# Application of Taylor's Mean Value Theorem and Formula

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***Abstract* — This paper mainly investigated the application of Taylor's mean value theorem and formula. The beginning of this article introduces the related concepts of mean value theorem and Taylor formula and its proof, then there is the key part, we mainly introduced the nine applications, presented in the form of examples. Then the Taylor mean value theorem and its formula in the limit of function, proving equality and inequality, discriminating convergence of series and generalized integral, estimating the boundary of derivative function and in the application of function equation are summarized.***

**Keywords —** *Taylor's formula, Taylor mean value theorem, the limit, inequality.*

## I. INTRODUCTION OF TAYLOR'S MEAN VALUE THEOREM AND FORMULA

***Definition 1.(Taylor formula)[1]*** Taylor's formula of function $f(x)$ expanded at point $a$ is generally expressed as:If $f(x)$ has a $n$ derivative at $a$ ,then $\forall x \in U(a)$ ,we have $f(x) = T_n(x) + o[(x-a)^n]$ .

Where $T_n(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$ , $R_n(x) = o[(x-a)^n]$ ,

$R_n(x)$ is a higher order infinitesimal of $R_n(x)$ .

In particular, when $a = 0$ ( $f(x)$ has a $n$ derivative at $0$ ), then

$f(x) = f(0) + f'(0)\cdot x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + o(x-a)^n$ is called McLaughlin's formula.

***Theorem 1.(Taylor's mean value theorem)[1]*** If the function $f(x)$ has a n+1 derivative in $U(a)$ ,then $\forall x \in \overset{0}{U}(a)$ ,the function $G(t)$ is continuous on the closed interval $I$ with endpoints $a$ and $x$ , and differentiable on the open interval.

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$$f(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{n+1}(c)}{n!G'(c)}(x-c)^n[G(x)-G(a)].$$

Where $$R_n(x) = \frac{f^{n+1}(c)}{n!G'(c)}(x-c)^n[G(x)-G(a)] .$$

## II. PROOF OF TAYLOR'S MEAN VALUE THEOREM AND FORMULA

### 1. *Proof of Taylor's formula*

***Analysis:*** Just use the definition of a higher-order infinitesimal[1] that we learned earlier, and we need to prove that $\lim_{x\to a}\frac{R_n(x)}{(x-a)^n} = \lim_{x\to a}\frac{f(x)-T_n(x)}{(x-a)^n} = 0$ .So this is obviously a $\frac{0}{0}$ undetermined type limit. L 'Hopital's rule[1] should be used to solve this problem, and note the number of times L 'Hopital's rule is used.

***Proof.*** Because

$$R_n(x) = f(x) - T_n(x)$$
$$= f(x) - [f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n].$$

so $$R_n'(x) = f'(x) - [f'(a) + f''(a)(x-a) + \cdots + \frac{f^n(a)}{(n-1)!}(x-a)^{n-1}.$$

$$R_n''(x) = f''(x) - [f''(a) + f'''(a)(x-a) + \cdots + \frac{f^n(a)}{(n-2)!}(x-a)^{n-2}.$$

$$\cdots$$

$$R_n^{(n-1)}(x) = f^{(n-1)}(x) - [f^{(n-1)}(a) + \frac{f^n(a)}{1!}(x-a)].$$

When $x \to a$ , we have $R_n(x)$ , $R_n'(x)$ , $\cdots$ , $R_n^{(n-1)}(x)$ and $(x-a)^k$ $(k \in N_+)$ are all infinitesimals. Then from L 'Hopital's rule, we get

$$\lim_{x\to a}\frac{R_n(x)}{(x-a)^n} = \lim_{x\to a}\frac{R_n'(x)}{n(x-a)^{n-1}} = \lim_{x\to a}\frac{R_n''(x)}{n(n-1)(x-a)^{n-2}} = \lim_{x\to a}\frac{R_n^{(n-1)}(x)}{n!(x-a)}$$
$$= \frac{1}{n!}\lim_{x\to a}[\frac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x-a} - f^n(a)] = \frac{1}{n!}[f^n(a) - f^n(a)] = 0.$$

This completes the proof.

***2. Proof of Taylor's mean value theorem***

***Proof.*** $\forall t \in I$ , let(Replace $a$ in the $n$ -th order Taylor polynomial $T_n(x)$ with $t$ )

$$F(t) = f(t) + f'(t)\cdot(x-t) + \frac{f''(t)}{2!}(x-t)^2 + \cdots + \frac{f^{(n)}(t)}{n!}(x-t)^n.$$

So

$$F'(t) = f'(t) - f'(t) + \frac{f''(t)}{1!}(x-t) - \frac{f''(t)}{1!}(x-t) + \frac{f'''(t)}{2!}(x-t)^2 - \cdots$$
$$-\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n = \frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

Because the function $F(t)$ and $G(t)$ are continuous on the closed interval $I$ , and differentiable on the open interval. At the same time, $G'(t) \neq 0$ .Satisfies the condition of Cauchy's mean value theorem, and is known by Cauchy's mean value theorem. There is at least a point $c$ between $a$ and $x$ , so that

$$F(x) - F(a) = \frac{f^{n+1}(c)}{n!G'(c)}(x-c)^n[G(x) - G(a)]. \tag{1}$$

When $x = t$ , we have

$$F(x) = f(x), F(a) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Substitute them into equation (1) and transfer the terms, and get

$$f(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{n+1}(c)}{n!G'(c)}(x-c)^n[G(x) - G(a)],$$

Where $$R_n(x) = \frac{f^{n+1}(c)}{n!G'(c)}(x-c)^n[G(x) - G(a)].$$

The proof is completed.

## III. APPLICATION OF TAYLOR'S MEAN VALUE THEOREM AND FORMULA

Taylor's mean value theorem and formula are widely used in higher mathematics. The following are specific applications:

***Case 1:*** Taylor's formula is used to calculate the limit

When it comes to limits, the most important thing is the calculation of limits. The infinitive limit is a common form in the limit calculation. For the calculation of the infinitive limit, we first think of using L 'Hopital's rule and equivalent infinitesimal substitution to calculate. But these two methods also have certain limitations. So you're going to look for some easy algorithms, and naturally you're going to come up with Taylor's formula. The concrete steps of applying Taylor formula to solve the limit are given below.

***Method:*** Step1：Look at the limit, and find a formula that you can expand out using Taylor's formula.

Step2：let's look at the rest of the formula, and decide how many degrees to expand it.

Step3：Expanding this formula out using Taylor's formula.

Step4：You just substitute the expansion into the original limit and solve it.

***Example 1：*** Calculate the limit: $\lim\limits_{x\to 0}(1+\frac{1}{x^2}-\frac{1}{x^3}\ln\frac{2+x}{2-x})$ .

***Analysis:*** By observing the limit formula, it is found that $\ln\frac{2+x}{2-x}$ is similar to $\ln x$ , and Taylor's formula can be used to calculate the limit.

***Solve.*** Because $\ln\frac{2+x}{2-x}=\ln\frac{1+\frac{x}{2}}{1-\frac{x}{2}}=\ln(1+\frac{x}{2})-\ln(1-\frac{x}{2})$ .

From the expansion of $\ln\frac{2+x}{2-x}$ ,we have

$$\ln\frac{2+x}{2-x}=[\frac{x}{2}-\frac{1}{2}(\frac{x}{2})^2+\frac{1}{3}(\frac{x}{2})^3+o(x)^3]+[\frac{x}{2}+\frac{1}{2}(\frac{x}{2})^2+\frac{1}{3}(\frac{x}{2})^3+o(x)^3]=x+\frac{1}{12}x^3+o(x)^3 .$$

then
$$1+\frac{1}{x^2}-\frac{1}{x^3}\ln\frac{2+x}{2-x}=1+\frac{1}{x^2}-\frac{1}{x^3}(x+\frac{1}{12}x^3)+\frac{o(x)^3}{x^3}=1-\frac{1}{12}+\frac{o(x)^3}{x^3} .$$

So
$$\lim_{x\to 0}(1+\frac{1}{x^2}-\frac{1}{x^3}\ln\frac{2+x}{2-x})=\lim_{x\to 0}[1-\frac{1}{12}+\frac{o(x)^3}{x^3}]=\frac{11}{12} .$$

***Example 2：*** Calculate the limit: $\lim\limits_{x\to 0}\frac{\cos x-e^{-\frac{x^2}{2}}}{x^4}$ .

***Analysis:*** By observing the limit formula, we find that both $\cos x$ and $e^{-\frac{x^2}{2}}$ can be expanded by Taylor's formula, which can simplify the calculation process and reduce the amount of computation. Therefore Taylor's formula is used to calculate the limit in this problem.

***Solve.*** Because the degree of the denominator is 4, so $\cos x$ and $e^{-\frac{x^2}{2}}$ go to the fourth power of $x$ .

Then $$\cos x = 1-\frac{x^2}{2}+\frac{x^4}{24}+o(x)^4 \quad , \quad e^{-\frac{x^2}{2}} = 1-\frac{x^2}{2}+\frac{x^4}{8}+o(x)^4 \quad .$$

We get

$$\lim_{x\to 0}\frac{\cos x - e^{-\frac{x^2}{2}}}{x^4} = \frac{1}{x^4}\lim_{x\to 0}\{[1-\frac{x^2}{2}+\frac{x^4}{24}+o(x)^4]-[1-\frac{x^2}{2}+\frac{x^4}{8}+o(x)^4]\} = \lim_{x\to 0}[-\frac{1}{12}+\frac{o(x)^4}{x^4}] = -\frac{1}{12}.$$

***Case 2:*** Taylor's mean value theorem is used to calculate the limit

The specific solving steps are the same as the above steps for calculating the limit with Taylor's formula.

***Example 3*:** Calculate the limit: $\lim_{x\to 0}\frac{x-\sin x}{x^3}$.

***Solve.*** Because the degree of the denominator is 3, so $\sin x$ go to the third power of $x$ .

Then $$\sin x = x-\frac{x^3}{3!}+o(x)^3 \quad .$$

We get $$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{\frac{x^3}{3!}+o(x)^3}{x^3} = \lim_{x\to 0}(\frac{1}{3!}+\frac{o(x)^3}{x^3}) = \frac{1}{6}.$$

***Example 4*:** Calculate the limit: $\lim_{x\to 0}\frac{\cos x - e^{-\frac{x^2}{2}}}{1-\cos x}$.

***Analysis:*** If you expand $\cos x$ and $e^{-\frac{x^2}{2}}$ to the second power, the numerator and denominator are both zero, that's type $\frac{0}{0}$ , and if you want to solve it, you have to use L 'Hopital's rule, which proves that you can expand it to $x^4$ .

***Solve.*** Because the degree of the denominator is 4, so $\cos x$ and $e^{-\frac{x^2}{2}}$ go to the fourth power of $x$ .

Then $$\cos x = 1-\frac{x^2}{2}+\frac{x^4}{24}+o(x)^4 \quad , \quad e^{-\frac{x^2}{2}} = 1-\frac{x^2}{2}+\frac{x^4}{8}+o(x)^4 \quad .$$

We get

$$\lim_{x\to 0}\frac{\cos x - e^{-\frac{x^2}{2}}}{1-\cos x} = \lim_{x\to 0}\frac{-\frac{1}{12}x^4+o(x)^4}{\frac{x^2}{2}-\frac{x^4}{24}+o(x)^4} = 0 \; .$$

***Case 3:*** Taylor's formula is used to prove the equality

***Example 5***: Let $f(x)$ be differentiable three times on the interval $[a,b]$ ,prove:

$$\exists c \in (a,b) \text{ makes } f(b) = f(a) + f'(\frac{a+b}{2})(b-a) + \frac{1}{24} f'''(c)(b-a)^3 \text{ .} \tag{1}$$

***Analysis:*** In the proof of equality, if there is a higher derivative in the problem, Taylor's formula and differential

mean value theorem are often used in combination.

***Proof.*** Let's say I have a real number $k$ , so that this is true

$$f(b) - f(a) - f'(\frac{a+b}{2})(b-a) - \frac{1}{24} k(b-a)^3 = 0 \tag{2}$$

In this way, we can turn the question into a proof: $\exists c \in (a,b)$ makes $k = f'''(c)$ (3)

Let $$g(x) = f(x) - f(a) - f'(\frac{a+x}{2})(x-a) - \frac{k}{24}(x-a)^3 \tag{4}$$

We have $$g(a) = g(b) = 0 \text{ .}$$

According to Rolle's theorem, we know that $\exists \zeta \in (a,b)$ makes $g'(\zeta) = 0$ .

Take $\zeta = x$ and get it

$$g(\zeta) = f(\zeta) - f(a) - f'(\frac{a+\zeta}{2})(\zeta - a) - \frac{k}{24}(\zeta - a)^3 \text{ .}$$

If we take the derivative of both sides of this equation, we have

$$g'(\zeta) = f'(\zeta) - f''(\frac{a+\zeta}{2})(\zeta - a) - f'(\frac{a+\zeta}{2}) - \frac{k}{8}(\zeta - a)^2 = 0 \text{ .}$$

So $$f'(\zeta) - f''(\frac{a+\zeta}{2})(\zeta - a) - f'(\frac{a+\zeta}{2}) - \frac{k}{8}(\zeta - a)^2 = 0 \tag{5}$$

If we expand $f'(\zeta)$ at the point $\frac{a+\zeta}{2}$ by Taylor's formula, we get

$$f'(\zeta) = f'(\frac{a+\zeta}{2}) + f''(\frac{a+\zeta}{2})\frac{\zeta - a}{2} + \frac{1}{2} f'''(c)(\frac{\zeta - a}{2})^2 \tag{6}$$

Where $c \in (a,b)$, by comparing (5) and (6), equation (3) can be obtained. That is $k = f'''(c)$.

The proof is completed.

***Case 4:*** Taylor's formula is used to prove the inequality

There are a lot of ways to prove this inequality, but the main thing we're going to talk about here is that when you have $f^{n}(x)(n \geq 2)$ on a closed interval, you usually use Taylor's formula to prove it.

***Example 6***: Let $f(x)$ be differentiable second times on the interval $[a,b]$, and $f'(a) = f'(b) = 0$, prove:

$$\exists \zeta \in (a,b) \text{ makes } \left|f''(c)\right| \geq \frac{4}{(b-a)^2}\left|f(b) - f(a)\right| .$$

***Proof.*** Because $f(x)$ is differentiable second times on the interval $[a,b]$, thus, expanding $f(x)$ at the point $x = a$ by Taylor's formula, we get

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(\zeta_1)}{2!}(x-a)^2 .(\zeta_1 \in (a,x))$$

If we expand $f(x)$ at the point $x = b$ by Taylor's formula, we get

$$f(x) = f(b) + f'(b)(x-b) + \frac{f''(\zeta_2)}{2!}(x-b)^2 .(\zeta_2 \in (x,b))$$

Because $f'(a) = f'(b) = 0$, so $f(x) = f(a) + \frac{f''(\zeta_1)}{2!}(x-a)^2$, $f(x) = f(b) + \frac{f''(\zeta_2)}{2!}(x-b)^2$.

Let $x = \frac{a+b}{2}$, we get

$$f(\frac{a+b}{2}) = f(b) + \frac{f''(\zeta_1)}{2!}(\frac{\text{b}-\text{a}}{2})^2 , \tag{7}$$

$$f(\frac{a+b}{2}) = f(b) + \frac{f''(\zeta_2)}{2!}(\frac{a-b}{2})^2 . \tag{8}$$

From $(8)-(7)$, we get $0 = f(b) - f(a) + \frac{1}{2}[\frac{f''(\zeta_2)}{2!}(\frac{a-b}{2})^2 - \frac{f''(\zeta_1)}{2!}(\frac{b-a}{2})^2]$,

Then

$$\left|f(b)-f(a)\right|=\frac{1}{2}\left|\frac{f''(\zeta_2)}{2!}\left(\frac{a-b}{2}\right)^2-\frac{f''(\zeta_1)}{2!}\left(\frac{b-a}{2}\right)^2\right|$$
$$\le\frac{1}{2}\left(\left|f''(\zeta_1)\right|\frac{(b-a)^2}{4}+\left|f''(\zeta_2)\right|\frac{(b-a)^2}{4}\right)=\frac{(b-a)^2}{8}\left(\left|f''(\zeta_1)\right|+\left|f''(\zeta_2)\right|\right). \quad (9)$$

Let $\left|f''(c)\right|=\max\{f''(\zeta_1), f''(\zeta_2)\}$ ,so $(9)\le\frac{(b-a)^2}{4}\left|f''(c)\right|$.

Therefore $\left|f''(c)\right|\ge\frac{4}{(b-a)^2}\left|f(b)-f(a)\right|$.

The proof is completed.

***Case 5:*** Taylor's mean value theorem is used to prove the inequality

***Method:*** Firstly, analyze the question, select the appropriate point, and then expand the known function here with Taylor's formula. Then the inequality to be proved can be obtained by proper deformation of the expansion and combining with the question.

***Example 7***: Let the function $f(x)$ be in the open interval $(a,b)$, which satisfies $f''(x)\ge 0$ .Prove:

For any two points $x_1$ and $x_2$ in the open interval $(a,b)$ ,we have $f\left(\frac{x_1+x_2}{2}\right)<\frac{f(x_1)+f(x_2)}{2}$.

***Proof.*** Expanding $f(x)$ at the point $x_0=\frac{x_1+x_2}{2}$ by Taylor's formula, we get

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(\zeta)}{2!}(x-x_0)^2.$$

Where $\zeta$ is between $x_0$ and $x$ .

Let $x=x_1$ and $x=x_2$ , respectively. We can obtain that

$$f(x_1)=f(x_0)+\frac{f'(x_0)}{1!}(x_1-x_0)+\frac{f''(\zeta_1)}{2!}(x_1-x_0)^2,\zeta_1\in(x_1,x_0)$$

$$f(x_2)=f(x_0)+\frac{f'(x_0)}{1!}(x_2-x_0)+\frac{f''(\zeta_2)}{2!}(x_2-x_0)^2,\zeta_2\in(x_0,x_2)$$

Add these two formulas, we have

$$f(x_1)+f(x_2)=2f(x_0)+\frac{[f''(\zeta_1)+f''(\zeta_2)]}{8}(x_1-x_0)^2$$

Because $f''(x)\geq 0$ ,then $f(x_1)+f(x_2)>2f(x_0)$ .

Therefore $$f(\frac{x_1+x_2}{2})<\frac{f(x_1)+f(x_2)}{2}.$$

The proof is completed.

***Case 6:*** Taylor's formula is used to judge the convergence and divergence of series

There are many ways to judge the convergence and divergence of series. This paper mainly focuses on the complicated form of general term expression of series in the title, which is composed of different types of functions. Taylor's formula is usually applied to simplify the general term of series and facilitate the final judgment.

***Example 8***: Discuss the convergence and divergence of series $\sum_{n=1}^{\infty}(\frac{1}{\sqrt{n}}-\sqrt{\ln\frac{n+1}{n}})$ .

***Analysis:*** It's hard to tell whether a series is positive or negative based directly on the general term. If we look carefully, we'll see $\ln\frac{n+1}{n}=\ln(1+\frac{1}{n})$ .If you expand it by Taylor's formula to a power of $\frac{1}{n}$ when you $,$

take it to the second power, it corresponds to $\frac{1}{\sqrt{n}}$, and that makes the discriminant easier.

***Solve.*** Because $ln\frac{n+1}{n}=ln(1+\frac{1}{n})=\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\cdots<\frac{1}{n}$, so $\sqrt{\ln\frac{n+1}{n}}<\frac{1}{\sqrt{n}}$

Let $u_n=\frac{1}{\sqrt{n}}-\sqrt{\ln\frac{n+1}{n}}$ ,we have $u_n>0$ .

So this series is a positive series.

From

$$\sqrt{\ln\frac{n+1}{n}}=\sqrt{\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}+o(\frac{1}{n^3})}>\sqrt{\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{4n^3}}=\sqrt{(\frac{1}{\sqrt{n}}-\frac{1}{2n^{\frac{3}{2}}})^2}=\frac{1}{\sqrt{n}}-\frac{1}{2n^{\frac{3}{2}}},$$

we get

$$u_n = \frac{1}{\sqrt{n}} - \sqrt{\ln \frac{n+1}{n}} < \frac{1}{\sqrt{n}} - (\frac{1}{\sqrt{n}} - \frac{1}{2n^{\frac{3}{2}}}) = \frac{1}{2n^{\frac{3}{2}}} .$$

Because $\sum_{n=1}^{\infty} \frac{1}{2n^{\frac{3}{2}}}$ converges .According to the comparison test of positive series, the original series converges.

***Case 7:*** Taylor's formula is used to judge convergence and divergence of generalized integrals

Generalized integrals include infinite integrals and even integrals. In the process of judging the convergence and divergence of infinite integrals, trigonometric functions are usually expanded by Taylor's formula at the point zero, and then judged by the generalized integral method.

***Example 9***: To distinguish convergence and divergence of generalized integral $\int_0^1 \frac{xsi\ nx}{x - si\ nx} dx$ .

***Solve.*** Because $\lim_{x \to 0^+} \frac{xsi\ nx}{x - si\ nx} = \infty$ , so $x = 0$ is a improper point.

By comparison, $\lim_{x \to 0^+} x^q f(x) = d$ ,$0 \le d \le +\infty$ ,we get

when $q < 1$ , $\int_0^1 f(x) dx$ converges;when $q < 1$ , $\int_0^1 f(x) dx$ diverges.

Because $$f(x) = \frac{xsi\ nx}{x - si\ nx} = \frac{x(x - \frac{1}{3!}x^3 + o(x)^4)}{x - (x - \frac{1}{3!}x^3 + o(x)^4)} = \frac{x^2(1 - \frac{1}{6}x^2 + o(x)^3)}{\frac{1}{6}x^3 + o(x)^4} ,$$

so $$\lim_{x \to 0^+} x \frac{xsi\ nx}{x - si\ nx} = 6 .$$

From $q = 1$ , we know that the generalized integrals $\int_0^1 \frac{xsi\ nx}{x - si\ nx} dx$ diverge.

***Case 8:*** Taylor's formula is used to estimate the bounds of the derivative

The first thing you do is you expand the function by Taylor at a particular point. And then I'm going to deform some of these expansions. Finally, with proper scaling, the desired bounds can be found.

***Example 10***: Let $f(x)$ be differentiable second times on the interval $[0,1]$ .If $0 \le x \le 1$ ,then $\left| f(x) \right| \le 1$ ,

$\left| f''(x) \right| < 2$ . Prove:When $0 \le x \le 1$ ,we have $\left| f'(x) \right| \le 3$ .

***Proof.*** Because $f(x)$ is differentiable second times on the interval $[0,1]$,thus, expanding $f(1)$ at the point $x$ by Taylor's formula, we get

$$f(1) = f(x) + f'(x)(1-x) + \frac{1}{2} f''(\zeta)(1-x)^2 .$$

Similarly

$$f(0) = f(x) + f'(x)(-x) + \frac{1}{2} f''(\eta)(-x)^2 .$$

Then

$$f(1) - f(0) = f'(x) + \frac{1}{2} f''(\zeta)(1-x)^2 - \frac{1}{2} f''(\eta) x^2 ,$$

Therefore

$$\left|f'(x)\right| \le \left|f(1)\right| + \left|f(0)\right| + \frac{1}{2}\left|f''(\zeta)\right|(1-x)^2 + \frac{1}{2}\left|f''(\eta)\right|x^2 \le 2 + (1-x)^2 + x^2 \le 2 + 1 = 3 .$$

***Case 8:*** Application of Taylor's formula in functional equation

***Example 11***: Let $f(x)$ be differentiable second times on the interval $(-1,1)$, $f(0) = f'(0) = 0$ and

$$\left|f''(x)\right| \le \left|f(x)\right| + \left|f'(x)\right| . \tag{10}$$

Prove: $\exists \delta > 0$ makes $f(x) \equiv 0$ within $(-\delta, \delta)$.

***Analysis:*** .To prove that $f(x)$ is constant zero in the neighborhood of $x = 0$, we can prove that the maximum value of $\left|f(x)\right|$ is equal to 0.Because $f(x)$ is differentiable second times on the interval,we know that $\left|f(x)\right|$ and $\left|f'(x)\right|$ are continuous at $(-1,1)$.There exists a closed interval contained in $(-1,1)$, so that $\left|f(x)\right|$ and $\left|f'(x)\right|$ have a maximum value on the closed interval.

***Proof.*** To prove that $f(x)$ is constant to zero in the neighborhood of $x = 0$, we need to expand $f(x)$ and $f'(x)$ on the right-hand side of equation (10) by Taylor's formula at $x = 0$. We get

$$f(x) = f(0) + f'(0)x + \frac{f''(\zeta)}{2} x^2 \quad , f'(x) = f'(0) + f''(\eta) x$$

Because $f(0) = f'(0) = 0$,

so $f(x) = f(0) + f'(0)x + \frac{f''(\zeta)}{2}x^2 = \frac{f''(\zeta)}{2}x^2$, $f'(x) = f'(0) + f''(\eta)x = f''(\eta)x$.

Then $$\left|f(x)\right| + \left|f'(x)\right| = \left|\frac{1}{2}f''(\zeta)x^2\right| + \left|f''(\eta)x\right| \tag{11}$$

If you take a $x \in [-\frac{1}{4}, \frac{1}{4}]$, you get $\left|f(x)\right| + \left|f'(x)\right|$ is continuous and bounded.

$\exists x_0 \in [-\frac{1}{4}, \frac{1}{4}]$ makes $\left|f(x_0)\right| + \left|f'(x_0)\right| = \max_{-\frac{1}{4} \le x \le \frac{1}{4}} \{\left|f(x)\right| + \left|f'(x)\right|\} \equiv M$ .

Now we just have to get $M = 0$.

Because

$$M = \left|f(x_0)\right| + \left|f'(x_0)\right| = \left|\frac{1}{2}f''(\zeta)x^2\right| + \left|f''(\eta)x\right| \le \frac{1}{4}(\left|f''(\zeta_0)\right| + \left|f''(\eta_0)\right|)$$
$$\le \frac{1}{4}(\left|f'(\zeta_0)\right| + \left|f(\zeta_0)\right| + \left|f'(\eta_0)\right| + \left|f(\eta_0)\right|) \le \frac{1}{4} \cdot 2M = \frac{1}{2}M$$

therefore $0 \le M \le \frac{1}{2}M$ ,then $M = 0$.

So $f(x) \equiv 0$ on $[-\frac{1}{4}, \frac{1}{4}]$.

## IV. CONCLUSIONS

Taylor's mean value theorem and formula are important contents in mathematical analysis. This paper first introduces the relevant concepts and their proofs, then introduces the application of Taylor's mean value theorem and formula, which are applied to the limit of function, proof of equality and inequality, discrimination of convergence of series and generalized integral, estimation of the boundary of derivative function and so on. This paper systematically introduces their application in the original knowledge system, and gives detailed examples of each application, and makes analysis and explanation, so that readers can deeply understand the essence of learning Taylor's mean value theorem and formula.

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