Novel method to find the derivative of a function

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Abstract — *It is possible to find the derivative of a function by constraining a line in such a way that the line is allowed to move as a tangent to that function. Then the equation of the slope of that tangent line gives the derivative equation of the function.*

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I. DERIVATIVE FOR SIMPLE FUNCTION: $f(x) = ax^2$

Let's use this method to find the derivative of a simple function $f(x) = ax^2$. Consider a Cartesian plane plotted with the given function $f(x)$ and a line which is free to move in the plane. Then the equations of given function and line are represented as:

$$y = ax^2 \quad (1)$$

$$y = m(x - k) \quad (2)$$

Where m is the slope of the line and k is the x-intercept of the line.
The line and function curve will intersect at points where the x and y coordinates of line are equal corresponding coordinates of function curve. Hence the intersecting points can be found out by substituting $eq(1)$ in $eq(2)$,which results in the following equation:

$$ax^2 = m(x - k)$$

$$ax^2 - mx + mk = 0 \quad (3)$$

$eq(3)$ gives the intersection points of the line and the curve.
Since, $eq(3)$ is a quadratic equation it will have two solutions which implies that the line will meet the function curve at two points. The type of solution depends upon the value of the discriminant. For $eq(3)$ the discriminant is:

$$\Delta = m^2 - 4amk$$

- If the discriminant is greater than 0, then there exists two roots and the line and curve will have two intersection points in real Cartesian plane.
- If the discriminant is less than 0, then there exists two imaginary roots and the line and curve will not have any intersection points in real Cartesian plane.
- If the discriminant is equal to 0 then there exists only one solution and line intersects the function curve at only one point. If a line intersects the function curve at only one point , then that line is a tangent to that function curve.

$$\Delta = 0$$

$$k = \frac{m}{4a} \quad (4)$$

Hence $eq(4)$ acts as a constraint equation which makes the line to move as a tangent to function curve. Since, slope of the tangent equals to the value of derivative to the function, the derivative of the function is obtained by imposing tangency constraint($eq(4)$) to intersection points equation($eq(3)$).Here value of k is replaced by m and the degree of freedom of line is reduced to one. By equating $eq(3)$ and $eq(4)$ we have:

$$ax^2 - mx + \frac{m^2}{4a} = 0$$

For given value of x , we find the value of slope(i.e, m).Solving the above equation and taking m as variable will results in the following equation:

$$m^2 - 4axm + 4a^2x^2 = 0$$

$$m = 2ax$$

As slope m represents the derivative of the function : $\frac{d}{dx}ax^2 = 2ax$.

The discriminant equation is complex for the functions with higher powers of x. So, it is required to incorporate another method to constraint the line to move as tangent to the function curve. This can be done by setting two solutions of an equation to one value. A simple quadratic equation consisting two equal real roots can be described as shown below:

$$(x-p)^2 = 0$$

Where p is the common solution, where the line meets the curve as tangent. For some further references we call the above equations as comparison equation. On expanding the above equation :

$$x^2 - 2px + p^2 = 0 \quad (5)$$

The line and function curve, intersection equation is:

$$ax^2 - mx + mk = 0$$

$$x^2 - \frac{m}{a}x + \frac{m}{a}k = 0 \quad (6)$$

Comparing the coefficients of $eq(6)$ with coefficients of $eq(5)$ will force the line curve intersection equation (i.e, $eq(6)$) to set its two of solutions to one value. There by making line to move as tangent to function curve. Comparing coefficients will result in the following linear equations with m and k as unknowns:

$$2p = \frac{m}{a} \quad \Rightarrow \quad p = \frac{m}{2a} \quad (7)$$

$$p^2 = \frac{mk}{a} \quad (8)$$

Solving the above two equations will results in:

$$k = \frac{m}{4a}$$

The above obtained equation is constraint equation which is same as $eq(4)$. So this new method can also be used to obtain constraint equation as an alternative of discriminant for higher powers. Substituting the above k value in $eq(6)$ will give a new equation and solving that new equation for m will give the value of the derivative of that function (and is equal to $2ax$).

This new process may seem simple for $f(x) = ax^2$ but will be complicated for higher powers of x. It is observed in $eq(8)$ that p is the point where the line meets the function curve as tangent. So at this x-coordinate we get the derivative on solving the equations . So, point p can be replaced with x.

By replacing p with x in $eq(7)$:

$$x = \frac{m}{2a} \quad \Rightarrow \quad m = 2ax$$

As slope m represents the derivative of the function: $\frac{d}{dx}ax^2 = 2ax$.

Consider the function equation and combine it with line equation having x- intercept and compare the the coefficients of this equation with comparison equation having p as tangent solution .After comparison solve for p. Replacing p with x will yield the derivative of the function.

II. DERIVATIVES OF FUNCTIONS WITH HIGHER POWERS OF x

Consider a simple third degree polynomial, $f(x) = ax^3$ and a line plotted in a Cartesian plane. Their equations in the plane are represented as:

$$y = ax^3 \quad (9)$$

$$y = m(x-k) \quad (10)$$

Where m is the slope of the line and k is the x-intercept of the line. Substituting $eq(9)$ in $eq(10)$ to get the intersection points of the line and the function curve:

$$ax^3 = m(x-k)$$

$$ax^3 - mx + mk = 0 \quad (11)$$

Since cubic equations consist three solutions. For the line to intersect the function curve as tangent , two of the solutions of cubic equation must be equal at the point of tangent (at $x = p$) and the remaining third solution can be any other arbitrary value (at $x = q$) .Then the comparison equation will be written as:

$$(x-p)^2(x-q) = 0$$

$$x^3 - (2p+q)x^2 + (p^2 + 2pq)x - p^2q = 0 \quad (12)$$

$eq(11)$ can be rewritten as:

$$x^3 - \frac{m}{a}x + \frac{mk}{a} = 0 \quad (13)$$

Any set of values of p and q which are satisfying this comparison equation will constraint the line in such a way that the line will always moves as tangent to function curve at the $x = p$.

Comparing coefficients of $eq(12)$ and $eq(13)$ will result in the following equations:

$$2p + q = 0$$

$$p^2 + 2pq = -\frac{m}{a}$$

On solving the above two equations for p :

$$p^2 = \frac{m}{3a}$$

Since, p is the x-coordinate at which line touches function curve as tangent (i.e, this is the point where slope of the function is obtained) , p can be replaced with x. On doing so, the above equation changes as:

$$x^2 = \frac{m}{3a}$$

$$m = 3ax^2$$

As slope m represents the derivative of the function:

$$\frac{d}{dx}(ax^3) = 3ax^2$$

For $f(x) = ax^n$, use the following comparison equation:

$$(x-p)^2(x-q_1)(x-q_2)\dots\dots(x-q_{n-2}) = 0$$

The above discussed procedure can be used to obtain derivatives of any other Algebraic functions.

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