Variance Swap Pricing under an Extension of Mean-Reverting Gaussian Model

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Abstract : *With a rapid development of financial markets, volatility has always been a core issue in the field of financial research. The essence of a variance swap is a forward contract whose value depends on the future volatility level of the underlying asset. This derivative with volatility as an underlying asset provides direct risk exposure to volatility. Based on the mean-reverting Gaussian volatility model, this paper modifies the differential equation that volatility obeys, and uses the risk-neutral pricing principle to derive the pricing formula of a variance swap under the new model.*

Keywords: *volatility, Mean-reverting Gaussian model, risk-neutral pricing, variance swap*

I. Background Introduction

Volatility derivatives include Gamma swaps, volatility swaps, (conditional) variance swaps, volatility (index) options, variance options, etc. Among them, volatility swaps and variance swaps are the most common and most easily accepted by financial practitioners. Therefore, this article conducts further research on volatility swaps and variance swaps.

Consider that the stock price S_t satisfies the following stochastic differential equation:

$$dS_t = rS_t dt + \sigma S_t dW_t . \qquad (1)$$

Among them, r represents the expected rate of the asset, σ represents the volatility of the asset, W_t is the standard Brownian motion, and both r and $\sigma > 0$ are constants. Applying Ito lemma, the solution of (1.1) can be obtained as follows:

$$S(t) = S(0)e^{\sigma W(t)+(r-\frac{1}{2}\sigma^2)t} . \qquad (2)$$

This is the asset price model used in the B-S-M option pricing formula[1,2], and S_t is called "Geometric Brownian Motion". In fact, the B-S-M model does not match the stock prices observed in the financial market. The literature by Turner and Weigl mentions that volatility is not a constant [3]. Gatheral analyzed the stock prices after the US stock market crash and found that the volatility of stock prices is very random [4]. Cont's literature shows that volatility is mean reverted, clustering, and has a long-term memory [5]. The volatility smile effect,etc. indicate that the volatility of the stock price should be related to its own level, time, and other factors [6].Therefore, many economists and financial scientists have relaxed the assumptions of the BSM model from various perspectives, improved the defect of "constant volatility" in the model, and proposed a variety of models, including time-dependent models, time and price dependent model and stochastic volatility model.For the time-dependent volatility model, the volatility σ is a unary function of time t,marked as $\sigma = \sigma(t)$ .For time and price dependent models, volatility σ is a binary function of time t and asset prices S_t ,marked as $\sigma = \sigma(t, S_t)$ .However, the stochastic volatility model assumes that the volatility process and asset price process are both random. The well-known stochastic volatility models include Hull-White model [7], Stein & Stein model [8], Heston model [9], and mean-reverting Gaussian volatility model [10].For the mean-reverting Gaussian volatility model, the price of the underlying asset S_t and its volatility σ_t satisfy the following stochastic differential equation:

$$\begin{cases} \dfrac{dS_t}{S_t} = rdt + \sigma_t dW_t^1 \\ d\sigma_t = \kappa(\theta - \sigma_t)dt + \nu dW_t^2 \end{cases} \quad (3)$$

where r represents the expected return rate of the asset, θ represents the long-term average value of volatility, κ represents the mean recovery speed parameter of volatility, ν represents the volatility of volatility (also known as the coefficient of variation of volatility) and it is used to describe volatility uncertainty. W_t^1 and W_t^2 represent two standard Brownian motions with a correlation coefficient of ρ .

After giving the foreshadowing of the above mean-reverting model, we modify the differential equation that the volatility σ_t in the original model obeys, and give the new model as follows:

$$\begin{cases} \dfrac{dS_t}{S_t} = rdt + \sigma_t dW_t \\ d\sigma_t = \kappa(\theta - \sigma_t)dt + \nu\sigma_t dW_\sigma \end{cases} \quad (4)$$

W_t and W_σ represent two standard Brownian motions[13] with a correlation coefficient of ρ .It is not difficult to find that in our new model,the volatility of volatility is not a constant any longer. Instead, we assume it depends on the level of volatility.Based on the new model, we give the pricing formulas for a variance swap under the new model.

II. Preliminary

This chapter is the basis for the subsequent derivation process. We will briefly introduce the definition of Ito process, Ito-Deblin formula about the Ito process and the Ito integral of non-random integrand. Even under the generalized mean-reverting Gaussian model, the following theorems still hold.

A. Ito process[12]

Assuming $W(t)$, $t \geq 0$ is Brownian motion, $F(t)$ is the corresponding basin flow. The Ito process is a random process of the following form:

$$X(t) = X(0) + \int_0^t \Delta(u)dW(u) + \int_0^t \Theta(u)du . \quad (5)$$

where $X(0)$ is not random, $\Delta(u)$ and $\Theta(u)$ are adaptive stochastic processes.

B. The Ito-Deblin formula about the Ito process[11]

Let $X(t)$, $t \geq 0$ be the Ito process described in Definition 2.1. The partial derivatives $f_t(t,x)$, $f_x(t,x)$ and $f_{xx}(t,x)$ of the function $f(t,x)$ are defined and continuous, then for each $T \geq 0$:

$$f(T, X(T)) = f(0, X(0)) + \int_0^T f_t(t, X(t))\,dt + \int_0^T f_x(t, X(t))\,dX(t)$$
$$+ \frac{1}{2}\int_0^T f_{xx}(t, X(t))\,d[X, X](t).$$

If the differential notation is used, the above formula can be rewritten as:

$$df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t)) + \frac{1}{2} f_{xx}(t, X(t))dX(t)dX(t). \quad (6)$$

In fact, $f(t, X(t))$ takes the Taylor expansion respect to variables t and $X(t)$,then the quadratic variation of variable t is 0 and the quadratic variation of $X(t)$ is non-zero, so we expand t to the first order and expand $X(t)$ to the second order to get the above formula.

C. Ito integral of non-random integrand[12]

Assuming $W(s)$, $s \geq 0$ is Brownian motion, $\Delta(s)$ is a non-random function of the time variable.Define $I(t) = \int_0^t \Delta(s)dW(s)$,for each $t \geq 0$, $I(t)$ is a normal random variable with an expected value of 0.

We note that the main consideration in random analysis is the Ito process in definition A. The traditional Ito-Deblin formula discusses the Brownian motion, but Theorem B discusses a more general random process than Brownian motion.We should also note that the assumption in Theorem C is that $\Delta(s)$ is not random. If $\Delta(s)$ is non-random, there is no reason to assert that $\int_0^t \Delta(s)dW(s)$ obeys a normal distribution.

III. PRICING ISSUES UNDER THE NEW MODEL

In this chapter, we will assume that volatility obeys a new type of mean-reverting model, and derive pricing formulas for a volatility swap and a variance swap using risk-neutral pricing on this basis. Suppose $\mathrm{E}[\bullet]$ represents the mathematical expectation under the risk-neutral measure, r represents the risk-free interest rate, and denote $\mathrm{E}[\bullet \mid F_t] = E_t[\bullet]$.

Theorem Suppose the price process of the underlying asset and the volatility process satisfies (4), then the value of a variance swap at time t can be obtained as follows:

$$V_t = e^{-r(T-t)}[\frac{1}{T}(\int_0^t \sigma_s^2 ds + \int_t^T \begin{pmatrix} \frac{2\kappa\theta}{\kappa-\nu^2} e^{\kappa(t-s)}(1-e^{(\nu^2-\kappa)(s-t)})(\sigma_t-\theta) \\ + \frac{2\kappa\theta^2}{\kappa-\nu^2}(1-e^{(\nu^2-2\kappa)(s-t)}) + \sigma_t^2 e^{(\nu^2-2\kappa)(s-t)} \end{pmatrix} ds)$$

$$+ \frac{2\kappa\theta(\sigma_0-\theta)}{T\kappa(\nu^2-\kappa)} e^{-\kappa s} - \frac{2\kappa\theta\sigma_0 e^{(\nu^2-2\kappa)s}}{T(\nu^2-2\kappa)(\kappa-\nu^2)} + \frac{\sigma_0^2 e^{(\nu^2-2\kappa)s}}{T(\nu^2-2\kappa)} - \frac{2\kappa\theta^2}{\nu^2-2\kappa}].$$

Proof. The profit and loss of the variance swap on the maturity date T is as follows:

$$\sigma_R^2 - K^{vol} = \frac{1}{T}\int_0^T \sigma_t^2 dt - K^{vol}.$$

We use σ_R^2 to represent the realized variance of the return, K^{vol} to represent the delivery price.According to the principle of risk-neutral pricing, the value of the time variance swap can be obtained:

$$\begin{aligned} V_t &= \mathrm{E}\left[e^{-r(T-t)}(\sigma_R^2 - K^{vol}) \mid F_t\right] \\ &= \mathrm{E}[e^{-r(T-t)}\frac{1}{T}(\sigma_R^2 - K^{vol}) \mid F_t] \\ &= e^{-r(T-t)}[\frac{1}{T}(\int_0^t \sigma_s^2 ds + \int_t^T \mathrm{E}_t[\sigma_s^2]ds) - K^{vol}]. \end{aligned} \quad (7)$$

Let $f(t,x) = e^{\kappa t}x$, by the Ito-Deblin formula

$$\begin{aligned} d(e^{\kappa t}\sigma_t) &= df(t,\sigma_t) \\ &= f_t(t,\sigma_t)dt + f_x(t,\sigma_t)d\sigma_t + \frac{1}{2}f_{xx}(t,\sigma_t)d\sigma_t d\sigma_t \\ &= \kappa e^{\kappa t}\sigma_t dt + e^{\kappa t}d\sigma_t \\ &= \kappa e^{\kappa t}\sigma_t dt + e^{\kappa t}(\kappa(\theta-\sigma_t)dt) + e^{\kappa t}\nu\sigma_t dW_\sigma \\ &= \kappa\theta e^{\kappa t}dt + \nu e^{\kappa t}\sigma_t dW_\sigma. \end{aligned}$$

For $s \geq t$, integrating both sides of the above formula at the same time, we can get:

$$e^{\kappa s}\sigma(s) = e^{\kappa t}\sigma(t) + \kappa\theta\int_t^s e^{\kappa u}du + \nu\int_t^s e^{\kappa u}\sigma(u)dW_\sigma$$

$$= e^{\kappa t}\sigma(t) + \theta(e^{\kappa s} - e^{\kappa t}) + v\int_t^s e^{\kappa u}\sigma(u)dW_\sigma .$$

According to Theorem C,

$$e^{\kappa s}\mathrm{E}_t\sigma(s) = e^{\kappa t}\sigma(t) + \theta(e^{\kappa s} - e^{\kappa t}),$$

$$\mathrm{E}_t\sigma(s) = e^{\kappa(t-s)}\sigma(t) + \theta(1 - e^{\kappa(t-s)}). \qquad (8)$$

Let $X(t) = e^{\kappa t}\sigma(t) = e^{\kappa t}\sigma_t$,

$$d(X^2(t)) = 2X(t)dX(t) + dX(t)dX(t)$$

$$= 2e^{\kappa t}\sigma_t(\kappa\theta e^{\kappa t}dt + ve^{\kappa t}\sigma_t dW_\sigma) + v^2 e^{2\kappa t}\sigma_t^2 dt .$$

Integrate both sides at the same time, we can get:

$$X^2(t) = X^2(0) + 2\kappa\theta\int_0^t e^{2\kappa u}\sigma_u du + v^2\int_0^t e^{2\kappa u}\sigma_u^2 du + 2v\int_0^t e^{2\kappa u}\sigma_u^2 dW_\sigma .$$

For $s \geq t$, we have

$$X^2(s) = X^2(0) + 2\kappa\theta\int_0^s e^{2\kappa u}\sigma_u du + v^2\int_0^s e^{2\kappa u}\sigma_u^2 du + 2v\int_0^s e^{2\kappa u}\sigma_u^2 dW_\sigma ,$$

$$X^2(s) - X^2(t) = 2\kappa\theta\int_t^s e^{2\kappa u}\sigma_u du + v^2\int_t^s e^{2\kappa u}\sigma_u^2 du + 2v\int_t^s e^{2\kappa u}\sigma_u^2 dW_\sigma .$$

According to Theorem C, the expected value of the Ito integral of the non-random integrand is 0, and the expected value is:

$$\mathrm{E}_t X^2(s) - X^2(t) = 2\kappa\theta\int_t^s e^{2\kappa u}\mathrm{E}_t[\sigma_u]du + v^2\int_t^s e^{2\kappa u}\mathrm{E}_t[\sigma_u^2]du .$$

Substitute $\mathrm{E}_t X^2(s) = \mathrm{E}_t[e^{2\kappa s}\sigma_s^2]$ and $\mathrm{E}_t X^2(s) = \mathrm{E}_t[e^{2\kappa s}\sigma_s^2]$ into the above formula:

$$e^{2\kappa s}\mathrm{E}_t[\sigma_s^2] = e^{2\kappa t}\sigma_t^2 + 2\kappa\theta\int_t^s e^{2\kappa u}\mathrm{E}_t[\sigma_u]du + v^2\int_t^s e^{2\kappa u}\mathrm{E}_t[\sigma_u^2]du . \qquad (9)$$

Substituting (8) into (9), and letting $x(s) = \mathrm{E}_t[\sigma_s^2]$, we can get the following differential equation:

$$(2\kappa - v^2)x(s) + x'(s) = 2\kappa\theta\,[e^{\kappa(t-s)}\sigma_t + \theta(1 - e^{\kappa(t-s)})]. \qquad (10)$$

Solve the equation:

$$x(s) = \frac{2\kappa\theta\sigma_t}{\kappa - v^2}e^{\kappa(t-s)} + \frac{2\kappa\theta^2}{2\kappa - v^2} - \frac{2\kappa\theta^2}{\kappa - v^2}e^{\kappa(t-s)} + ce^{(v^2-2\kappa)s}, \qquad (11)$$

where c is a constant. Let $s = t$, the boundary conditions of the differential equation (10) can be obtained as follows:

$$c = e^{(2\kappa - v^2)t}\left(-\frac{2\kappa\theta(\sigma_t - \theta)}{\kappa - v^2} - \frac{2\kappa\theta^2}{2\kappa - v^2} + \sigma_t^2\right). \qquad (12)$$

Substituting (12) into (11), we can get:

$$x(s) = \frac{2\kappa\theta}{\kappa - v^2}e^{\kappa(t-s)}[(1 - e^{(v^2-\kappa)(s-t)})(\sigma_t - \theta)]$$

$$+ \frac{2\kappa\theta^2}{2\kappa - v^2}(1 - e^{(v^2-2\kappa)(s-t)}) + \sigma_t^2 e^{(v^2-2\kappa)(s-t)}. \qquad (13)$$

Substitute (13) into (7) to get:

$$V_t = e^{-r(T-t)}\Big[\frac{1}{T}\Big(\int_0^t \sigma_s^2 ds + \int_t^T \begin{pmatrix} \frac{2\kappa\theta}{\kappa-\nu^2} e^{\kappa(t-s)}(1-e^{(\nu^2-\kappa)(s-t)})(\sigma_t-\theta) \\ + \frac{2\kappa\theta^2}{\kappa-\nu^2}(1-e^{(\nu^2-2\kappa)(s-t)}) + \sigma_t^2 e^{(\nu^2-2\kappa)(s-t)} \end{pmatrix} ds\Big) - K^{vol}\Big].$$

According to the principle of no arbitrage, the initial value of variance swap is 0. Therefore, let $t = 0$ to get

$$\frac{1}{T}\Big[\int_0^T \frac{2\kappa\theta}{\kappa-\nu^2} e^{-\kappa s}(1-e^{(\nu^2-\kappa)s})(\sigma_0-\theta) + \frac{2\kappa\theta^2}{\kappa-\nu^2}(1-e^{(\nu^2-2\kappa)s}) + \sigma_0^2 e^{(\nu^2-2\kappa)s} ds\Big] - K^{vol} = 0.$$

$$K^{vol} = \frac{2\kappa\theta(\sigma_0-\theta)}{T\kappa(\nu^2-\kappa)} e^{-\kappa s} - \frac{2\kappa\theta\sigma_0 e^{(\nu^2-2\kappa)s}}{T(\nu^2-2\kappa)(\kappa-\nu^2)} + \frac{\sigma_0^2 e^{(\nu^2-2\kappa)s}}{T(\nu^2-2\kappa)} - \frac{2\kappa\theta^2}{\nu^2-2\kappa}.$$

Therefore, the final variance swap pricing formula is as follows:

$$V_t = e^{-r(T-t)}\Big[\frac{1}{T}\Big(\int_0^t \sigma_s^2 ds + \int_t^T \begin{pmatrix} \frac{2\kappa\theta}{\kappa-\nu^2} e^{\kappa(t-s)}(1-e^{(\nu^2-\kappa)(s-t)})(\sigma_t-\theta) \\ + \frac{2\kappa\theta^2}{\kappa-\nu^2}(1-e^{(\nu^2-2\kappa)(s-t)}) + \sigma_t^2 e^{(\nu^2-2\kappa)(s-t)} \end{pmatrix} ds\Big)$$
$$+ \frac{2\kappa\theta(\sigma_0-\theta)}{T\kappa(\nu^2-\kappa)} e^{-\kappa s} - \frac{2\kappa\theta\sigma_0 e^{(\nu^2-2\kappa)s}}{T(\nu^2-2\kappa)(\kappa-\nu^2)} + \frac{\sigma_0^2 e^{(\nu^2-2\kappa)s}}{T(\nu^2-2\kappa)} - \frac{2\kappa\theta^2}{\nu^2-2\kappa}\Big].$$

IV. SUMMARY AND OUTLOOK

In this article, we first assume that the underlying asset and its volatility behavior obey an extension of the mean-reverting Gaussian model, and then we use the risk-neutral pricing theory under the new model to give the pricing formulas of a volatility swap and a variance swap respectively. So far, we have discussed the pricing of a volatility swap and a variance swap under the extended stochastic volatility model. In addition, we have other related issues to study, such as: the pricing of Gamma swaps ,variance options ,volatility index options and other volatility derivatives under the extended model.How to conduct detailed numerical tests under the new model is also a further research direction.

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