L-fuzzy ideals of Semilattices

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Abstract

In this paper the notion of an L-fuzzy ideal of a semilattice is introduced and proved certain important structural properties of these. 0-distributive semilattices are characterized in terms of L-fuzzy ideals and prime L-fuzzy filters. The Stones's version separation theorem on prime filters of distributive semilattices is extended to prime L-fuzzy filters. Furthermore, the notions of prime(maximal) L-fuzzy ideals of bounded semilattices are introduced and characterized.

Keywords: 0-distributive semilattice; L-fuzzy ideal; L-fuzzy filter; prime L-fuzzy filter; prime L-fuzzy ideal; frame; meet-prime element.

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1 Introduction

Ever since Zadeh [10] introduced fuzzy set theory, several researchers fuzzified various notions/results of abstract algebra with membership function assuming truth values in the unit interval $[0, 1]$ of real numbers. Rosenfield [1] formulated the concept of a fuzzy subgroup of a group. Kuroki [11] investigated the properities of fuzzy ideals of a semigroup. Malik and Moderson [5] worked on fuzzy subrings and ideals of rings. Liu [17] introduced fuzzy invarient subgroups and fuzzy ideals, and so on.

In [2, 3], Sundar Raj, Subrahmanyam and Swamy have introduced and studied the notions of fuzzy filters and prime(maximal) L-fuzzy filters of meet-semilattices having truth values in a complete lattice L satisfying the infinite meet distributive law. This type of lattice is called a frame. In this paper we introduce the concept of an L-fuzzy ideal of a semilattice S having truth values in a frame L and prove certain important structural properties of these. The class of distributive semilattices is an important subclass of semilattices; for details we refere to [7, 8, 9, 18]. In particular, the class of 0-distributive semilattices is a natural generalzation of the class of psedo-complemented semilattices. Here, mainly we characterize 0-distributive semilattices in terms of L-fuzzy ideals and prime L-fuzzy filters. Also, a general theory of prime L-fuzzy filters of bounded semilattices [2, 3] is applied to extend the Stone's version separation theorem on prime filters of distributive semilattices to prime L-fuzzy filters. Finally, prime(maximal) L-fuzzy ideals of a bounded semilattice S are determined by obtaining a one-to-one correspondence between prime(maximal) L-fuzzy ideals of S and the pairs (P, α), where P is a prime ideal of S and α a meet-prime(dual atom) of L.

Throughout this paper, L stands for a non-trivial frame $(L, \wedge, \vee, 0, 1)$; i.e., a complete lattice satisfying the infinite meet distributive law:

$$\alpha \wedge \Big(\bigvee_{\beta \in M} \beta \Big) = \bigvee_{\beta \in M} (\alpha \wedge \beta)$$

for all $\alpha \in L$ and $M \subseteq L$. Here the operations $\vee$ and $\wedge$ are, respectively, $l.u.b$ and $g.l.b$ in the lattice L. Also S stands for a semilattice $(S, \wedge)$ unless otherwise stated. As usual, by an L-fuzzy subset of S, we mean a mapping of S into L. If $L = [0, 1]$, the unit intervel of real numbers, these are the usual fuzzy subsets of S originally introduced by L. A. Zadah [10]. An element $\alpha(\neq 1)$ of L is said to be meet-prime if, for any $a, b \in L$, $a \wedge b \leq \alpha$ implies $a \leq \alpha$ or $b \leq \alpha$. An element $\alpha(\neq 1)$ in L is said to be a dual atom of L if there is no $\beta \in L$ such that $\alpha < \beta < 1$. Note that, α is dual atom if and only if α is maximal in $L - \{1\}$. It is known that any dual atom in a bounded distributive lattice is meet-prime.

2 Preliminaries

Throughout this article the word semilattice will mean meet-semilattice i.e., a non-empty set S together with an idempotent, commutative and associative binary operation $\wedge$ on S. If we define $x \leq y$ if and only if $x \wedge y = x$ for all $x, y \in S$, then $(S, \leq)$ becomes a partial ordered set in which for any $x, y \in S$, $x \wedge y$ is the $g.l.b$ of $\{x, y\}$ in S. This is said to be the partial order induced by $\wedge$ on S. If the $l.u.b$ of any $x, y \in S$ exsits, then we say that $x \vee y$ exists in S. If $x \vee y$ exists in S for all $x, y \in S$, then S is said to be a lattice.

A subset X of S is said to be directed above if for any $x, y \in X$, there exists $z \in X$ such that $z \geq x, y$. A non-empty subset I of S is said to be an initial(final) segment of S, if for any $x \in I$, $y \in S$, $y \leq x$ ($x \leq y$ respectively) implies $y \in I$. An inital segment I of S is said to be an ideal of S, if I is directed above. A final segment F of S is said to be a filter of S, if $x \in F$, $y \in F$ implies $x \wedge y \in F$. An ideal(filter) I of S is said to be proper if $I \neq S$.

For any $X \subseteq S$, the filter generated by X of S is given by $[X) = \{y \in S : \bigwedge_{i=1}^{n} x_i \leq y$ for some $x_i \in X\}$. Inparticular, for any $x \in S$, $[x) = \{y \in S : y \geq x\}$ called the principal filter generated by x. A proper filter F of S is said to be prime if, whenever two filters F_1 and F_2 are such that $\phi \neq F_1 \cap F_2 \subseteq F$ then $F_1 \subseteq F$ or $F_2 \subseteq F$ (or, equivalently, if, for any $x, y \in S$, $x \notin F$ and $y \notin F$ imply the existence of $z \in S$ such that $x \leq z$, $y \leq z$ and $z \notin F$).

A proper ideal(filter) I of S is said to be maximal if the only ideal(filter) strictly containing I is S. For any non-empty subset X of a semilattice S with smallest element 0, the set $X^* = \{y \in S : x \wedge y = 0 \text{ for all } x \in X\}$ is called the annihilator of X. For any $x \in S$ we write $(x]$ for the principal ideal generated by x. Note that $(x] = \{y \in S : y \leq x\}$ and $(x]^* = \{x\}^* = \{y \in S : x \wedge y = 0\}$. A semilattice S with 0 is called pseudo-complemented if, for each $x \in S$, there exists $x^* \in S$ such that $x \wedge x^* = 0$ and $x \wedge y = 0 \Rightarrow y \leq x^*$, for all $y \in S$; this element x^* is called the pseudo-complement of x. A semilattice S is said to be distributive if for any $a, b, c \in S$, $a \wedge b \leq c$ implies the existence of $x, y \in S$ such that $x \geq a$, $y \geq b$ and $x \wedge y = c$.

On the other hand, we present some necessary definitions and results concerning the notions of L-fuzzy filters and prime L-fuzzy filters of bounded semilattices and L-fuzzy ideals of lattices mostly taken from [2, 3, 14] which will be used later on.

Definition 2.1. An L-fuzzy subset A of a semilattice $(S, \wedge)$ is said to be an L-fuzzy filter(simply, fuzzy filter) of S if

$$A(x_0) = 1 \text{ for some } x_0 \in S \text{ and } A(x \wedge y) = A(x) \wedge A(y) \text{ for all } x, y \in S.$$

Theorem 2.2. *The following are equivalent to each other for any L-fuzzy subset A of S*

(1) *A is an L-fuzzy filter of S*

(2) $A(x_0) = 1$ *for some* $x_0 \in S$,
$A(x \wedge y) \geq A(x) \wedge A(y)$ *and* $x \leq y \Rightarrow A(y) \geq A(x)$

(3) *A_α is a filter of S for all $\alpha \in L$*

Lemma 2.3. Let A be a fuzzy filter of S and X a non-empty subset of S, and $x, y \in S$. We have

(1) $x \in [X) \Rightarrow A(x) \geq \bigwedge_{i=1}^{m} A(a_i)$ for some $a_1, a_2, \ldots a_m \in X$

(2) $x \in [y) \Rightarrow A(x) \geq A(y)$

(3) If S is bounded then $A(0) < 1$ and $A(1) = 1$

Theorem 2.4. *Let $\mathcal{F}F(S)$ denote the set of all fuzzy filters of a meet-semilattice $(S, \wedge)$ with greatest element 1. Then $(\mathcal{F}F(S), \leq)$ is a complete lattice in which, for any family $\{A_i : i \in \Delta\}$ of fuzzy filters of S, the g.l.b and l.u.b are given by*

$$\bigwedge_{i\in\Delta} A_i = \textit{The point-wise infimum of } A_i's,$$
$$\bigvee_{i\in\Delta} A_i = \textit{The point- wise infimum of } \{A \in \mathcal{F}F(S) : A_i \leq A \textit{ for all } i \in \Delta\}.$$

Theorem 2.5. *Let A be an L-fuzzy subset of S. Then the fuzzy filter $\bar{A}$ generated by A is given by*

$$\bar{A}(x_0) = 1 \textit{ for some } x_0 \in S,$$
$$\bar{A}(x) = \bigvee \Big\{ \bigwedge_{i=1}^{n} A(a_i) : a_1, a_2, \ldots a_n \in S, \quad \bigwedge_{i=1}^{n} a_i \leq x \Big\} \quad \textit{for any } x_0 \neq x \in S.$$

Corollary 2.6. Let $\{A_i\}_{i\in\Delta}$ be a class of fuzzy filters of S. Then the supremum $\bigvee\limits_{i\in\Delta} A_i$ of $\{A_i\}_{i\in\Delta}$ in $\mathcal{F}\big(F(S)\big)$ is given by

$$\Big(\bigvee_{i\in\Delta} A_i\Big)(x) = \bigvee \Big\{ \bigwedge_{a\in X} B(a) : x \in [X],\ X \text{ is a non-empty finite subset of } S\Big\},$$
$$\text{where} \quad B(x) = \bigvee \Big\{A_i(x) : i \in \Delta\Big\} \quad \text{(i.e., the point-wise supremum of } A_i's)$$

Corollary 2.7. For any fuzzy filters A and B of S, the supremum $A \vee B$ is given by

$$(A \vee B)(x) = \bigvee \Big\{ \bigwedge_{a\in X} \big(A(a) \vee B(a)\big) : x \in [X],\ X \text{ is a non-empty finite subset of } S\Big\}$$

Theorem 2.8. *Let $(S, \wedge)$ be a semilattice with greatest element 1. Then the following are equivalent to each other:*

(1) $\mathcal{F}\big(F(S)\big)$ is a distributive lattice

(2) $F(S)$ is a distributive lattice

(3) S is distributive.

Definition 2.9. Let I be a non-empty subset of S and for any $\alpha \in L$, define $A_\alpha^I : S \to L$ by

$$A_\alpha^I(x) = \begin{cases} 1 & \text{if } x \in I \\ \alpha & \text{otherwise.} \end{cases}$$

It can be easily seen that for any $\beta \in L$, the β-cut of A_α^I is given by

$$\big(A_\alpha^I\big)_\beta = \begin{cases} S & \text{if } \beta \leq \alpha \\ I & \text{if } \beta \nleq \alpha. \end{cases}$$

In particular, when $\alpha = 0$, $A_\alpha^I = \chi_{_I}$, the characteristic map which is defined by

$$\chi_{_I}(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Also,} \quad A_\alpha^I \leq A_\beta^J \Longleftrightarrow I \subseteq J \text{ and } \alpha \leq \beta,\ A_\alpha^{I\cap J} = A_\alpha^I \wedge A_\alpha^J$$

for any non-empty subsets I and J of S and $\alpha, \beta \in L$.

Theorem 2.10. *Let P be a filter of S and $\alpha \in L$. Then A_α^P is a prime L-fuzzy filter of S iff P is a prime filter of S and α is a meet-prime element in L.*

Theorem 2.11. *Let A be an L-fuzzy fileter of S. Then A is prime iff the following are satisfied.*

(1) $\mid Im(A) \mid = 2$; that is A is two-valued

(2) For any $x \in S$, either $A(x) = 1$ or $A(x)$ is meet-prime element in L.

(3) A_1; the 1-cut of A is prime filter of S.

Theorem 2.12. *Let A be an L-fuzzy filter of S. Then A is a prime L-fuzzy filter of S iff there exists a prime filter P of S and a meet-prime element α in L such that $A = A_\alpha^P$.*

Theorem 2.13. *Let A be an L-fuzzy filter of S. Then A is maximal iff $A = A_\alpha^M$ for some maximal filter M of S and a dual atom α in L.*

Definition 2.14. Let $\mathcal{C}$ be a class of fuzzy subsets of a set X. A subclass $\{A_i\}_{i \in \Delta}$ of $\mathcal{C}$ is called directed above if, for any $i, j \in \Delta$ there is $k \in \Delta$ such that $A_i \leq A_k$ and $A_j \leq A_k$. $\mathcal{C}$ is said to be an algebric fuzzy system if, $\mathcal{C}$ is closed under point-wise infimums and point-wise supremums of directed above subclasses.

Theorem 2.15. *Let $(S, \wedge)$ be a semilattice with greatest element 1. Then the class $\mathcal{F}\big(F(S)\big)$ of all fuzzy filters of S is an algebraic.*

Definition 2.16. An L-fuzzy subset A of a bounded lattice D is said to be an L-fuzzy ideal of D if A_α is an ideal of D for all $\alpha \in L$, where $A_\alpha = \{x \in D : \ \alpha \leq A(x)\}$; the α-cut of A.

Theorem 2.17. *Let A be an L-fuzzy subset of a bounded lattice D. Then A is an L-fuzzy ideal of D if and only if any one of the following sets of conditions is satisfied:*

(1) $A(0) = 1$ and $A(x \vee y) = A(x) \wedge A(y)$ for all $x, y \in D$,

*(2) $A(0) = 1$ and $A(x \vee y) \geq A(x) \wedge A(y)$ and
$A(x \wedge y) \geq A(x) \vee A(y)$ for all $x, y \in D$.*

3 L-fuzzy ideals.

An ideal I of a semilattice S is a non-empty subset of S such that I is an initial segment and directed above. Let $\mathcal{I}(S)$ denote the set of all ideals of S. Then it can be easily proved that $\mathcal{I}(S)$ is a meet-semilattice under the usual set inclusion ordering. Let us recall that for any $\alpha \in L$, the α-cut of an L-fuzzy subset A of S is denoted by simply A_α, i.e., $A_\alpha = \{x \in S : \ \alpha \leq A(x)\}$. Now we introduce L-fuzzy ideals.

Definition 3.1. An L-fuzzy subset A of S is said to be an L-fuzzy ideal of S if A_α is an ideal of S for all $\alpha \in L$.

The following results facilitates to identify any (crisp) ideal of S with an L-fuzzy ideal of S.

Theorem 3.2. *Let S be a semilattice directed above and L a frame. Then A^I_α is an L-fuzzy ideal of S iff I is an ideal of S. In particular, χ_I is an L-fuzzy ideal of S iff I is an ideal of S.*

Before characterize L-fuzzy ideals, we introduce the following.

Definition 3.3. An L-fuzzy subset A of S is said to be

(i) an antitone if for any $x, y \in S$, $x \leq y$ implies $A(y) \leq A(x)$

(ii) directed above if for any $x, y \in S$ there exists $z \in S$ such that $z \geq x, y$ and $A(x) \wedge A(y) \leq A(z)$.

Theorem 3.4. *Let A be an L-fuzzy subset of S. Then A is an L-fuzzy ideal of S iff the following conditions are satisfied:*

(1) $A(x_0) = 1$ *for some* $x_0 \in S$

(2) A *is an antitone*

(3) A *is directed above.*

Proof. Suppose A is an L-fuzzy ideal of S. Then A_α is an ideal of S for all $\alpha \in L$.
(1). Since A_1 is non-empty, there exists $x_0 \in A_1$ so that $A(x_0) = 1$.
(2). Let $x, y \in S$ with $x \leq y$. Put $\alpha = A(y)$. Then $y \in A_\alpha$ and hence $x \in A_\alpha$ since A_α is an initial segment . So that $\alpha \leq A(x)$ and hence $A(y) \leq A(x)$. Therefore A is an antitone.
(3). Let $x, y \in S$. Put $\alpha = A(x) \wedge A(y)$. Then $\alpha \leq A(x)$ and $A(y)$ so that $x, y \in A_\alpha$. As A_α is an ideal of S, there exists $z \in A_\alpha$ such that $z \geq x, y$. So $\alpha \leq A(z)$ and hence $A(x) \wedge A(y) \leq A(z)$. Thus A is directed above.

Conversely suppose the given conditions are satisfied. Let $\alpha \in L$. By (1), $A(x_0) = 1 \geq \alpha$ for some $x_0 \in S$. Hence $x_0 \in A_\alpha$ so that A_α is a non-empty subset of S. By (2),

$$\begin{aligned} y \leq x \text{ and } x \in A_\alpha &\Rightarrow A(x) \leq A(y) \text{ and } \alpha \leq A(x) \\ &\Rightarrow \alpha \leq A(y) \Rightarrow y \in A_\alpha. \end{aligned}$$

Further, let $x, y \in A_\alpha$. Then $\alpha \leq A(x)$ and $A(y)$ so that $\alpha \leq A(x) \wedge A(y)$. By (3), there exists $z \in S$ such that $z \geq x, y$ and $A(x) \wedge A(y) \leq A(z)$ which implies $\alpha \leq A(z)$ so $z \in A_\alpha$. Therefore A_α is an ideal of S for all $\alpha \in L$. Thus A is an L-fuzzy ideal of S. □

By Theorem 3.2(1), it can be observed that if S is bounded below by 0 and A is an L-fuzzy ideal of S then $A(0) = 1$. For any L-fuzzy subsets A and B of S, define an L-fuzzy subset $A \wedge B$ of S by

$$(A \wedge B)(x) = A(x) \wedge B(x)$$

$$\text{and define} \quad A \leq B \text{ if and only if } A(x) \leq B(x) \text{ for all } x \in S.$$

It can be easily verified that $\leq$ is a partial order on the set of all L-fuzzy subsets of S and is called the point-wise ordering. In the following we discuss certain properities of L-fuzzy ideals. Let us denote the set of all L-fuzzy ideals of S by $\mathcal{F}_L(\mathcal{I}(S))$.

Theorem 3.5. *$\mathcal{F}_L(\mathcal{I}(S))$ is a meet-semilattice.*

Proof. Let $A, B \in \mathcal{F}_L(\mathcal{I}(S))$. Then $A(x_0) = 1 = B(y_0)$ for some $x_0,\ y_0 \in S$. As A and B are antitones, it follows that $A(x_0 \wedge y_0) = 1 = B(x_0 \wedge y_0)$ so that $(A \wedge B)(x_0 \wedge y_0) = 1$. For any $x, y \in S$,

$$\begin{aligned} x \leq y &\Rightarrow A(y) \leq A(x) \text{ and } B(y) \leq B(x) \\ &\Rightarrow A(y) \wedge B(y) \leq A(x) \wedge B(x) \\ &\Rightarrow (A \wedge B)(y) \leq (A \wedge B)(x). \end{aligned}$$

Therefore $A \wedge B$ is an antitone. Further, as A and B are directed above, there exist z_1 and $z_2 \in S$ such that

$$\begin{aligned} &z_1 \geq x, y \text{ and } A(x) \wedge A(y) \leq A(z_1) \\ \text{and } \ &z_2 \geq x, y \text{ and } B(x) \wedge B(y) \leq B(z_2). \end{aligned}$$

Now $z_1 \wedge z_2$ is an $u.b$ of $\{x, y\}$. As A and B are antitones, $A(z_1) \leq A(z_1 \wedge z_2)$ and $B(z_2) \leq B(z_1 \wedge z_2)$. Now,

$$\Big(A(x) \wedge B(x)\Big) \wedge \Big(A(y) \wedge B(y)\Big) \leq A(z_1) \wedge B(z_2) \leq A(z_1 \wedge z_2) \wedge B(z_1 \wedge z_2)$$

which implies $(A \wedge B)(x) \wedge (A \wedge B)(y) \leq (A \wedge B)(z_1 \wedge z_2)$. Therefore $A \wedge B$ is directed above. Hence $A \wedge B$ is an L-fuzzy ideal of S. Further, it can be easily seen that $A \wedge B$ is the $g.l.b$ of $\{A, B\}$ with respect to the point-wise ordering. Thus $\mathcal{F}_L\big(\mathcal{I}(S)\big)$ is a meet-semilattice. □

Theorem 3.6. *Let S be a semilattice directed above and L a frame. Then the following statements are equivalent:*

(1) S is a lattice bounded below

(2) $\mathcal{I}(S)$ is a complete lattice

(3) $\mathcal{F}_L(\mathcal{I}(S))$ is a complete lattice.

Proof. (1) $\Rightarrow$ (2). It is well-known result.

(2) $\Rightarrow$ (3). Let $\{A_i : i \in \Delta\}$ be a class of L-fuzzy ideals of S. Define $A : S \to L$ by

$$A(x) = \bigwedge_{i\in\Delta} A_i(x) = \text{The point-wise infimum of } A_i's \text{ in } L$$

Let I be the smallest ideal of S then there exists $x_0 \in S$ such that $I = (x_0]$ and x_0 is the smallest element in S. Then $A(x_0) = 1$ since each $A_i(x_0) = 1$. Also, for any $x, y \in S$,

$$\begin{aligned} x \leq y &\Rightarrow A_i(y) \leq A_i(x) \qquad \text{(since each } A_i \text{ is an antitone)} \\ &\Rightarrow \bigwedge_{i\in\Delta} A_i(y) \leq \bigwedge_{i\in\Delta} A_i(x) \\ &\Rightarrow A(y) \leq A(x). \end{aligned}$$

Terefore A is an antitone. Again, let $x, y \in S$. Then, since each A_i is an L-fuzzy ideals of S, there exists $z_i \in S$ such that $z_i \geq x, y$ and $A_i(x) \wedge A_i(y) \leq A_i(z_i)$. By (2), we have $\bigcap_{i\in\Delta} (z_i]$ is an ideal of S belonging x, y. Hence there exists $z \in S$ such that $x \leq z$, $y \leq z$ and $z \in \bigcap_{i\in\Delta} (z_i]$. Now

$$\begin{aligned} A(x) \wedge A(y) &= \Big(\bigwedge_{i\in\Delta} A_i(x)\Big) \wedge \Big(\bigwedge_{i\in\Delta} A_i(y)\Big) \\ &= \bigwedge_{i\in\Delta} \Big(A_i(x) \wedge A_i(y)\Big) \\ &\leq \bigwedge_{i\in\Delta} A_i(z_i) \\ &\leq \bigwedge_{i\in\Delta} A_i(z) = A(z) \end{aligned}$$

Therefore A is directed above and hence A is an L-fuzzy ideal of S. Also, A is the $g.l.b$ of $\{A_i : i \in \Delta\}$ under the point-wise ordering. Therefore every subset of $\mathcal{F}_L(\mathcal{I}(S))$ has $g.l.b.$ Thus $\mathcal{F}_L(\mathcal{I}(S))$ is a complete lattice under the point-wise ordering, in which χ_I and χ_J are respectively, the smallest and greatest element in the lattice $\mathcal{F}_L(\mathcal{I}(S))$ corresponding to the smallest element I and greatest element $J(= S)$ in the lattice $\mathcal{I}(S)$.

(3) $\Rightarrow$ (1). Suppose $\mathcal{F}_L(\mathcal{I}(S))$ is a complete lattice under point-wise ordering. Let $a, b \in S$. Then the characteristic functions $\chi_{(a]}$ and $\chi_{(b]}$ are L-fuzzy ideals of S and hence by(3), their $l.u.b$ $\chi_{(a]} \vee \chi_{(b]}$ is an L-fuzzy ideal of S, say A. As A is directed above, there exists $x \in S$ such that $x \geq a, b$ and $A(a) \wedge A(b) \leq A(x)$. Let $y \in S$ with $y \geq a, b$. Then $(a] \subseteq (y]$ and $(b] \subseteq (y]$ so that $\chi_{(a]} \leq \chi_{(y]}$ and $\chi_{(b]} \leq \chi_{(y]}$. Therefore $A = \chi_{(a]} \vee \chi_{(b]} \leq \chi_{(y]}$. In particular, $A(x) \leq \chi_{(y]}(x)$. Since $\chi_{(a]} \leq A$ and $\chi_{(b]} \leq A$, it follows that $A(a) = 1 = A(b)$ which implies $1 = A(a) \wedge A(b) \leq \chi_{(y]}(x)$ and hence $\chi_{(y]}(x) = 1$ so that $x \leq y$. Therefore x is the $l.u.b$ of {a, b}. Hence S is a lattice. Further, let B be the smallest L-fuzzy ideal of S. Then $B_1 = \{x \in S : B(x) = 1\}$ is the smallest ideal of S; for let I be an ideal of S. Then χ_I is an L-fuzzy ideal of S and hence $B \leq \chi_I$. For any $x \in B_1$, $B(x) = 1 = \chi_I(x)$ so that $x \in I$.

Therefore $B_1 \subseteq I$. Hence there exists an element x_0 in S such that $B_1 = (x_0]$ and x_0 is the smallest element in S. Hence S is bounded below. □

The following is an application of Zorn's lemma which allow us to denote the existence of maximal L-fuzzy filters. This can be proved easily by Theorem (2.9).

Lemma 3.7. Let S be a semilattice with greatest element 1 and B a non-constant L-fuzzy filter of S. Then there exists a maximal L-fuzzy filter A of S such that $B \leq A$.

Let us furnish a characterization of 0-distributive semilattices interms of both crisp and fuzzy maximal filters. According to Verlet [9], a semilattice with smallest element 0 is called 0-distributive, if for any $a, b, c \in S$ such that $a \wedge b = 0 = a \wedge c$ implies $a \wedge d = 0$ for some $d \geq b, c$ (or, equivalently, for any $x \in S$, the annihilator $\{x\}^*$ is an ideal of S). A proper filter F is maximal if and only if, for any $x \in S - F$, $\{x\}^* \cap F \neq \phi$.

Theorem 3.8. *Let S be a bounded semilattice and L a frame. Then the following statements are equivalent :*

(1) S is 0-distributive

(2) Every maximal filter of S is prime

(3) Every maximal L-fuzzy filter of S is prime.

Proof. (1) $\Rightarrow$ (2). Let M be a maximal filter of S which is not prime. Then, there exist two filters F and G of S such that $F \cap G \subseteq M$, but $F \nsubseteq M$ and $G \nsubseteq M$. So there exist $x \in F - M$ and $y \in G - M$. As M is maximal, there exist f and g in M such that $x \wedge f = 0 = y \wedge g$. Now $x \wedge f \wedge g = 0 = y \wedge f \wedge g$. By hypothesis, we can find z in S such that $z \geq x, y$ and $z \wedge (f \wedge g) = 0$. Since z, f, g all belong to M it implies $0 \in M$, a contradiction. Hence M is prime.

(2) $\Rightarrow$ (3). Let A be a maximal L-fuzzy filter of S. Then $A = A_\alpha^M$ for some maximal filter M of S and α a dual atom in L (by Theorem 2.8). By hypothesis, M is a prime filter of S. Since L is a distributive lattice, α is meet-prime. So A_α^M is a prime L-fuzzy filter of S (by Theorem 2.7). Thus A is prime.

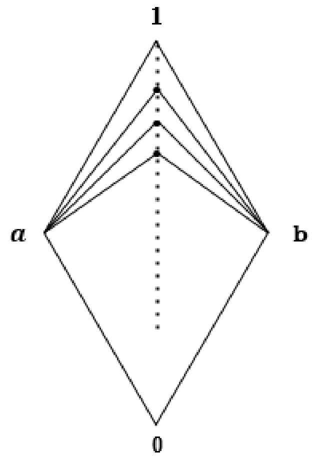
(3) $\Rightarrow$ (1). Let $a \wedge b = 0 = a \wedge c$ for any $a, b, c \in S$. Then it can be easily seen that the set $F = \{x \in S : x \geq b, c\}$ is a filter of S. Also, the set $G = \{x \in S : x \geq a \wedge f, f \in F\}$ is a filter of S containg F. Suppose G is proper filter of S. Then the characteristic function χ_G is a non-constant L-fuzzy filter of S. By above lemma, there exists a maximal L-fuzzy filter say A of S such that $\chi_G \leq A$. By hypothesis A is prime so that $A = A_\alpha^P$ for some prime filter P of S and a meet-prime $\alpha \in L$ (by Theorem 2.7). Therefore $\chi_G \leq A_\alpha^P$ which implies $G \subseteq P$. Since $[b) \cap [c) \subseteq G \subseteq P$ and P is prime, $[b) \subseteq P$ or $[c) \subseteq P$ which implies $0 \in P$, a contradiction. Hence $G = S$, in particular $0 \in G$, there exists $f \in F$ such that $a \wedge f = o$. Hence b and c have an upper bound f such that $a \wedge f = 0$. Thus S is 0-distributive. □

Pawar and Thakare [18] have characterized 0-distributive semilattices by assuming that the set of all ideals of a semilattice form a lattice, but it is not true. In general, in a semilattice S the set $\mathcal{I}(S)$ of all ideals of S may not form a lattice; for consider the following example.

Example 3.9. Let $S = \{a, b, 0\} \cup C$, where $C = (0, 1]$ the interval of real numbers x such that $0 < x \leq 1$. On S, we define the partial ordering as follows:

$$0 < a < x, \ 0 < b < x \text{ for all } x \in C$$

and together with usual ordering of real numbers on C.



Then S is a bounded distributive semilattice which is not a lattice. Let $I = \{0, a\}$ and $J = \{0, b\}$. Then I and J are ideals of S. For any $x \in C$, let $K = \{0, a, b\} \cup (0, x]$. Then K is an ideal of S and also an upperbound of I and J. Since the chain C has no smallest element, it follows that $I \vee J$ does not exist in $\mathcal{I}(S)$ so that $\mathcal{I}(S)$ is not a lattice.

Infact, we have that for any semilattice S, $\mathcal{I}(S)$ is a lattice if and only if S is a lattice itself. Now we obtain a set of equivalent conditions for a semilattice S with 0 to be 0-distributive.

Theorem 3.10. *Let L be a frame and S be a bounded semilattice such that $\mathcal{I}(S)$ is a lattice. Then the following statements are equivalent:*

(1) S is 0-distributive

(2) $\{x\}^*$ is an ideal for all $x \in S$

(3) X^* is an ideal for all $X \subseteq S$

(4) $\mathcal{I}(S)$ is pseudo-complemented

(5) $\mathcal{I}(S)$ is 0-distributive

(6) Every maximal filter of S is prime

(7) Every maximal L-fuzzy filter of S is prime.

(8) $\mathcal{F}_L(\mathcal{I}(S))$ *is* 0*-distributive.*

Proof. $(1) \Rightarrow (2)$. It is clear obviously.

$(2) \Rightarrow (3)$. It follows from the facts that $X^* = \bigcap_{x \in X} \{x\}^*$, for all $X \subseteq S$, and $\mathcal{I}(S)$ is a complete lattice.

$(3) \Rightarrow (4)$. Let I be an ideal of S. Then I^* is an ideal of S (by(3)). By the definition, I^* will be the pseudo-complement of I in the lattice $\mathcal{I}(S)$. Hence $\mathcal{I}(S)$ is pseudo-complemented.

$(4) \Rightarrow (5)$. As every pseudo-complemented lattice is 0-distributive, we get $\mathcal{I}(S)$ is 0-distributive.

$(5) \Rightarrow (6)$. Let M be a maximal filter of S and $x \notin M$, $y \notin M$. As M is maximal, we get $x \wedge f = 0 = y \wedge g$ for some $f, g \in M$. Now $x \wedge (f \wedge g) = 0 = y \wedge (f \wedge g)$ and hence $(f \wedge g] \cap (x] = (0] = (f \wedge g] \cap (y]$. By 0-distributivity of $\mathcal{I}(S)$, there exists an ideal I containing both $(x]$ and $(y]$ such that $(f \wedge g] \cap I = (0]$. As I is directed above and $x, y \in I$, there exists $z \in I$ such that $z \geq x, y$. Now $(f \wedge g] \cap (z] = (0]$ so that $(f \wedge g) \wedge z = 0$. As $(f \wedge g) \wedge z \notin M$ and $f \wedge g \in M$ we have $z \notin M$. Hence M is prime.

$(6) \Rightarrow (7)$. It is already proved in Theorem 3.5.

$(7) \Rightarrow (8)$. Let $A, B, C \in \mathcal{F}_L\big(\mathcal{I}(S)\big)$ such that $A \wedge B = \chi_{(0]} = A \wedge C$. As $\mathcal{F}_L\big(\mathcal{I}(S)\big)$ is a lattice, $B \vee C$ exists in $\mathcal{F}_L\big(\mathcal{I}(S)\big)$. Now we shall prove that $A \wedge (B \vee C) = \chi_{(0]}$. If $A \wedge (B \vee C) \neq \chi_{(0]}$ then $\chi_{(0]} < A \wedge (B \vee C)$. Let F denote the L-fuzzy filter generated by $A \wedge (B \vee C)$. Then F is non-constant; for, if $F(x) = 1$ for all $x \in S$, then $\chi_{(0]}(0) < \big(A \wedge (B \vee C)\big)(0) \leq F(0) < 1$ which implies $1 < 1$, a contradiction. Hence by Lemma (3.1), there exists a maximal L-fuzzy filter, say M of S such that $F \leq M$. By hypothesis, M is prime and hence $M = A_\alpha^P$ for some prime filter P of S and meet-prime element α of L (by Theorem 2.7). So that $A \wedge (B \vee C) \leq F \leq A_\alpha^P$. If $A \wedge B$ and $A \wedge C \nleq A_\alpha^P$ then we can find $x, y \in S$ such that $(A \wedge B)(x) \nleq A_\alpha^P(x)$ and $(A \wedge C)(y) \nleq A_\alpha^P(y)$ which implies $(A \wedge B)(x) \nleq \alpha$ and $(A \wedge C)(y) \nleq \alpha$ and both $x, y \notin P$. Hence $x \vee y \notin P$ since P is prime. As α is meet-prime, we get $A \wedge B)(x) \wedge (A \wedge C)(y) \nleq \alpha$. Now,

$$
\begin{aligned}
\big(A \wedge (B \vee C)\big)(x \vee y) &= A(x \vee y) \wedge (B \vee C)(x \vee y) \\
&= A(x) \wedge A(y) \wedge (B \vee C)(x) \wedge (B \vee C)(y) \\
&\geq A(x) \wedge A(y) \wedge B(x) \wedge C(y) \\
&= \big(A(x) \wedge B(x)\big) \wedge \big(A(y) \wedge C(y)\big) \\
&= (A \wedge B)(x) \wedge (A \wedge C)(y).
\end{aligned}
$$

Therefore $\big(A \wedge (B \vee C)(x \vee y) \nleq \alpha = A_\alpha^P(x \vee y)$. So that $A \wedge (B \vee C) \nleq A_\alpha^P$ which is a contradiction. Therefore $A \wedge B$ or $A \wedge C \leq A_\alpha^P$ and hence $\chi_{(0]} \leq A_\alpha^P$ so that $0 \in P$ which leeds to contradiction and hence $A \wedge (B \vee C) = \chi_{(0]}$. Thus $\mathcal{F}_L(\mathcal{I}(S))$ is 0-distributive.

$(8) \Rightarrow (1)$. Let $a \wedge b = 0 = a \wedge c$. Then $\chi_{(a \wedge b]} = \chi_{(0]} = \chi_{(a \wedge c]}$ which implies $\chi_{(a]} \wedge \chi_{(b]} = \chi_{(0]} = \chi_{(a]} \wedge \chi_{(c]}$. By hypothesis, there exsits an L-fuzzy ideal say A of S such that $\chi_{(b]} \leq A$, $\chi_{(c]} \leq A$ and $\chi_{(a]} \wedge A = \chi_{(0]}$. Now $A(b) = 1 = A(c)$. Therefore $b, c \in A_1$; the 1-cut of A. Since A_1 is an ideal of S, there exists $d \in A_1$ such that $d \geq b, c$. As A is an antitone, $A(d) = 1$ and it follows that $A(a \wedge d) = 1$. Now

$$\chi_{(0]}(a \wedge d) = \chi_{(a]}(a \wedge d) \wedge A(a \wedge d) = 1 \wedge 1 = 1.$$

Then $a \wedge d = 0$. Thus S is 0-distributive. □

Finally in this section, we extend an important M. H. Stone's version separation therom on prime filters of distributive semilattice to prime L-fuzzy filters. First let us recall the Stone's version separation theorem [9]: For any filter F and any ideal I of a distribution semilattice S such that $F \cap I = \phi$, there exist a prime filter P of S such that $F \subseteq P$ and $P \cap I = \phi$.

In the following, S is assumed to be a bounded distributive semilattice and $\bar{\alpha}$ denote the constant L-fuzzy subset attaining the value α in L.

Theorem 3.11. *Let α be a meet-prime element in L, A be an L-fuzzy filter of S and B be an L-fuzzy ideal of S such that $A \wedge B \leq \bar{\alpha}$. Then, there exists a prime L-fuzzy filter A^P_α such that*

$$A \leq A^P_\alpha \quad \textit{and} \quad A^P_\alpha \wedge B \leq \bar{\alpha}.$$

Proof. We are given that $A(x) \wedge B(x) \leq \alpha$ for all $x \in S$. Put $F = \{x \in S : A(x) \nleq \alpha\}$ and $I = \{x \in S : B(x) \nleq \alpha\}$, since $A(1) = B(0) = 1 \nleq \alpha$, $1 \in F$ and $0 \in I$ and hence F, I are non-empty. For any $x, y \in S$.

$$\begin{aligned}
x, y \in F &\Rightarrow A(x) \nleq \alpha \text{ and } A(y) \nleq \alpha \\
&\Rightarrow A(x \wedge y) = A(x) \wedge A(y) \nleq \alpha \\
&\Rightarrow x \wedge y \in F \\
\text{Also,} \quad y \geq x \in F &\Rightarrow A(y) \geq A(x) \text{and } A(x) \nleq \alpha \\
&\Rightarrow A(y) \nleq \alpha \\
&\Rightarrow y \in F. \text{ Therefore F is a filter of S. Further} \\
\text{Again,} \quad y \leq x \in I &\Rightarrow B(x) \leq B(y) \text{ and } B(x) \nleq \alpha \\
&\Rightarrow B(y) \nleq \alpha \\
&\Rightarrow y \in I. \\
\text{Also,} \quad x, y \in I &\Rightarrow B(x) \nleq \text{ and } B(y) \nleq \alpha \\
&\Rightarrow B(x) \wedge B(y) \nleq \alpha.
\end{aligned}$$

Since B is directed above, there exists $z \in S$ such that

$$z \geq x, y \quad \text{and} \quad B(x) \wedge B(y) \leq B(z).$$

If $B(z) \leq \alpha$ then $B(x) \wedge B(y) \leq \alpha$, a contradiction. Therefore $B(z) \nleq \alpha$ and hence $z \in I$. So I is an ideal of S. Since α is meet-prime and $A(x) \wedge B(x) \leq \alpha$, it follows that $A(x) \leq \alpha$ or $B(x) \leq \alpha$ so that $x \notin F$ or $x \notin I$. Therefore $F \cap I = \phi$. Hence, there exists a prime filter P of S such that $F \subseteq P$ and $P \cap I = \phi$. Now A_α^P is a prime L-fuzzy filter of S and $A \leq A_\alpha^P$ (since, $x \notin P \Rightarrow x \notin F \Rightarrow A(x) \leq \alpha = A_\alpha^P(x)$) and $A_\alpha^P \wedge B \leq \bar{\alpha}$ (since, $x \in P \Rightarrow x \notin I$ and hence $B(x) \leq \alpha$). □

4 Prime and Maximal L-fuzzy ideals

Swamy and Raju [13, 15] have generalised the notions of fuzzy prime(maximal) ideals of a ring [12, 16] by introducing the notions of prime(maximal) fuzzy $\mathcal{S}$-subsets of a non-empty set X corresponding to an algebraic closure set system $\mathcal{S}$ on X. Here we extend these notions to L-fuzzy ideals of a bounded semilattice S, even though $\mathcal{I}(S)$ is not a closure set system on S. Let us recall that a proper ideal P of a semilattice S is said to be prime if for any ideals I and J of S, $\phi \neq I \cap J \subseteq P$ implise $I \subseteq P$ or $J \subseteq P$ (or, equivalently, if for any $x, y \in S$, $x \wedge y \in P$ implise $x \in P$ or $y \in P$).
Analogously, a non-constant L-fuzzy ideal A of S is said to be prime if, for any L-fuzzy ideals B and C of S,

$$B \wedge C \leq A \text{ implise } B \leq A \text{ or } C \leq A.$$

The following is a characterization of prime L-fuzzy ideals.

Theorem 4.1. *Let P be a prime ideal of a bounded semilattices S and α a meet-prime element in L. Then, A_α^P is a prime L-fuzzy ideal of S. Conversely any prime L-fuzzy ideal of S is of the form A_α^P for some prime ideal P of S and a meet-prime element α of L.*

Proof. Since P is prime, we have A_α^P is non-constant L-fuzzy ideal of S. Let A and B be any L-fuzzy ideals of S such that $A \nleq A_\alpha^P$ and $B \nleq A_\alpha^P$. Then, there exist $x, y \in S$ such that $A(x) \nleq A_\alpha^P(x)$ and $B(y) \nleq A_\alpha^P(y)$ which implies that $A_\alpha^P(x) = \alpha = A_\alpha^P(y)$ so that $x \notin P$ and $y \notin P$. Hence $x \wedge y \notin P$ as P is prime. Also, since α is meet-prime and $A(x) \nleq \alpha$ and $B(y) \nleq \alpha$, it follows that $A(x) \wedge B(y) \nleq \alpha$. As A and B are isotones, we get

$$(A \wedge B)(x \wedge y) = A(x \wedge y) \wedge B(x \wedge y) \geq A(x) \wedge B(y)$$

Therefore $\alpha \ngeq (A \wedge B)(x \wedge y)$. Hence $(A \wedge B)(x \wedge y) \nleq A_\alpha^P(x \wedge y)$ so that $A \wedge B \nleq A_\alpha^P$. Thus A_α^P is prime.

Conversely, let A be a prime L-fuzzy ideal of S. First we prove that A is two-valued. Now A assumes at least two values; for otherwise A is constant. As $A(0) = 1$, 1 is necessarily a value of A. Let $y, z \in S$ such that $A(y) = \alpha < 1$ and $A(z) = \beta < 1$. Now, define L-fuzzy subsets B and C of S by

$$B(x) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x \nleq y \end{cases} \quad \text{and} \quad C(x) = \begin{cases} 1 & \text{if } x = 0 \\ \alpha & \text{if } x \neq 0 \end{cases}$$

Then $B = \chi_{(y]}$ and $C = A_\alpha^{[0]}$ and hence B, C are L-fuzzy ideal of S. Now $(B \wedge C)(x) \leq A(x)$ for all $x \in S$; for

$$\begin{aligned} & x = 0 \Rightarrow B(x) \wedge C(x) = 1 = A(x) \\ & 0 \neq x \leq y \Rightarrow B(x) \wedge C(x) = 1 \wedge \alpha = \alpha = A(y) \leq A(x) \\ \text{and} \quad & x \nleq y \Rightarrow B(x) \wedge C(x) = 0 \wedge \alpha = 0 \leq A(x). \end{aligned}$$

Therefore $B \wedge C \leq A$. As A is prime, $B \leq A$ or $C \leq A$. Since $B(y) = 1$ and $A(y) = \alpha$, we get $B(y) \nleq A(y)$ so that $B \nleq A$ and hence $C \leq A$; in particular, $C(z) \leq A(z) = \beta$. Since $A(z) \neq A(0), z \neq 0$ and hence $C(z) = \alpha$. Therefore $\alpha \leq \beta$. By symmetry $\beta \leq \alpha$. Hence $\alpha = \beta$. Thus A assumes exactly one value, say α other than 1. Let $P = \{x \in S : A(x) = 1\}$. As A is two-valued and $P = A_1$; the 1-cut of A, it follows that P is a proper ideal of S. Hence for any $x \in S$,

$$A(x) = \begin{cases} 1 & \text{if } x \in P \\ \alpha & \text{otherwise.} \end{cases}$$

So that $A = A_\alpha^P$. Finally, we prove that α is meet-prime and P is prime. Let $a, b \in L$ such that $a \wedge b \leq \alpha$. Then A_a^P and A_b^P are L-fuzzy ideals of S and $A_a^P \wedge A_b^P \leq A$. As A is prime, $A_a^P \leq A$ or $A_b^P \leq A$. Therefore, for any $x \notin P$, either $A_a^P(x) \leq A(x)$ or $A_b^P(x) \leq A(x)$ so that $a \leq \alpha$ or $b \leq \alpha$ and hence α is a meet-prime element in L. Let I and J be two ideals of S such that $I \cap J \subseteq P$. Then $\chi_I \wedge \chi_J = \chi_{I \cap J} \leq \chi_P \leq A$. As A is prime, $\chi_I \leq A$ or $\chi_J \leq A$ which implies $I \subseteq P$ or $J \subseteq P$. Thus P is a prime ideal of S. Hence the theorem. □

A non-constant L-fuzzy ideal A of S is said to be maximal L-fuzzy ideal of S if A is a maximal element in the set of all non-constant L-fuzzy ideals of S under the point-wise ordering. Before characterize maximal L-fuzzy ideals, we need the following which can be proved easily.

Lemma 4.2. Let A be an L-fuzzy subset of a bounded semilattice S and $\alpha \in L$. Let $A \vee \alpha$ denote an L-fuzzy subset of S defined by

$$(A \vee \alpha)(x) = A(x) \vee \alpha \text{ for all } x \in S.$$

Then $A \vee \alpha$ is an L-fuzzy ideal of S iff A is an L-fuzzy ideal of S, for all $\alpha \in L$.

Theorem 4.3. *Let A be an L-fuzzy subset of a bounded semilattice S. Then A is a maximal L-fuzzy ideal of S iff $A = A_\alpha^M$ for some maximal ideal M of S and a dual atom α of L.*

Proof. Let M be a maximal ideal of S and α a dual atom in L such that $A = A_\alpha^M$. Then A is non-constant L-fuzzy ideal of S. Let B be a non-constant L-fuzzy ideal of S such that $A \leq B$. Then $M \subseteq \{x \in S : B(x) = 1\} \subsetneq S$. By the maximality of M, we get $M = \{x \in S : B(x) = 1\}$ and for any $x \notin M$, $\alpha = A(x) \leq B(x) \neq 1$. As α is a dual atom in L, $A(x) = B(x)$ so that $A = B$. Thus A is a maximal L-fuzzy ideal of S.

Conversely suppose A is a maximal L-fuzzy ideal of S. Then $M = \{x \in S : A(x) = 1\}$ is a proper ideal of S. Now $A(0) = 1$, we shall prove that A is two-valued. Let $x, y \in S$ such that $A(x) < 1$ and $A(y) < 1$. Put $A(x) = \alpha$ and $A(y) = \beta$. Now $A \vee \alpha$ and $A \vee \beta$ are L-fuzzy ideals of S and also, $(A \vee \alpha)(x) = A(x) \vee \alpha = \alpha \vee \alpha = \alpha < 1$ as well as $(A \vee \beta)(y) = \beta < 1$. Hence $A \leq A \vee \alpha < 1$ and $A \leq A \vee \beta < 1$. By the maximality of A, $A = A \vee \alpha = A \vee \beta$. In particular, $A(y) = A(y) \vee \alpha$ and $A(x) = A(x) \vee \beta$ which implies $\beta \geq \alpha$ and $\alpha \geq \beta$ so that $\alpha = \beta$. Therefore A assumes exactly one value, say $\alpha \neq 1$. Now α is a dual atom; for, suppose $\alpha < \beta \in L$ then an L-fuzzy subset B of S defined by

$$B(x) = \begin{cases} 1 & \text{if } x \in M \\ \beta & \text{otherwise} \end{cases}$$

Then $B = A_P^M$, so that B is an L-fuzzy ideal of S. Clearly $A < B$. As A is maximal, B is constant which implies $\beta = 1$. Finally, let N be a proper ideal of S suchthat $M \subseteq N$. Then $A = A_\alpha^M \leq A_\alpha^N \neq 1$. Hence $A_\alpha^M = A_\alpha^N$ by the maximality of A. So that $M = N$ and hence M is a maximal ideal of S. Thus $A = A_\alpha^M$, where M is a maximal ideal of S and α is a dual atom of L. Hence the theorem. $\square$

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