# Double Power of 2 Decomposition [DPo2D] of Graphs

S. Asha[1] and V.G. Smilin Shali[2]
[1]*Assistant Professor*
[2]*Research Scholar, Reg. No. 12609*
*Research Department of Mathematics, Nesamony Memorial Christian College, Marthandam, Kanyakumari District - 629 165, Tamil Nadu, India.*
*Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli - 627 012, Tamil Nadu, India.*

**ABSTRACT -** *Let G be a finite, connected simple graph with p vertices and q edges. If $G_1, G_2, \ldots, G_n$ are connected edge-disjoint subgraphs of G with $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_n)$, then $\{G_1, G_2, \ldots, G_n\}$ is said to be a decomposition of G. In this paper we introduce a new concept called Double power of 2 Decomposition of graphs. A graph G is said to have Double Power of 2 Decomposition if G can be decomposed into subgraphs $\{2G_1, 2G_2, \ldots, 2G_n\}$ such that each $G_{2^i}$ is connected and $|E(G_i)| = 2^i$, for $1 \le i \le n$. Clearly, $q = 4[2^n - 1]$. In this paper, we investigate the necessary and sufficient condition for graphs such as J(m , 3) , $L_m$ , $T_m$ and $H_m$ to accept Double Power of 2 Decomposition.*

**KEYWORDS :** *Decomposition of Graph, Power of 2 Decomposition, Double Power of 2 Decomposition.*

## I. INTRODUCTION

Let G be a simple, connected graph with p vertices and q edges. If $G_1,G_2,\ldots,G_n$ are connected edge-disjoint subgraphs of G with $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_n)$, then $\{G_1,G_2,\ldots,G_n\}$ is said to be a Decomposition of G. Different type of decomposition of G have been studied in the literature by imposing suitable conditions on the subgraphs $G_i$. In this paper we introduce a new concept called Double power of 2 Decomposition of graphs. A graph G is said to have Double Power of 2 Decomposition if G can be decomposed into subgraphs $\{2G_1, 2G_2, \ldots, 2G_n\}$ such that each $G_{2^i}$ is connected and $|E(G_i)| = 2^i$, for $1 \le i \le n$. Clearly, $q = 4[2^n - 1]$. In this paper, we investigate the necessary and sufficient condition for graphs such as J(m , 3) , $L_m$ , $T_m$ and $H_m$ to accept Double Power of 2 Decomposition. Terms not defined here are used in the sense of Harary [2] .

## II. PRELIMINARIES

**Definition 2.1.** Let G be a simple graph of order p and size q. If $G_1,G_2,\ldots,G_n$ are edge-disjoint subgraphs of G such that $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_n)$, then $\{G_1,G_2,\ldots,G_n\}$ is said to be a Decomposition of G.

**Definition 2.2.** A graph G is said to have Power of 2 Decomposition if G can be decomposed into n subgraphs $\{G_1, G_2, \ldots, G_n\}$ such that each $G_i$ is connected and $|E(G_i)| = 2^i$, for $1 \le i \le n$. Clearly $q = 2[2^n - 1]$ is the sum of $2, 2^2, 2^3, \ldots, 2^n$. Thus we denote the Power of 2 Decomposition as $\{G_2, G_4, G_8, \ldots, G_{2^n}\}$.

**Theorem 2.3.** A graph G admit Power of 2 Decomposition $\{G_2, G_4, G_8, \ldots, G_{2^n}\}$ if and only if $q = 2[2^n - 1]$ for each $n \in N$.

**Definition 2.4.** The Jelly Fish graph denoted by J(m, n) is a graph obtained from a 4-cycle $(v_1, v_2, v_3, v_4)$ together with an edge $v_1v_3$ and appending m pendant edges to $v_4$ and n pendant edges to $v_2$.

**Definition 2.5.** The Ladder graph $L_m$ is defined as the cartesian product of path $P_m$ with a complete graph $K_2$.

**Definition 2.6.** A triangular snake graph , denoted by Tm , is a graph obtained from a path $u_1u_2 \ldots u_n$ by joining $u_i$ and $u_{i+1}$ to a new vertex $v_i$ , $1 \le i \le n-1$.

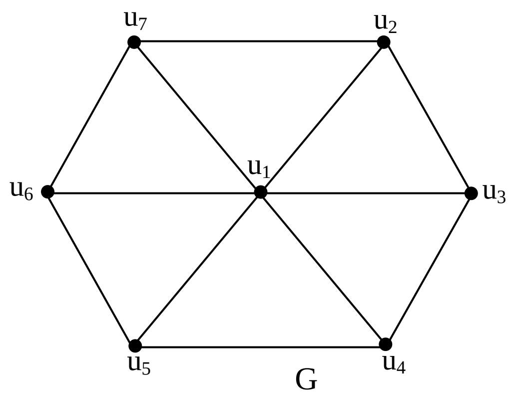
**Definition 2.7.** The Helm $H_m$ is the graph obtained from wheel $W_m$ by attaching pendant edges to each of its rim vertices.

## III. DOUBLE POWER OF 2 DECOMPOSITION OF GRAPHS

**Definition 3.1.** A graph G is said to have Double Power of 2 Decomposition [DPo2D] if G can be decomposed into 2n subgraphs $\{2G_1, 2G_2, \ldots, 2G_n\}$ such that each $G_i$ is connected and $|E(G_i)| = 2^i$, $1 \leq i \leq n$.

Clearly, $q = 4[2^n - 1]$. We denote the Double Power of 2 Decomposition [DPo2D] as $\{2G_2, 2G_2^2, \ldots, 2G_2^n\}$.

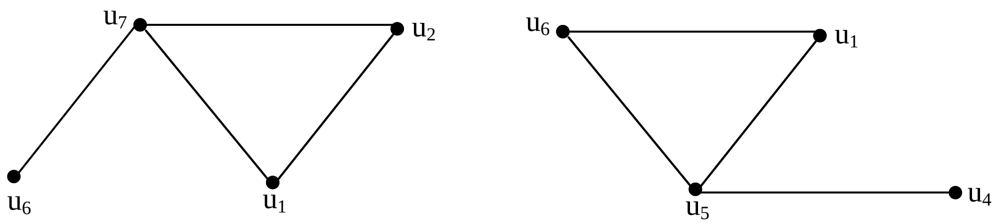
**Example 3.2.** Consider the graph G given in the following figure.




The graph G admit Double Power of 2 Decomposition. The DPo2D of G is given in the following figure.

$G_2$:


$G_2^2$ :




## IV. DOUBLE POWER OF 2 DECOMPOSITION OF J(m, 3)

**Lemma 4.1.** Let $n \equiv 0 \pmod 2$. Then G can be decomposed into $\{2G_2, 2G_2^2, \ldots, 2G_2^n\}$. Here $4[2^n - 1] = m + 8$.

**Proof .** We have $n \equiv 0 \pmod 2$. Then $n = 2r$, $r \geq 1$ and $r \in \mathbb{Z}$. Proof is by induction on r. when $r = 1$, $n = 2$. Then $m + 8 = 12$ can be decomposed into $\{2G_2, 2G_2^2\}$. Hence the result is true for $r = 1$.

Assume that the result is true for $r - 1$. Then $n = 2r - 2$. Thus $q' = m + 8 = 4[2^{2r-2} - 1]$ can be decomposed into $\{2G_2, 2G_2^2, \ldots, 2G_2^{2r-2}\}$.

Now, to prove the result is true for r. We have to prove that $q = m + 8 = 4[2^{2r} - 1]$ can be decomposed into $\{2G_2, 2G_2^2, \ldots, 2G_2^{2r}\}$. Define $q = q' \cup (2r-1) \cup (2r)$. Then $q = m + 8 = q' + 2[2^{2r-1} + 2^{2r}] = 4[2^{2r} - 1]$ can be decomposed into $\{2G_2, 2G_2^2, \ldots, 2G_2^{2r}\}$. Hence by induction hypothesis, the lemma is proved for all r.

**Lemma 4.2.** Let n - 1 ≡ 0 (mod 2). Then G can be decomposed into $\{2G_2, 2G_2^2, \ldots 2G_2^n\}$. Here $4[2^n -1] = m + 8$.

**Proof.** We have n - 1 ≡ 0 (mod 2). Then n = 2r +1, r ≥ 1 and r ∈ ℤ. Proof is by induction on r. When r = 1, n = 3. Then m + 8 = 28 can be decomposed into $\{2G_2, 2G_2^2, 2G_2^3\}$. Hence the result is true for r = 1.

Assume that the result is true for r - 1. Then n = 2r -1. Thus q' = m + 8 = $4[2^{2r-1} - 1]$ can be decomposed into$\{2G_2, 2G_2^2, \ldots, 2G_2^{2r-1}\}$.

Now, to prove the result is true for r. We have to prove that $q = m + 8 = 4[2^{2r+1} - 1]$ can be decomposed into $\{2G_2, 2G_2^2, \ldots, 2G_2^{2r+1}\}$. Define $q = q' \cup (2r) \cup (2r + 1)$. Then $q = m + 8 = q' + 2[2^{2r} + 2^{2r+1}] = 4[2^{2r+1} -1]$ can be decomposed into $\{2G_2, 2G_2^2, \ldots, 2G_2^{2r+1}\}$. Hence by induction hypothesis, the lemma is proved for all r.

**Theorem 4.3.** For an even integer m, the Jelly fish graph J(m, 3) admit Double Power of 2 Decomposition $\{2G_2, 2G_2^2, \ldots, 2G_2^n\}$ if and only if there exists an integer n satisfying the following properties :

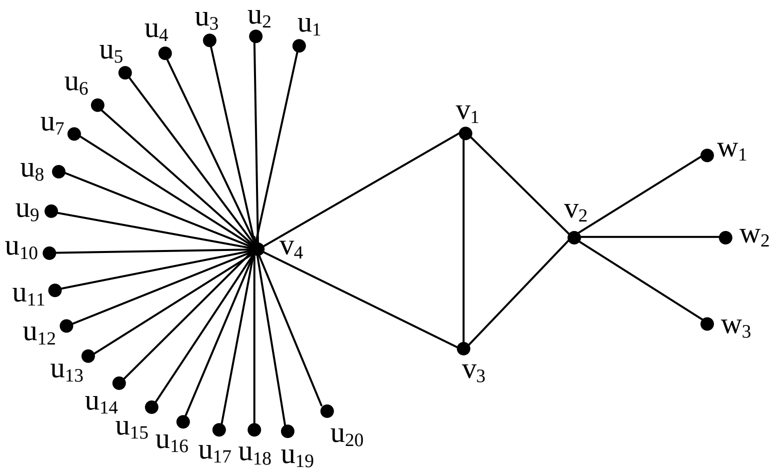
(a) n = 2r or 2r+1, r ≥ 1 and r ∈ ℤ

(b) $4[2^n – 1] = m + 8$

**Proof.** Let G = J(m, 3). By the definition of G, q = m + 8. Assume that G admit Double Power of 2 Decomposition. By the definition, $q = 4[2^n -1]$. Hence $4[2^n -1] = m+8$. Clearly, m is an even integer. Now, $2^n = \frac{m + 12}{4}$. This implies n = 2r or 2r + 1, r ≥ 1 and r ∈ ℤ.

Conversely, assume that n = 2r or 2r+1, r ≥ 1 and r ∈ ℤ. Also, $4[2^n -1] = m+8$. This implies that m is always even. By lemma 3.1 and 3.2 , $q = m + 8 = 4[2^n -1]$ can be decomposed into $\{G_2, G_2^2, \ldots, G_2^n\}$. Hence G admit Double Power of 2 Decomposition.

**Illustration 4.4.** As an illustration, let us decompose the Jelly Fish J(20, 3). The graph J(20, 3) is given in figure.

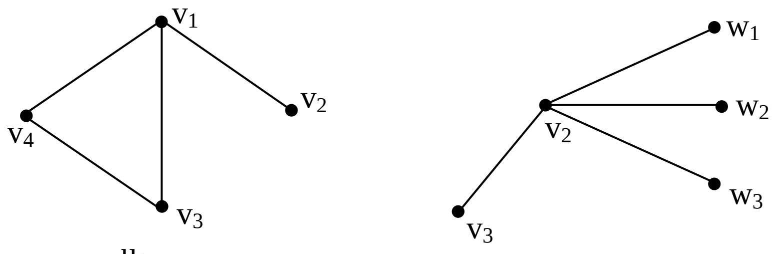



**Jelly Fish Graph J(20, 3)**

Here m = 20. Thus, n = 3. Hence there will be two copies of three decompositions. The DPo2D of J(20, 3) is $\{2G_2, 2G_2^2, 2G_2^3\}$ and the decompositions are given in the following figure.

$G_2$ :


$G_2^2$ :




$G_2^3$ :

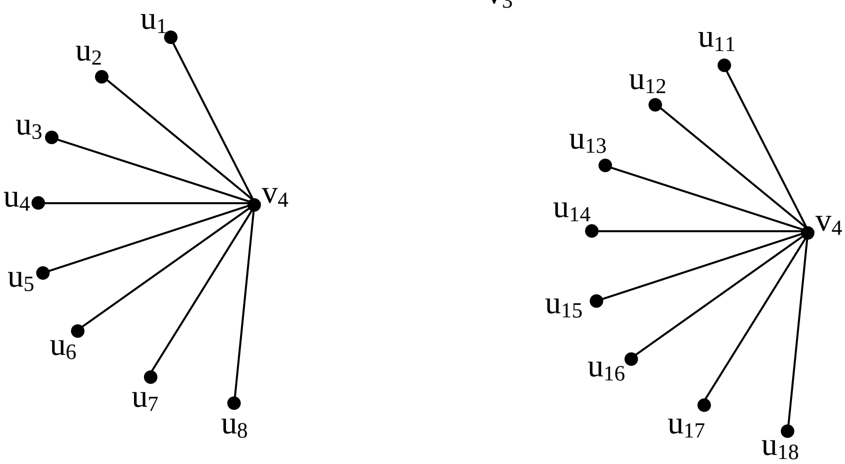



**Table 4.5.** List of first 10 J(m, 3)'s which accept DPo2D and their decompositions are given in the following table.

| m | DPo2D |
|---|---|
| 4 | $2G_2, 2G_2^2$ |
| 20 | $2G_2, 2G_2^2, 2G_2^3$ |
| 52 | $2G_2, 2G_2^2, \ldots, 2G_2^4$ |
| 116 | $2G_2, 2G_2^2, \ldots, 2G_2^5$ |
| 244 | $2G_2, 2G_2^2, \ldots, 2G_2^6$ |
| 500 | $2G_2, 2G_2^2, \ldots, 2G_2^7$ |
| 1012 | $2G_2, 2G_2^2, \ldots, 2G_2^8$ |
| 2036 | $2G_2, 2G_2^2, \ldots, 2G_2^9$ |
| 4084 | $2G_2, 2G_2^2, \ldots, 2G_2^{10}$ |
| 8180 | $2G_2, 2G_2^2, \ldots, 2G_2^{11}$ |

## V. DOUBLE POWER OF 2 DECOMPOSITION OF $L_m$

**Theorem 5.1.** For an even integer m, the ladder graph $L_m$ admit Double Power of 2 Decomposition $\{2G_2, 2G_2^2, \ldots, 2G_2^n\}$ if and only if there exists an integer n satisfying the following properties.

1. $n = 2r - 1$, $r \geq 1$ and $r \in \mathbb{Z}$
2. $4[2^n - 1] = 3m - 2$

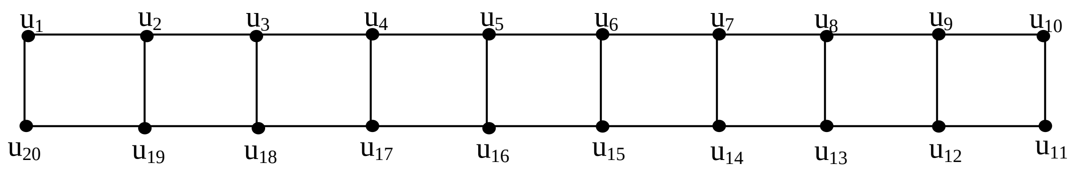
**Proof .** Let $G = L_m$. By the definition of G, q = 3m-2. Assume that G admit Double Power of 2 Decomposition. By the definition, $q = 4[2^n - 1]$. Hence $4[2^n - 1] = 3m-2$. Clearly, m is an even integer Now, $2^n = \frac{3m+2}{4}$. This implies n = 2r - 1, $r \geq 1$ and $r \in \mathbb{Z}$.

Conversely, assume that n = 2r - 1, $r \geq 1$ and $r \in \mathbb{Z}$. Also, $4[2^n - 1] = 3m-2$. We have to prove that G accept Double Power of 2 Decomposition. We prove this by induction on r. when r = 1, n = 1. Then 3m-2 = 4 can be decomposed into $\{2G_2\}$. Hence the result is true for r = 1.

Assume that the result is true for r-1. Then n = 2r-3. Thus $q' = 3m-2 = 4[2^{2r-3} - 1]$ can be decomposed into $\{2G_2, 2G_2^2, \ldots, 2G_2^{2r-3}\}$.

Now, to prove the result is true for r. We have to prove that $q = 3m-2 = 4[2^{2r-1} – 1]$ can be decomposed into $\{2G_2, 2G_2^2, \ldots, 2G_2^{2r-1}\}$. Define $q = q' \cup (2r-2) \cup (2r-1)$. Then $q = 3m-2 = q' + 2[2^{2r-2} + 2^{2r-1}] = 4[2^{2r-1} - 1]$ can be decomposed into $\{2G_2, 2G_2^2, \ldots, 2G_2^{2r-1}\}$. Hence by induction hypothesis, G can be decomposed into $\{2G_2, 2G_2^2, \ldots, 2G_2^n\}$ where n = 2r-1, $r \geq 1$ and $r \in \mathbb{Z}$. Thus G admit Double Power of 2 Decomposition.

**Illustration 5.2.** As an illustration, let us decompose the ladder graph $L_{10}$. The graph $L_{10}$ is given in the following figure.




**Ladder Graph $L_{10}$**

Here m = 10. Then n = 3. Hence there will be two copies of three decompositions. The DPo2D of $L_{10}$ is $\{2G_2, 2G_2^2, 2G_2^3\}$ and the decompositions are given in the following figure.

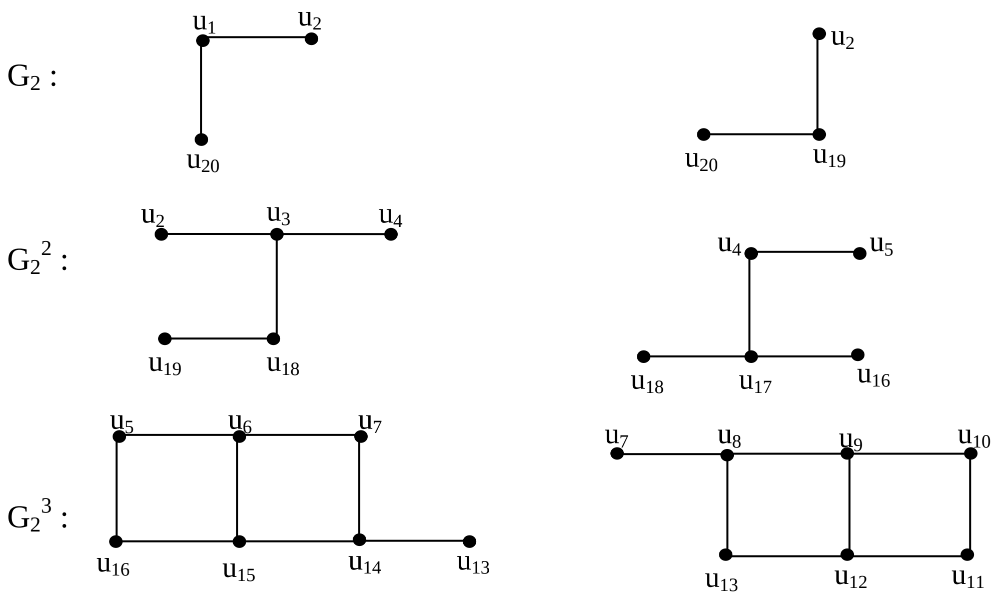


**Table 5.3.** List of first 10 $L_m$'s that accept DPo2D and their decompositions are given in the following table.

| m | DPo2D |
|---|---|
| 2 | $2G_2$ |
| 10 | $2G_2, 2G_2^2, 2G_2^3$ |
| 42 | $2G_2, 2G_2^2, \ldots, 2G_2^5$ |
| 170 | $2G_2, 2G_2^2, \ldots, 2G_2^7$ |
| 682 | $2G_2, 2G_2^2, \ldots, 2G_2^9$ |
| 2730 | $2G_2, 2G_2^2, \ldots, 2G_2^{11}$ |
| 10922 | $2G_2, 2G_2^2, \ldots, 2G_2^{13}$ |
| 43690 | $2G_2, 2G_2^2, \ldots, 2G_2^{15}$ |
| 174762 | $2G_2, 2G_2^2, \ldots, 2G_2^{17}$ |
| 699050 | $2G_2, 2G_2^2, \ldots, 2G_2^{19}$ |

## VI. DOUBLE POWER OF 2 DECOMPOSITION OF $T_m$

**Theorem 6.1.** For an odd integer m, the triangular snake graph $T_m$ accept Double Power of 2 Decomposition $\{2G_2, 2G_2^2, \ldots, 2G_2^n\}$ if and only if there exists an integer n satisfying the following properties.

1. $n = 2r$, $r \geq 1$ and $r \in \mathbb{Z}$
2. $4[2^n - 1] = 3m - 3$

**Proof .** Let $G = T_m$. By the definition of G, q = 3m-3. Assume that G admit Double Power of 2 Decomposition. By the definition, $q = 4[2^n - 1]$. Hence $4[2^n - 1] = 3m - 3$. Clearly, m is an odd integer. Now, $2^n = \frac{3m+1}{4}$. This implies $n = 2r$, $r \geq 1$ and $r \in \mathbb{Z}$.

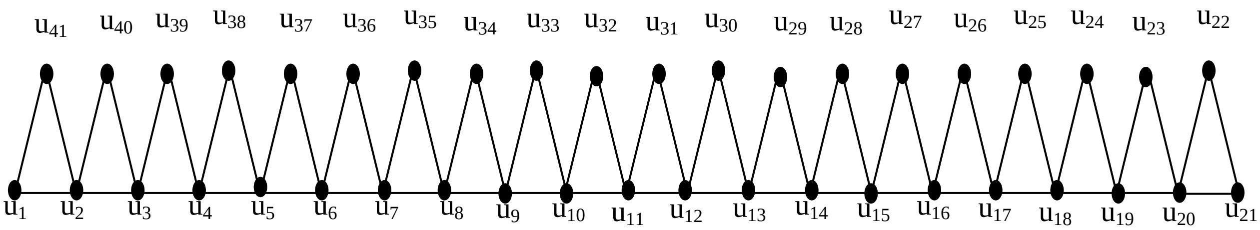
Conversely, assume that $n = 2r$, $r \geq 1$ and $r \in \mathbb{Z}$. Also, $4[2^n - 1] = 3m - 3$. We have to prove that G accept Double Power of 2 Decomposition. We prove this by induction on r.

When r = 1, n = 2. Then 3m - 3 = 12 can be decomposed into $\{2G_2, 2G_2^2\}$. Hence the result is true for r = 1.

Assume that the result is true for r - 1. Then n = 2r-2. Thus $q' = 3m-3 = 4[2^{2r-2} - 1]$ can be decomposed into $\{2G_2, 2G_2^2, \ldots, 2G_2^{2r-2}\}$.

Now, to prove the result is true for r. We have to prove that $q = 3m-3 = 4[2^{2r} - 1]$ can be decomposed into $\{2G_2, 2G_2^2, \ldots, 2G_2^{2r}\}$. Define $q = q' \cup (2r-1) \cup (2r)$. Then $q = 3m-3 = q' + 2[2^{2r-1} + 2^{2r}] = 4[2^{2r} - 1]$ can be decomposed into $\{2G_2, 2G_2^2, \ldots, 2G_2^{2r}\}$. Hence by induction hypothesis, G can be decomposed into $\{2G_2, 2G_2^2, \ldots, 2G_2^n\}$ where $n = 2r$, $r \geq 1$ and $r \in \mathbb{Z}$. Thus G admit Double Power of 2 Decomposition.

**Illustration 6.2.** As an illustration, let us decompose the triangular snake graph $T_{21}$. The graph $T_{21}$ is given in the following figure.




**Triangular Snake Graph $T_{21}$**

Here m = 21. Then n = 4. Thus there will be two copies of four decompositions. The DPo2D of $T_{21}$ is $\{2G_2, 2G_2^2, 2G_2^3, 2G_2^4\}$ and the decomposition are given in the following figure.

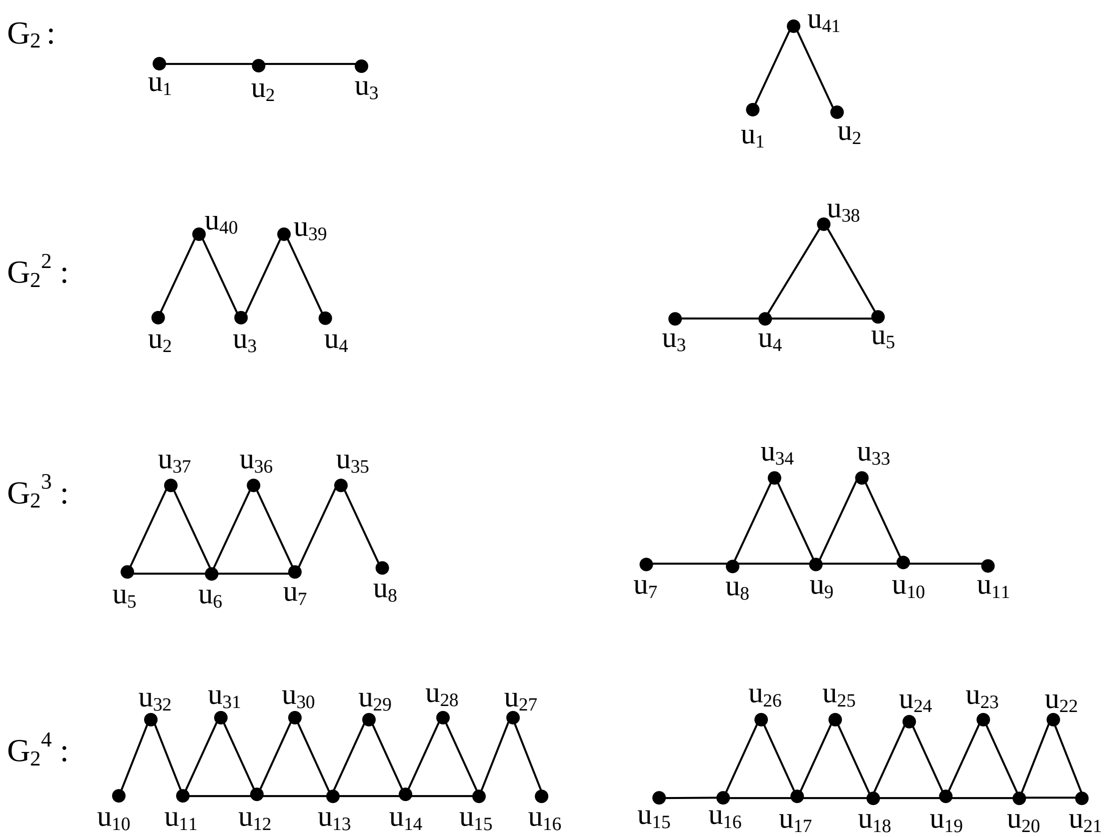



**Table 6.3.** List of first 10 Tm's that accept Double Power of 2 Decomposition and their decompositions are given in the following table.

| m | DPo2D |
|---|---|
| 5 | $2G_2, 2G_2^2$ |
| 21 | $2G_2, 2G_2^2, 2G_2^3, 2G_2^4$ |
| 85 | $2G_2, 2G_2^2, \ldots, 2G_2^6$ |
| 341 | $2G_2, 2G_2^2, \ldots, 2G_2^8$ |

| | |
|---|---|
| 1365 | $2G_2, 2G_2^2, \ldots, 2G_2^{10}$ |
| 5461 | $2G_2, 2G_2^2, \ldots, 2G_2^{12}$ |
| 21845 | $2G_2, 2G_2^2, \ldots, 2G_2^{14}$ |
| 87381 | $2G_2, 2G_2^2, \ldots, 2G_2^{16}$ |
| 349525 | $2G_2, 2G_2^2, \ldots, 2G_2^{18}$ |
| 1398101 | $2G_2, 2G_2^2, \ldots, 2G_2^{20}$ |

## VII. DOUBLE POWER OF 2 DECOMPOSITION OF $H_m$

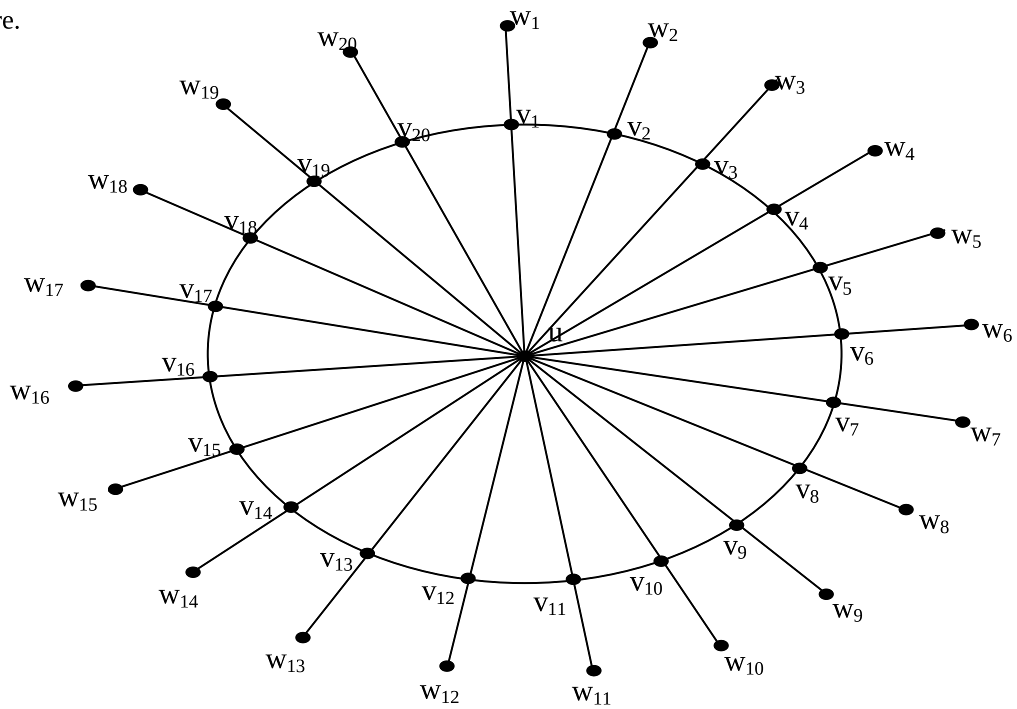
**Theorem 7.1.** For an even integer m, the helm $H_m$ accept Double Power of 2 Decomposition $\{2G_2, 2G_2^2, \ldots, 2G_2^n\}$ if and only if there exists an integer n satisfying the following properties :

1. $n = 2r$, $r \geq 1$ and $r \in \mathbb{Z}$
2. $4[2^n - 1] = 3m$

**Proof.** Let $G = H_m$. By the definition of G, q = 3m. Assume that G accept Double Power of 2 Decomposition. By the definition, $q = 4[2^n - 1]$. Hence $4[2^n - 1] = 3m$. Clearly, m is an even integer. Now, $2^n = \frac{3m+4}{4}$. This implies $n = 2r$, $r \geq 1$ and $r \in \mathbb{Z}$.

Conversely, assume that $n = 2r$, $r \geq 1$ and $r \in \mathbb{Z}$. Also, $4[2^n - 1] = 3m$. We have to prove that G accept Double Power of 2 Decomposition. We prove this by induction on r. When r = 1, n = 2. Then 3m = 12 can be decomposed into $\{2G_2, 2G_2^2\}$. Hence the result is true for r = 1. Assume that the result is true for r-1. Then n = 2r-2. Thus $q' = 3m = 4[2^{2r-2} - 1]$ can be decomposed into $\{2G_2, 2G_2^2, \ldots, 2G_2^{2r-2}\}$. Now, to prove the result is true for r. We have to prove that $q = 3m = 4[2^{2r} - 1]$ can be decomposed into $\{2G_2, 2G_2^2, \ldots, 2G_2^{2r}\}$. Define $q = q' \cup (2r-1) \cup (2r)$. Then $q = 3m = q' + 2[2^{2r-1} + 2^{2r}] = 4[2^{2r} - 1]$ can be decomposed into $\{2G_2, 2G_2^2, \ldots, 2G_2^{2r}\}$. Hence by induction hypothesis, G can be decomposed into $\{2G_2, 2G_2^2, \ldots, 2G_2^n\}$ where $n = 2r$, $r \geq 1$ & $r \in \mathbb{Z}$. Thus G admit Double Power of 2 Decomposition.

**Illustration 7.2.** As an illustration, let us decompose the helm $H_{20}$. The graph $H_{20}$ is given in the following figure.



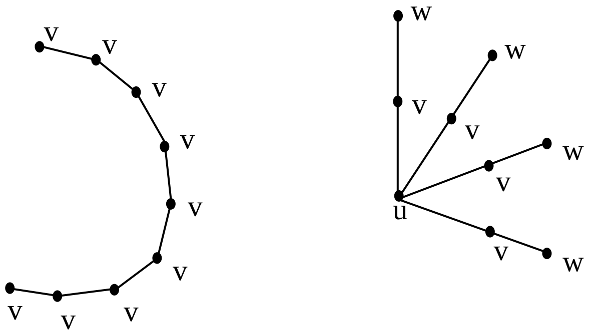

**The Graph $H_{20}$**

Here m = 20. Hence n = 4. Thus there will be two copies of four decompositions. The DPo2D of $H_{20}$ is $\{2G_2, 2G_2^2, 2G_2^3, 2G_2^4\}$ and the decompositions are given in the following figure.

$G_2$:


$G_2^2$:


$G_2^3$:




$G_2^4$ :

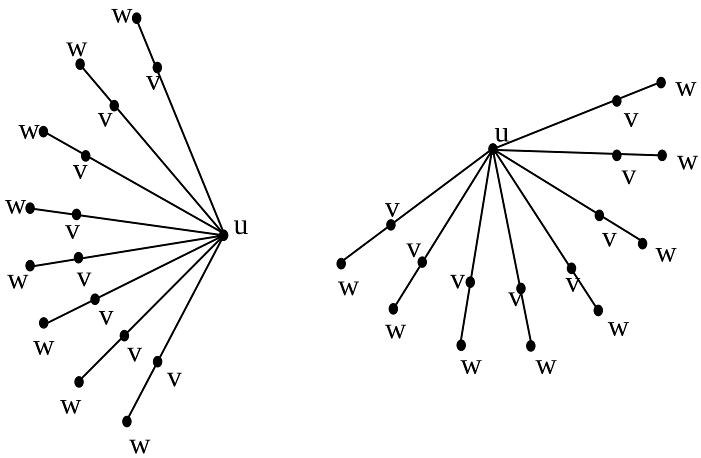



**Table 7.3.** List of first 10 $H_m$'s that accept Double Power of 2 Decomposition and their decompositions are given in the following table.

| m | DPo2D |
|---|---|
| 4 | $2G_2, 2G_2^2$ |
| 20 | $2G_2, 2G_2^2, 2G_2^3, 2G_2^4$ |
| 84 | $2G_2, 2G_2^2, \ldots, 2G_2^6$ |
| 340 | $2G_2, 2G_2^2, \ldots, 2G_2^8$ |
| 1364 | $2G_2, 2G_2^2, \ldots, 2G_2^{10}$ |
| 5460 | $2G_2, 2G_2^2, \ldots, 2G_2^{12}$ |
| 21844 | $2G_2, 2G_2^2, \ldots, 2G_2^{14}$ |
| 87380 | $2G_2, 2G_2^2, \ldots, 2G_2^{16}$ |
| 349524 | $2G_2, 2G_2^2, \ldots, 2G_2^{18}$ |
| 1398100 | $2G_2, 2G_2^2, \ldots, 2G_2^{20}$ |

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