^{*}(s) = D(\rho_1^{*},1)^{T} e^{i\omega_0\bar{\tau}s}$ are respectively the eigenvectors corresponding to the eigenvalues $i\omega_0\overline{\tau}$ and $-i\omega_0\overline{\tau}$ of $A(0)$ and $A^{*}(0)$, and

$$< q^{*}, q >= 1 , \quad < q^{*}, \bar{q} >= 0 ,$$

where

$$(1, \rho_1)^T = (1, \frac{i\omega_0 e^{i\omega_0\bar{\tau}} + a_{11}x_0}{a_{12}x_0})^T, (\rho_1^{*}, 1)^T = (\frac{a_{22}y_0 e^{i\omega_0\bar{\tau}} - i\omega_0}{a_{21}y_0 e^{i\omega_0\bar{\tau}}}, 1)^T$$

$$\bar{D} = [\rho_1 + \bar{\rho}_1^{*} + \bar{\tau} e^{-i\omega_0\bar{\tau}} (\bar{\rho}_1^{*}, 1) B (1, \rho_1)^T]^{-1}$$

proof. $\pm i\omega_0\bar{\tau}$ are the eigenvalues of $A(0)$, so they are also the eigenvalues of $A^{*}(0)$ .In order to determine the standard form of the operator $A(0)$,we assume that the eigenvectors $q(\theta)$ and $q_1^{*}(s)$ are respectively the eigenvectors corresponding to the eigenvalues $i\omega_0\bar{\tau}$ and $-i\omega_0\bar{\tau}$ of $A(0)$ and $A^{*}(0)$ .We can obtain

$$\begin{cases} A(0)q(\theta) = i\omega_0\bar{\tau}q(\theta) \\ A^{*}(0)q_1^{*}(s) = -i\omega_0\bar{\tau}q_1^{*}(s) \end{cases} \tag{46}$$

From (39) and (41), (46) can be written as

$$\begin{aligned} &\frac{dq(\theta)}{d\theta} = i\omega_0\bar{\tau}q(\theta), && \theta \in [-1,0). \\ &L_0 q(0) = i\omega_0\bar{\tau}q(0), && \theta = 0. \end{aligned} \tag{47}$$

therefore $$q(\theta) = q(0)e^{i\omega_0\bar{\tau}\theta}, \qquad \theta \in [-1,0].$$

Where $q(0) = (q_1(0), q_2(0))^T \in C^2$ is a constant vector, obtained from (37), (46)

$$Be^{-i\omega_0\bar{\tau}} q(0) = i\omega_0 Iq(0)$$

By direct calculate, we get $$q(0) = \begin{pmatrix} 1 \\ \rho_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{i\omega_0 e^{i\omega_0\bar{\tau}} + a_{11}x_0}{a_{12}x_0} \end{pmatrix}$$

then $$q(\theta) = (1, \rho_1)^T e^{i\omega_0\bar{\tau}\theta}$$

For non-zero vectors $q_1^{*}(s)$, $s \in [0,1]$, we have $Be^{i\omega_0\bar{\tau}} q_1^{*}(0) = -i\omega_0 Iq_1^{*}(0)$

Similarly $q_1^*(0) = \begin{pmatrix} \rho_1^* \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{a_{22} y_0 e^{i\omega_0\bar{\tau}} - i\omega_0}{a_{21} y_0 e^{i\omega_0\bar{\tau}}} \\ 1 \end{pmatrix}$,

then $q_1^*(s) = (\rho_1^*, 1)^T e^{i\omega_0\bar{\tau}s}$,we make $q^*(s) = D(\rho_1^*, 1)^T e^{i\omega_0\bar{\tau}s}$,

Now let's prove that $< q^*, q >= 1$ and $< q^*, q >= 1$, from equation (45), we get

$$\begin{aligned}
&\langle q^*, q\rangle \\
&= \bar{q}^*(0)^T q(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \bar{q}^{*T}(\xi-\theta)\, d\eta(\theta) q(\xi)\, d\xi. \\
&= \bar{D}[(\bar{\rho}_1^*, 1)(1, \rho_1)^T - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} (\bar{\rho}_1^*, 1) e^{i\omega_0\bar{\tau}(\theta-\xi)} d\eta(\theta)(1, \rho_1)^T e^{i\omega_0\bar{\tau}\xi} d\xi] \\
&= \bar{D}[\rho_1 + \bar{\rho}_1^* - (\bar{\rho}_1^*, 1)\int_{-1}^{0} \theta e^{i\omega_0\bar{\tau}\theta} d\eta(\theta)(1, \rho_1)^T] \\
&= \bar{D}[\rho_1 + \bar{\rho}_1^* + \bar{\tau} e^{-i\omega_0\bar{\tau}} (\bar{\rho}_1^*, 1) B (1, \rho_1)^T]
\end{aligned} \tag{48}$$

Let $\bar{D} = [\rho_1 + \bar{\rho}_1^* + \bar{\tau} e^{-i\omega_0\bar{\tau}} (\bar{\rho}_1^*, 1) B (1, \rho_1)^T]^{-1}$, we can get $< q^*, q >= 1$ .By

$\langle \psi, A\varphi \rangle = \langle A^*\psi, \varphi \rangle$, we obtain

$$-i\omega_0\bar{\tau}\langle q^*, \bar{q}\rangle = \langle q^*, A\bar{q}\rangle = \langle A^* q^*, \bar{q}\rangle = \langle -i\omega_0\bar{\tau} q^*, \bar{q}\rangle = i\omega_0\bar{\tau}\langle q^*, \bar{q}\rangle. \tag{49}$$

So $< q^*, \bar{q} >= 0$. The proof is completed.

Next, we will use the same notations as in Hassard et al., we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Define

$$z(t) = < q^*, u_t > \tag{50}$$

and

$$W(t, \theta) = u_t(\theta) - 2\,\mathrm{Re}\{z(t) q(\theta)\}. \tag{51}$$

On the center manifold C_0, we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta) \tag{52}$$

Where $$W(z,\bar{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots.$$

For the central epidemic C_0, z and $\bar{z}$ respectively represent the local coordinates of the central epidemic in the direction of q and q^*. Note that W is real if u_t is real, therefore we only real solutions. Since $\mu = 0$, it is easy to see that

$$\dot{z}(t) = i\omega_0\bar{\tau} z(t) + \bar{q}^*(0) f(W(z,\bar{z},0) + 2\operatorname{Re}\{z(t)q(0)\})$$

$$\stackrel{\Delta}{=} i\omega_0\bar{\tau} z(t) + \bar{q}^{*T} f_0(z,\bar{z}). \tag{53}$$

Let $$\dot{z}(t) = i\omega_0\bar{\tau} z + g(z,\bar{z}), \tag{54}$$

Where $$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots, \tag{55}$$

from (43) and (55), we have

$$\dot{W} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = \begin{cases} AW - 2\operatorname{Re}\bar{q}^{*T}(0) f_0(z,\bar{z})q(\theta), & \theta \in [-1,0), \\ AW - 2\operatorname{Re}\{\bar{q}^{*T}(0) f_0(z,\bar{z})q(\theta)\} + f_0(z,\bar{z}), & \theta = 0. \end{cases} \tag{56}$$

Which can be rewritten as

$$\dot{W} = AW + H(z,\bar{z},\theta) \tag{57}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \tag{58}$$

On the other hand, on C_0,

$$\dot{W} = W_z\dot{z} + W_{\bar{z}}\dot{\bar{z}} \tag{59}$$

Using (53) and (55) to replace W_z and $\dot{z}$ and their conjugates by their power series expansions, we obtain

$$\dot{W} = i\omega_0\bar{\tau} W_{20}(\theta)z^2 - i\omega_0\bar{\tau} W_{02}(\theta)\bar{z}^2 + \cdots. \tag{60}$$

Comparing the coefficients of the above equation with those of (58) and (60), we get

$$\begin{cases} (A - 2i\omega_0\bar{\tau} I)W_{20}(\theta) = -H_{20}(\theta), \\ AW_{11}(\theta) = -H_{11}(\theta), \\ (A + 2i\omega_0\bar{\tau} I)W_{02}(\theta) = -H_{02}(\theta). \end{cases} \quad (61)$$

Notice that $u_t(\theta) = W(z(t), \bar{z}(t), \theta) + zq + \overline{zq}$ and $q(\theta) = (1, \rho_1)^T e^{i\omega_0\bar{\tau}\theta}$, we get

$$u_t(\theta) = \begin{pmatrix} W^{(1)}(z, \bar{z}, \theta) \\ W^{(2)}(z, \bar{z}, \theta) \end{pmatrix} + z\begin{pmatrix} 1 \\ \rho_1 \end{pmatrix} e^{i\omega_0\bar{\tau}\theta} + \bar{z}\begin{pmatrix} 1 \\ \bar{\rho}_1 \end{pmatrix} e^{-i\omega_0\bar{\tau}\theta}. \quad (62)$$

so

$$\phi_1(0) = z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots$$

$$\phi_2(0) = z\rho_1 + \bar{z}\bar{\rho}_1 + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots$$

$$\phi_1(-1) = ze^{-i\omega_0\bar{\tau}} + \bar{z}e^{i\omega_0\bar{\tau}} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + \cdots$$

$$\phi_2(-1) = z\rho_1 e^{-i\omega_0\bar{\tau}} + \bar{z}\bar{\rho}_1 e^{i\omega_0\bar{\tau}} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + \cdots$$

From (38), we obtain

$$f_0(z, \bar{z}) = \bar{\tau}\begin{pmatrix} K_1 z^2 + K_2 z\bar{z} + K_3 \bar{z}^2 + K_4 z^2\bar{z} \\ K_5 z^2 + K_6 z\bar{z} + K_7 \bar{z}^2 + K_8 z^2\bar{z} \end{pmatrix}$$

where

$$K_1 = -a_{11}e^{-i\omega_0\bar{\tau}} + a_{12}\rho_1 e^{-i\omega_0\bar{\tau}} \quad ,$$

$$K_2 = -a_{11}(e^{-i\omega_0\bar{\tau}} + e^{i\omega_0\bar{\tau}}) + a_{12}(\bar{\rho}_1 e^{i\omega_0\bar{\tau}} + \rho_1 e^{-i\omega_0\bar{\tau}}) \quad ,$$

$$K_3 = -a_{11}e^{i\omega_0\bar{\tau}} + a_{12}\bar{\rho}_1 e^{i\omega_0\bar{\tau}} \; ,$$

$$K_4 = -a_{11}[\frac{e^{i\omega_0\bar{\tau}}}{2}W_{20}^{(1)}(0) + \frac{1}{2}W_{20}^{(1)}(-1) + e^{-i\omega_0\bar{\tau}}W_{11}^{(1)}(0) + W_{11}^{(1)}(-1)] + \\ a_{12}[\frac{\overline{\rho}_1 e^{i\omega_0\bar{\tau}}}{2}W_{20}^{(1)}(0) + \frac{1}{2}W_{20}^{(2)}(-1) + \rho_1 e^{-i\omega_0\bar{\tau}}W_{11}^{(1)}(0) + W_{11}^{(2)}(-1)],$$

$$K_5 = a_{21}\rho_1 e^{-i\omega_0\bar{\tau}} - a_{22}\rho_1^2 e^{-i\omega_0\bar{\tau}},$$

$$K_6 = a_{21}(\rho_1 e^{i\omega_0\bar{\tau}} + \overline{\rho}_1 e^{-i\omega_0\bar{\tau}}) - a_{22}(\rho_1\overline{\rho}_1 e^{i\omega_0\bar{\tau}} + \rho_1\overline{\rho}_1 e^{-i\omega_0\bar{\tau}}),$$

$$K_7 = a_{21}\overline{\rho}_1 e^{i\omega_0\bar{\tau}} - a_{22}\overline{\rho}_1^2 e^{i\omega_0\bar{\tau}},$$

$$K_8 = a_{21}[\frac{e^{i\omega_0\bar{\tau}}}{2}W_{20}^{(2)}(0) + \frac{1}{2}\overline{\rho}_1 W_{20}^{(1)}(-1) + e^{-i\omega_0\bar{\tau}}W_{11}^{(2)}(0) + \rho_1 W_{11}^{(1)}(-1)] - \\ a_{22}[\frac{\overline{\rho}_1 e^{i\omega_0\bar{\tau}}}{2}W_{20}^{(2)}(0) + \frac{1}{2}\overline{\rho}_1 W_{20}^{(2)}(-1) + \rho_1 e^{-i\omega_0\bar{\tau}}W_{11}^{(2)}(0) + \rho_1 W_{11}^{(2)}(-1)].$$

From $\overline{q}^{*T}(0) = \overline{D}(\overline{\rho}_1^*, 1)$, we obtain

$$\begin{aligned} g(z,\bar{z}) &= \overline{q}^{*T}(0) f_0(z,\bar{z}) \\ &= \bar{\tau}\overline{D}(\overline{\rho}_1^*, 1)\begin{pmatrix} K_1 z^2 + K_2 z\bar{z} + K_3\bar{z}^2 + K_4 z^2\bar{z} \\ K_5 z^2 + K_6 z\bar{z} + K_7\bar{z}^2 + K_8 z^2\bar{z} \end{pmatrix} \\ &= \bar{\tau}\overline{D}[(\overline{\rho}_1^* K_1 + K_5)z^2 + (\overline{\rho}_1^* K_2 + K_6)z\bar{z} + (\overline{\rho}_1^* K_3 + K_7)\bar{z}^2 + (\overline{\rho}_1^* K_4 + K_8)z^2\bar{z}] \end{aligned}$$

Comparing the cofficients of the above equation with those in (55), we get

$$\begin{aligned} g_{20} &= 2\bar{\tau}\overline{D}(\overline{\rho}_1^* K_1 + K_5), g_{11} = \bar{\tau}\overline{D}(\overline{\rho}_1^* K_2 + K_6), \\ g_{02} &= 2\bar{\tau}\overline{D}(\overline{\rho}_1^* K_3 + K_7), g_{20} = 2\bar{\tau}\overline{D}(\overline{\rho}_1^* K_4 + K_8). \end{aligned} \tag{63}$$

In order to determine the value of g_{21}, we also need to compute the values of $W_{20}(\theta)$ and $W_{11}(\theta)$,from $\theta \in [-1,0)$,we obtain

$$\begin{aligned} H(z,\bar{z},\theta) &= -2\,\mathrm{Re}[\overline{q}^{*T}(0) f_0(z,\bar{z})q(\theta)] \\ &= -(g_{20}(\theta)\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots)q(\theta) \\ &\quad -(\overline{g}_{20}(\theta)\frac{z^2}{2} + \overline{g}_{11}z\bar{z} + \overline{g}_{02}\frac{\bar{z}^2}{2} + \cdots)\overline{q}(\theta). \end{aligned} \tag{64}$$

Comparing the coefficients with (58), we gives that

$$\begin{aligned} H_{20}(\theta) &= -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta), \\ H_{11}(\theta) &= -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta). \end{aligned} \tag{65}$$

When $\theta = 0$, we have

$$\begin{aligned} H(z,\overline{z},0) &= -2\,\mathrm{Re}[\overline{q}^{*T}(0) f_0(z,\overline{z})q(0)] + f_0(z,\overline{z}) \\ &= -(g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + \cdots)q(0) \\ &\quad - (\overline{g}_{20}(\theta)\frac{z^2}{2} + \overline{g}_{11}z\overline{z} + \overline{g}_{02}\frac{\overline{z}^2}{2} + \cdots)\overline{q}(0) + \overline{\tau}\begin{pmatrix} K_1 z^2 + K_2 z\overline{z} + K_3\overline{z}^2 + K_4 z^2\overline{z} \\ K_5 z^2 + K_6 z\overline{z} + K_7\overline{z}^2 + K_8 z^2\overline{z} \end{pmatrix}. \end{aligned}$$

Comparing the coefficients with (58), we have

$$\begin{aligned} H_{20}(0) &= -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + 2\overline{\tau}\begin{pmatrix} K_1 \\ K_5 \end{pmatrix}, \\ H_{11}(0) &= -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + \overline{\tau}\begin{pmatrix} K_2 \\ K_6 \end{pmatrix}. \end{aligned} \tag{66}$$

Using (61), (65), we obtain

$$\begin{aligned} W_{20}(\theta) &= \frac{ig_{20}}{\omega_0\overline{\tau}}q(0)e^{i\omega_0\overline{\tau}\theta} + \frac{i\overline{g}_{02}}{3\omega_0\overline{\tau}}\overline{q}(0)e^{-i\omega_0\overline{\tau}\theta} + E_1 e^{2i\omega_0\overline{\tau}\theta}, \\ W_{11}(\theta) &= -\frac{ig_{11}}{\omega_0\overline{\tau}}q(0)e^{i\omega_0\overline{\tau}\theta} + \frac{i\overline{g}_{11}}{\omega_0\overline{\tau}}\overline{q}(0)e^{-i\omega_0\overline{\tau}\theta} + E_2. \end{aligned} \tag{67}$$

Where $E_1 \in R^2, E_2 \in R^2$ are two two-dimensional vectors.

From the definition of $A(0)$ and (61) , we have

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0\overline{\tau}W_{20}(0) - H_{20}(0)$$

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0)$$

and

$$(i\omega_0\overline{\tau}I - \int_{-1}^{0} e^{i\omega_0\overline{\tau}\theta}d\eta(\theta))q(0) = 0$$

$$(-i\omega_0\overline{\tau}I - \int_{-1}^{0} e^{-i\omega_0\overline{\tau}\theta}d\eta(\theta))\overline{q}(0) = 0.$$

Hence, we can get

$$(2i\omega_0\bar{\tau}I - \int_{-1}^{0} e^{2i\omega_0\bar{\tau}\theta} d\eta(\theta))E_1 = 2\bar{\tau}\begin{pmatrix} K_1 \\ K_5 \end{pmatrix}$$

$$(\int_{-1}^{0} d\eta(\theta))E_2 = -\bar{\tau}\begin{pmatrix} K_2 \\ K_6 \end{pmatrix}$$

Therefore, we have

$$\begin{cases} \begin{pmatrix} 2i\omega_0 + a_{11}x_0 e^{-2i\omega_0\bar{\tau}} & -a_{12}x_0 e^{-2i\omega_0\bar{\tau}} \\ -a_{21}y_0 e^{-2i\omega_0\bar{\tau}} & 2i\omega_0 + a_{22}y_0 e^{-2i\omega_0\bar{\tau}} \end{pmatrix} E_1 = 2\begin{pmatrix} K_1 \\ K_5 \end{pmatrix} \\ \begin{pmatrix} -a_{11}x_0 & a_{12}x_0 \\ a_{21}y_0 & -a_{22}y_0 \end{pmatrix} E_2 = -\begin{pmatrix} K_2 \\ K_6 \end{pmatrix} \end{cases} \tag{68}$$

By calculation we can get

$$\begin{aligned} E_1 &= 2\begin{pmatrix} 2i\omega_0 + a_{11}x_0 e^{-2i\omega_0\bar{\tau}} & -a_{12}x_0 e^{-2i\omega_0\bar{\tau}} \\ -a_{21}y_0 e^{-2i\omega_0\bar{\tau}} & 2i\omega_0 + a_{22}y_0 e^{-2i\omega_0\bar{\tau}} \end{pmatrix}^{-1} \begin{pmatrix} K_1 \\ K_5 \end{pmatrix} \\ E_2 &= -\begin{pmatrix} -a_{11}x_0 & a_{12}x_0 \\ a_{21}y_0 & -a_{22}y_0 \end{pmatrix}^{-1} \begin{pmatrix} K_2 \\ K_6 \end{pmatrix}. \end{aligned} \tag{69}$$

Based on the above analysis, we can get the following parameter values:

$$\begin{aligned} C_1(0) &= \frac{i}{2\omega_0\bar{\tau}}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{Re\{C_1(0)\}}{Re\{\lambda'(\bar{\tau})\}}, \\ \beta_2 &= 2Re\{C_1(0)\}, \\ T_2 &= -\frac{Im\{C_1(0)\} + \mu_2(Im\{\lambda'(\bar{\tau})\})}{\omega_0\bar{\tau}}. \end{aligned} \tag{70}$$

Theorem 4. In the case of system (3), the conclusion holds

a) The direction of the Hopf bifurcation is determined by the parameter μ_2. If $\mu_2 > 0$, the Hopf bifurcation is supercritical . If $\mu_2 < 0$, the Hopf bifurcation is subcritical .

b) β_2 determines the stability of the bifurcating periodic solution. If $\beta_2 < 0$, the bifurcating periodic solutions is stable; if $\beta_2 > 0$, the bifurcating periodic solutions is unstable.

c) The period of the bifurcating periodic solution is decided by the parameter T_2. If $T_2 > 0\,(< 0)$, the period increases(decreases).

IV. NUMERICAL SIMULATION

In this section, focus on case 2,we present numerical results to confirm the analytical predictions obtained in the previous section.

For system (3) , We take the parameters: $\gamma_1 = 3.6, \gamma_2 = 4.5, N_1 = 1, N_2 = 3.5, \sigma_1 = 0.5, \sigma_2 = 1.2$.

According to the previous analysis, we get that the equilibrium point of the system (3) is $E = (1.25, 1.75)$ .And these coefficients satisfy conditions (H_1) and $A^2 - B = 7.3406 > 0$,then it can be calculated from the formula of lemma 6:

$$\omega_0 = 0.6656\ , \tau_{(2)}^0 = 2.3598\ , \tau_{(2)}^1 = 11.7991\ , \tau_{(2)}^2 = 21.2384\ , \tau_{(2)}^3 = 30.6777\ , \cdots\cdots$$

Figure 1 shows that when $\tau < \tau_{(2)}^0$, the equilibrium point $E = (1.25, 1.75)$ of the system (3) is locally asymptotically stable.Figure 2 shows that when $\tau > \tau_{(2)}^0$, the positive equilibrium point $E = (1.25, 1.75)$ becomes unstable and generates a periodic solution with a small amplitude, which vibrates near the positive equilibrium point.

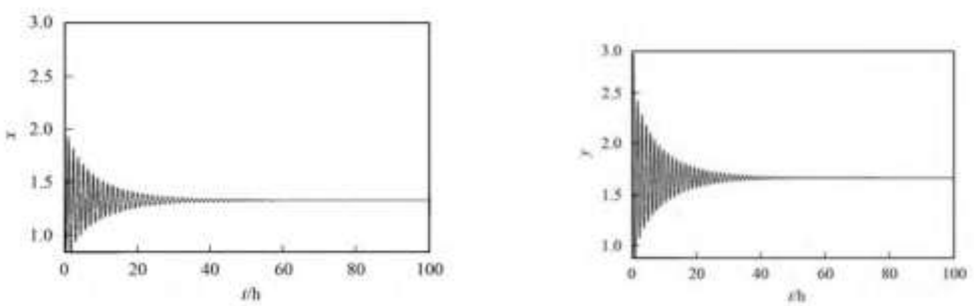

Figure 1 Waveform of system (3) when $\tau = 2 < 2.3598 = \tau_{(2)}^0$

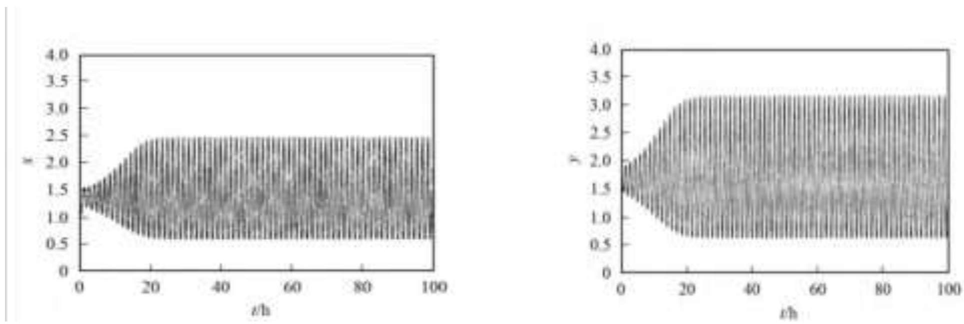

Figure 2 Waveform of system (3) when $\tau = 2.6 > 2.3598 = \tau_{(2)}^0$

V. CONCLUSIONS

In this paper, a partial profit model with three discrete time delays is studied.When (H_1) is satisfied,that is, there is a unique positive equilibrium point in model (3).Firstly,if $\tau = \tau_1 + \tau_2 > 0$, If the coefficients in model (3) satisfy condition (H_2) ,the sufficient conditions for the equilibrium stability of the system are obtained, and the existence conditions of the linear stability region and Hopf branch of the system are given.Then, if $\tau_1 = \tau_2 = \tau > 0$,when the hysteresis τ reaches a certain critical value, it will affect the stability of the equilibrium point of the model and lead to the emergence of branches.It can be seen that the partial profit cooperative system of the two groups can maintain a dynamic ecological balance under certain conditions, that is, the two groups can coexist and their numbers will oscillate periodically as the delay τ increases. This indicates that the stability of the system may be damaged by the introduction of time delay τ or by the excessively large time delay τ .Therefore, the existence of periodic solutions under certain conditions has important practical significance, which enables species in nature to coexist for a long time without extinction.

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