International Journal of Mathematics Trends and Technology (IJMTT) - Volume 66 Issue 9 - Sep 2020

Hybrid power mean of $2k^{th}$ power inversion of L-functions, trigonometric sums and general quartic Gauss sums

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Abstract

In this paper we calculate the hybrid power mean of $2k^{th}$ power inversion of L-functions, twisted trigonometric sums and general quartic Gauss sums. We also discuss its asymptotic formula with the help of the properties of Gauss sums and Dirichlet characters.

Keywords: Dirichlet L-functions, quartic Gauss sum, trigonometric sum, asymptotic formula.
MSC: 11L05, 11L07.

1 Introduction

Trigonometric Sums over primes have been extensively studied by Mordell in article [1], later Hua [4] and Min [5] extended this result to the case of two variables and established the formula

$$\sum_{x=1}^{p-1}\sum_{y=1}^{p-1} e\Big(\frac{f(x,y)}{p}\Big) \ll p^{2-\frac{2}{k}}, \tag{1.1}$$

where $f(x,y)$ is a k^{th}-degree polynomial with two variable x and y, but can not be transformed into one variable.
A Dirichlet L-function is defined by the series

$$L(l,\psi) = \sum_{m=1}^{\infty} \frac{\psi(m)}{m^s},$$

where $s = \rho + iw$ is a complex number and $\rho > 1$.
Let $r \geq 3$ be an integer and ψ a Dirichlet character modulo r. For any positive integer m, the general k-th Gauss sum $G(m,k,\psi;r)$ is defined as

$$G(m,k,\psi;r) = \sum_{b=1}^{r} \psi(b) e\left(\frac{mb^k}{r}\right).$$

where $e(t) = e^{2it\pi}$.
The analytical properties of the k^{th}-Gauss sums, exponential sums and related other sums, has been explored in the literature [2, 12, 13, 14, 15], for example - Li Xiaoxue and Hu Jiayuan [6] studied on the computational problem of one of kind fourth hybrid power mean of the quartic Gauss sums and Kloosterman sums and gave an estimate

$$\Big(\sum_{b=1}^{p-1}\Big|\sum_{a=0}^{p-1}e\Big(\frac{ba^4}{p}\Big)\Big|^2\Big)\Big(\Big|\sum_{c=0}^{p-1}e\Big(\frac{bc+\bar{c}}{p}\Big)\Big|^2\Big) = 3p^3 + O(p^2).$$

Zhang [9] studied the $2k^{th}$-power mean value of inversion of L-functions and described some identities. Zhang and Deng [2] described the $2k^{th}$-power mean of the inversion of Dirichlet L-functions with the weight of general quadratic Gauss sums and gave the formula

$$\sum_{p\le Q}\frac{1}{p(p-1)^2}\sum_{\psi \bmod p}\frac{|G(m,\psi;p)|^4}{|L(1,\psi)|^{2k}} = 3C(k)\pi(Q) + O(Q^{\frac{1}{2}+\epsilon}).$$

R. Ma, J. Zhang and Y. Zhang [8] summarized the $2m^{th}$-power mean of L-functions, trigonometric sums and gave the asymptotic formula,

$$\sum_{\psi\ne\psi_0}\Big|\sum_{a=1}^{p-1}\psi(a)e\Big(\frac{f(a)}{p}\Big)\Big|^2|L(1,\psi)|^{2m} = p^2\zeta^{2k-1}(2)\prod_{p_0}\left(1-\frac{1-C_{2k-2}^{k-1}}{p_0^2}\right) + O(p^{2-\frac{1}{k}+\epsilon}).$$

The main purpose of this article is to study the $2k^{th}$-power mean of inversion of L-functions, square of two variables trigonometric sums and general quartic Gauss sums, which has been state as below.

Theorem 1.1 *Let p be an odd prime with $p \le \Upsilon$. Then for any positive integer k, ϑ with $(\vartheta, p) = 1$ and $|\vartheta| \le \Upsilon$, we have*

$$\sum_{p\le\Upsilon}\ \sum_{\psi \bmod p}\left|\sum_{a=1}^{p-1}\ \sum_{b=1}^{p-1}\psi(ab)e\left(\frac{f(a,b)}{p}\right)\right|^2|G(\vartheta,4,\psi;p)|^2\frac{1}{|L(1,\psi)|^{2k}} = O\Big(\Upsilon^{5+\epsilon}\Big),$$

where Υ is any positive number with $\Upsilon \ge 2$, ψ is a Dirichlet character (mod p), *ϵ is any fixed positive number.*

2 Preliminaries:

In this section we discuss some properties of Gauss sums, Dirichlet character [7,11,10] and some lemmas which can be used in the proofs of the main results.

Lemma 2.1 *Let $y \ge 2$ be a real number, k a positive integer. Then for*

$$r(c) = \sum_{c_1c_2\ldots c_k=c}\mu(c_1)\ldots\mu(c_k),$$

we have

$$\sum_{p\leq \Upsilon}\sup\left|\sum_{\substack{c\leq x\\ c\equiv a(p)}} r(c)\right| \ll_{A,k} y(\ln y)^{-A}+y^{1-\frac{1}{2^k}}\Upsilon(\ln(y\Upsilon))^4,$$

where $\mu(c)$ is the M$\overset{''}{o}$bius function, A any positive number, $\ll_{A,k}$ denote the constants implied by the symbols $\ll$ depend only on parameter A and k.

Proof. See [9].

Lemma 2.2 *Let p be an odd prime with $p\equiv 3 \pmod 4$, ϑ be a positive integer with $(\vartheta,p)=1$. Then for any non-principal even character ψ mod p, we get*

$$\mid G(\vartheta,4,\psi;p)\mid^2+\mid G(\vartheta,4,\overline{\psi};p)\mid^2=4p.$$

Proof: See [3].

3 Mean value of trigonometric sum :

Lemma 3.1 *Let $f(a,b)$ be a l^{th}- degree polynomial with integer coefficients, ψ a Dirichlet character modulo p, we have*

$$\Big|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\psi(ab)e\Big(\frac{f(a,b)}{p}\Big)\Big|^2=(p-1)^2+\sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\psi(ab)\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\Big(\frac{g(u,v,a,b)}{p}\Big)+\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\sum_{b=2}^{p-1}\psi(b)e\Big(\frac{g(1,b,u,v)}{p}\Big)$$
$$+\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\sum_{a=2}^{p-1}\psi(a)e\Big(\frac{g(a,1,u,v)}{p}\Big).$$

Proof: Here we take $f(a,b)=a_0+a_1(ab)+a_2(a^2b^2)+a_3(a^3b^3)...a_d(a^lb^l)$, $1\leq u,v\leq p$, $gcd(u,p)=1$ and $gcd(v,p)=1$.

$$\Big|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\psi(ab)e\Big(\frac{f(a,b)}{p}\Big)\Big|^2=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\psi(ab)\overline{\psi}(uv)e\Big(\frac{f(a,b)-f(u,v)}{p}\Big)$$
$$=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\psi(abuv)\overline{\psi}(uv)e\Big(\frac{f(au,bv)-f(u,v)}{p}\Big)$$
$$=\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\psi(ab)e\Big(\frac{f(au,bv)-f(u,v)}{p}\Big).$$

Here we denote, $g(a,b,u,v)=f(au,bv)-f(u,v)$

$$= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\psi(ab)e\Big(\frac{g(a,b,u,v)}{p}\Big)$$

$$= \sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\psi(1)e\Big(\frac{g(1,1,u,v)}{p}\Big) + \sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\psi(ab)e\Big(\frac{g(a,b,u,v)}{p}\Big) + \sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\sum_{b=2}^{p-1}\psi(b)e\Big(\frac{g(1,b,u,v)}{p}\Big) + \sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\sum_{a=2}^{p-1}\psi(a)e\Big(\frac{g(a,1,u,v)}{p}\Big)$$

$$= \sum_{u=1}^{p-1}\sum_{v=1}^{p-1}1 + \sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\psi(ab)e\Big(\frac{g(a,b,u,v)}{p}\Big) + \sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\sum_{b=2}^{p-1}\psi(b)e\Big(\frac{g(1,b,u,v)}{p}\Big) + \sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\sum_{a=2}^{p-1}\psi(a)e\Big(\frac{g(a,1,u,v)}{p}\Big)$$

$$= (p-1)^2 + \sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\psi(ab)\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\Big(\frac{g(a,b,u,v)}{p}\Big) + \sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\sum_{b=2}^{p-1}\psi(b)e\Big(\frac{g(1,b,u,v)}{p}\Big) + \sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\sum_{a=2}^{p-1}\psi(a)e\Big(\frac{g(a,1,u,v)}{p}\Big)$$

Lemma 3.2 *Let $f(a,b)$ be a polynomial with integer coefficients and $g(a,b,u,v) = f(au,bv) - f(u,v) = \sum_{j=0}^{l} a_j(a^j b^j - 1)u^j v^j$. Then we have following identity,*

$$\Big|\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\Big(\frac{g(a,b,u,v)}{p}\Big)\Big| = \begin{cases} \ll p^{2-\frac{2}{l}} & \text{if } p \nmid (t_0, t_1, t_2 ... t_l), \\ (p-1)^2 & \text{if } p \mid (t_0, t_1, t_2 ... t_l). \end{cases}$$

Where $t_j = a_j(a^j b^j - 1), \quad j = 0,1,2,3,...,l.$

Proof. See [4].

4 Mean value of $2k^{th}$ power inversion of L- function:

Lemma 4.1 *Let p be an odd prime with $p \le \Upsilon$ and ψ a character modulo p. Then for any positive integer k, we have an estimate*

$$\sum_{p\le\Upsilon} p \sum_{a=2}^{p-1}\Big|\sum_{\psi \bmod p}\frac{\psi(a)}{|L(1,\psi)|^{2k}}\Big| = O(\Upsilon^{3+\epsilon}),$$

where ϵ is any fixed positive number.

Proof: For convenience, firstly we write

$$E(\psi,y) = \sum_{\frac{p}{a}<c\le y}\psi(c)r(c), \quad F(\psi,y) = \sum_{p<c\le y}\psi(c)r(c),$$

where a is positive integer $(1 \le a < p)$.
If $Re(w) > 1$ and $\psi \ne \psi_0$ (principal character mod p) then we have,

$$\frac{1}{L^k(w,\psi)} = \sum_{c=1}^{\infty}\frac{\psi(c)r(c)}{c^w}.$$

Now using Abel's identity we get,

$$\begin{aligned}
\frac{1}{L^k(w,\psi)} &= \sum_{1\leq c<\frac{p}{a}} \frac{\psi(c)r(c)}{c^w} + w\int\limits_{\frac{p}{a}}^{\infty} \frac{E(\psi,y)}{y^{w+1}}dy \\
&= \sum_{1\leq c<p} \frac{\psi(c)r(c)}{c^w} + w\int\limits_{p}^{\infty} \frac{F(\psi,y)}{y^{w+1}}dy.
\end{aligned}$$

As $w = 1$, $L(1,\psi) \neq 0$, we have

$$\begin{aligned}
\sum_{\psi \bmod p} \psi(a)\frac{1}{|L(1,\psi)|^{2k}} &= \sum_{\psi \bmod p}{}' \psi(a)\Big|\sum_{c=1}^{\infty}\frac{\psi(c)r(c)}{c}\Big|^2 \\
&= \sum_{\psi \bmod p}{}' \psi(a)\Big(\sum_{c=1}^{\infty}\frac{\psi(c)r(c)}{c}\Big)\Big(\sum_{m=1}^{\infty}\frac{\overline{\psi}(m)r(m)}{m}\Big) \\
&= \sum_{\psi \bmod p}{}' \psi(a)\Bigg(\sum_{1\leq c<p/a}\frac{\psi(c)r(c)}{c} + \int\limits_{p/a}^{\infty}\frac{E(\psi,y)}{y^2}dy\Bigg) \\
&\qquad \Bigg(\sum_{1\leq m<p}\frac{\overline{\psi}(m)r(m)}{m} + \int\limits_{p}^{\infty}\frac{F(\overline{\psi},y)}{y^2}dy\Bigg) \\
&= \sum\nolimits_{\psi \bmod p}{}' \psi(a)\Big(\sum_{1\leq c<p/a}\frac{\psi(c)r(c)}{c}\Big)\Big(\sum_{1\leq m<p}\frac{\overline{\psi}(m)r(m)}{m}\Big) \\
&\quad + \sum_{\psi \bmod p}{}' \Bigg(\psi(a)\Big(\sum_{1\leq c<p/a}\frac{\psi(c)r(c)}{c}\Big)\int\limits_{p}^{\infty}\frac{F(\overline{\psi},y)}{y^2}dy\Bigg) \\
&\quad + \sum_{\psi \bmod p}{}' \psi(a)\Big(\sum_{1\leq m<p}\frac{\overline{\psi}(m)r(m)}{m}\Big)\Bigg(\int\limits_{p/a}^{\infty}\frac{E(\psi,y)}{y^2}dy\Bigg) \\
&\quad + \sum_{\psi \bmod p}{}' \psi(a)\Bigg(\int\limits_{p/a}^{\infty}\frac{E(\psi,y)}{y^2}dy\Bigg)\Bigg(\int\limits_{p}^{\infty}\frac{F(\overline{\psi},y)}{y^2}dy\Bigg) \\
&= Z_1 + Z_2 + Z_3 + Z_4
\end{aligned}$$

where $\sum_{\psi \bmod p}'$ denotes the summation over all non-principal characters mod p. Here we shall use the estimate

$$\sum_{\psi \bmod p} \psi(c)\overline{\psi}(m) = \begin{cases} p-1, & \text{if } c \equiv m (\bmod\ p), \\ 0 & \text{otherwise.} \end{cases}$$

Now,

$$\begin{aligned}
Z_1 &= \sum_{\psi\neq\psi_0}{}'\psi(a)\Big(\sum_{1\le c<p/a}\frac{\psi(c)r(c)}{c}\Big)\Big(\sum_{1\le m<p}\frac{\overline{\psi}(m)r(m)}{m}\Big)\\
&= \sum_{\psi \mod p}\ \sum_{1\le c<p/a}\ \sum_{1\le m<p}\frac{\psi(ac)\overline{\psi}(m)r(c)r(m)}{cm}+O(p^{\epsilon})\\
&= \phi(p)\sum_{1\le c<p/a}\ \sum_{1\le m<p}\frac{r(c)r(m)}{cm}+O(p^{\epsilon})\\
&= \phi(p)\frac{r(a)}{a}\sum_{1\le c<p/a}\frac{r^2(c)}{c^2}+O(p^{\epsilon})\\
&= \phi(p)\frac{r(a)}{a}\sum_{c=1}^{\infty}\frac{|r(c)|^2}{c^2}+O(p^{\epsilon}).
\end{aligned}$$

Therefore,

$$Z_1=\phi(p)\frac{r(a)}{a}C+O(p^{\epsilon}),\qquad \text{where}\ \ C=\sum_{c=1}^{\infty}\frac{|r(c)|^2}{c^2}.$$

$$\sum_{p\le\Upsilon}p\sum_{a=2}^{p-1}|Z_1|=\sum_{p\le\Upsilon}p^2\sum_{a=2}^{p-1}\frac{r(a)}{a}+\sum_{p\le\Upsilon}p\sum_{a=2}^{p-1}p^{\epsilon}\ll\Upsilon^{3+\epsilon}. \tag{4.2}$$

$$\sum_{p\le\Upsilon}p\sum_{a=2}^{p-1}|Z_2| = \sum_{p\le\Upsilon}p\sum_{a=2}^{p-1}\Big|\sum_{\psi \mod p}{}'\psi(a)\Big(\sum_{c\le\frac{p}{a}}\frac{\psi(c)r(c)}{c}\Big)\int_p^{\infty}\frac{F(\overline{\psi},y)}{y^2}dy\Big|$$

$$\begin{aligned}
\sum_{p\le\Upsilon}p\sum_{a=2}^{p-1}|Z_2| \le{}& \sum_{p\le\Upsilon}p\sum_{a=2}^{p-1}\Big|\sum_{\psi \mod p}{}'\psi(a)\Big(\sum_{c\le\frac{p}{a}}\frac{\psi(c)r(c)}{c}\Big)\Big(\int_p^{\Upsilon^{2^k}}\sum_{p<m\le y}\frac{\overline{\psi}(m)r(m)}{y^2}dy\Big)\Big|+\\
&\sum_{p\le\Upsilon}p\sum_{a=2}^{p-1}\Big|\sum_{\psi \mod p}{}'\psi(a)\Big(\sum_{c\le\frac{p}{a}}\frac{\psi(c)r(c)}{c}\Big)\Big(\int_{\Upsilon^{2^k}}^{\infty}\sum_{p<m\le y}\frac{\overline{\psi}(m)r(m)}{y^2}dy\Big)\Big|
\end{aligned}$$

$$
\begin{aligned}
&\ll \sum_{p\le \Upsilon} p\sum_{a=2}^{p-1}\int_{p}^{\Upsilon^{2^k}} \frac{1}{y^2}\Big|\sum_{c\le \frac{p}{a}}\sum_{p<m\le y}\frac{r(c)r(m)}{c}\sum_{\psi \mod p}\psi(ac)\overline{\psi}(m)\Big|dy + \\
&O\Big(\Upsilon^{3+\epsilon}\int_{p}^{\Upsilon^{2^k}}\frac{1}{y^2}\Big(y(\log y)^{-A}+y^{1-\frac{1}{2^k}}\zeta\log(y\Upsilon)^y\Big)dy\Big)+O(\Upsilon^{3+\epsilon})\\
&= O\Big(\sum_{p\le\Upsilon}p^2\sum_{a=2}^{p-1}\sum_{c<p/a}\frac{1}{c}\int_{p}^{\Upsilon^{2^k}}\frac{y^{\frac{1}{p}}p^{\epsilon}}{y^2}dy\Big)+O(\Upsilon^{3+\epsilon})\\
&= O(\Upsilon^{3+\epsilon}). \qquad (4.3)
\end{aligned}
$$

Similarly, we find

$$
\sum_{p\le\Upsilon}p\sum_{a=2}^{p-1}|Z_3| = \sum_{p\le\Upsilon}p\sum_{a=2}^{p-1}|Z_4| = O(\Upsilon^{3+\epsilon}). \qquad (4.4)
$$

Adding the equations (4.2), (4.3), (4.4) we get,

$$
\sum_{p\le\Upsilon}p\sum_{a=2}^{p-1}\Big|\sum_{\psi \mod p}\frac{\psi(a)}{|L(1,\psi)|^{2k}}\Big| = O(\Upsilon^{3+\epsilon}). \qquad \blacksquare
$$

Lemma 4.2 *Let p be an odd prime with $p\le\Upsilon$ and ψ a character modulo p. Then for any positive integer k, we have an estimate*

$$
\sum_{p\le\Upsilon}p\Big|\sum_{\psi \mod p}\frac{\psi(ab)}{|L(1,\psi)|^{2k}}\Big| = O\big(\frac{r(ab)}{ab}.\Upsilon^{3+\epsilon}\big),
$$

where ϵ is any fixed positive number.

Proof: Let $ab=d$,

For convenience, firstly we write

$$
E(\psi,y)=\sum_{\frac{p}{d}<c\le y}\psi(c)r(c), \quad F(\psi,y)=\sum_{p<c\le y}\psi(c)r(c),
$$

If $Re(w)>1$ and $\psi\ne\psi_0$ (principal character mod p) then we have,

$$
\frac{1}{L^k(w,\psi)} = \sum_{c=1}^{\infty}\frac{\psi(c)r(c)}{c^w}.
$$

Now using Abel's identity we get,

$$\begin{aligned}\frac{1}{L^k(w,\psi)} &= \sum_{1\le c<\frac{p}{d}} \frac{\psi(c)r(c)}{c^w} + w\int_{\frac{p}{d}}^{\infty} \frac{E(\psi,y)}{y^{w+1}}dy \\ &= \sum_{1\le c<p} \frac{\psi(c)r(c)}{c^w} + w\int_{p}^{\infty} \frac{F(\psi,y)}{y^{w+1}}dy.\end{aligned}$$

As $w=1$, $L(1,\psi)\neq 0$, we have

$$\begin{aligned}\sum_{\psi \bmod p} \psi(d)\frac{1}{|L(1,\psi)|^{2k}} &= \sum_{\psi \bmod p}{}' \psi(d)\Big|\sum_{c=1}^{\infty}\frac{\psi(c)r(c)}{c}\Big|^2 \\ &= \sum_{\psi \bmod p}{}' \psi(d)\Big(\sum_{c=1}^{\infty}\frac{\psi(c)r(c)}{c}\Big)\Big(\sum_{m=1}^{\infty}\frac{\overline{\psi}(m)r(m)}{m}\Big) \\ &= \sum_{\psi \bmod p}{}' \psi(d)\left(\sum_{1\le c<p/d}\frac{\psi(c)r(c)}{c} + \int_{p/d}^{\infty}\frac{E(\psi,y)}{y^2}dy\right) \\ &\qquad \left(\sum_{1\le m<p}\frac{\overline{\psi}(m)r(m)}{m} + \int_{p}^{\infty}\frac{F(\overline{\psi},y)}{y^2}dy\right).\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{\psi \bmod p} \psi(d)\frac{1}{|L(1,\psi)|^{2k}} &= \sum_{\psi \bmod p}{}' \psi(d)\Big(\sum_{1\le c<p/d}\frac{\psi(c)r(c)}{c}\Big)\Big(\sum_{1\le m<p}\frac{\overline{\psi}(m)r(m)}{m}\Big) \\ &\quad + \sum_{\psi \bmod p}{}' \left(\psi(d)\Big(\sum_{1\le c<p/d}\frac{\psi(c)r(c)}{c}\Big)\int_{p}^{\infty}\frac{F(\overline{\psi},y)}{y^2}dy\right) \\ &\quad + \sum_{\psi \bmod p}{}' \psi(d)\Big(\sum_{1\le m<p}\frac{\overline{\psi}(m)r(m)}{m}\Big)\left(\int_{p/d}^{\infty}\frac{E(\psi,y)}{y^2}dy\right) \\ &\quad + \sum_{\psi \bmod p}{}' \psi(d)\left(\int_{p/d}^{\infty}\frac{E(\psi,y)}{y^2}dy\right)\left(\int_{p}^{\infty}\frac{F(\overline{\psi},y)}{y^2}dy\right) \\ &= G_1+G_2+G_3+G_4,\end{aligned}$$

where $\sum_{\psi \bmod p}'$ denotes the summation over all non-principal characters mod p. Here we shall use the estimate

$$\sum_{\psi \bmod p} \psi(c)\overline{\psi}(m) = \begin{cases} p-1, & \text{if } c\equiv m (\text{mod } p), \\ 0 & \text{otherwise.}\end{cases}$$

Now,

$$\begin{aligned} G_1 &= \sum_{\psi\neq\psi_0}{}' \psi(d)\Big(\sum_{1\leq c<p/d}\frac{\psi(c)r(c)}{c}\Big)\Big(\sum_{1\leq m<p}\frac{\overline{\psi}(m)r(m)}{m}\Big) \\ &= \sum_{\psi \mod p}\sum_{1\leq c<p/d}\sum_{1\leq m<p}\frac{\psi(dc)\overline{\psi}(m)r(c)r(m)}{cm} + O(p^{\epsilon}) \\ &= \phi(p)\sum_{1\leq c<p/d}\sum_{1\leq m<p}\frac{r(c)r(m)}{cm} + O(p^{\epsilon}) \\ &= \phi(p)\frac{r(d)}{d}\sum_{1\leq c<p/d}\frac{r^2(c)}{c^2} + O(p^{\epsilon}) \\ &= \phi(p)\frac{r(d)}{d}\sum_{c=1}^{\infty}\frac{|r(c)|^2}{c^2} + O(p^{\epsilon}). \end{aligned}$$

Therefore,

$$G_1 = \phi(p)\frac{r(d)}{d}C + O(p^{\epsilon}), \quad \text{where } C = \sum_{c=1}^{\infty}\frac{|r(c)|^2}{c^2}.$$

$$\sum_{p\leq\Upsilon} p|G_1| = \sum_{p\leq\Upsilon} p^2\frac{r(d)}{d} + \sum_{p\leq\Upsilon} pp^{\epsilon} \ll \frac{r(d)}{d}.\Upsilon^{3+\epsilon}. \tag{4.5}$$

$$\begin{aligned} \sum_{p\leq\Upsilon} p|G_2| &= \sum_{p\leq\Upsilon} p\Big|\sum_{\psi \mod p}{}'\psi(d)\Big(\sum_{c\leq\frac{p}{d}}\frac{\psi(c)r(c)}{c}\Big)\int_p^{\infty}\frac{F(\overline{\psi},y)}{y^2}dy\Big| \\ &\leq \sum_{p\leq\Upsilon} p\Big|\sum_{\psi \mod p}{}'\psi(d)\Big(\sum_{c\leq\frac{p}{d}}\frac{\psi(c)r(c)}{c}\Big)\Big(\int_p^{\Upsilon^{2^k}}\sum_{p<m\leq y}\frac{\overline{\psi}(m)r(m)}{y^2}dy\Big)\Big| + \\ &\quad \sum_{p\leq\Upsilon} p\Big|\sum_{\psi \mod p}{}'\psi(d)\Big(\sum_{c\leq\frac{p}{d}}\frac{\psi(c)r(c)}{c}\Big)\Big(\int_{\Upsilon^{2^k}}^{\infty}\sum_{p<m\leq y}\frac{\overline{\psi}(m)r(m)}{y^2}dy\Big)\Big| \end{aligned}$$

$$\begin{aligned} \sum_{p\leq\Upsilon} p|G_2| &\ll \sum_{p\leq\Upsilon} p\int_p^{\Upsilon^{2^k}}\frac{1}{y^2}\Big|\sum_{c\leq\frac{p}{d}}\sum_{p<m\leq y}\frac{r(c)r(m)}{c}\sum_{\psi \mod p}\psi(dc)\overline{\psi}(m)\Big|dy + \\ &\quad O\Big(\Upsilon^{2+\epsilon}\int_p^{\Upsilon^{2^k}}\frac{1}{y^2}\Big(y(\log y)^{-A} + y^{1-\frac{1}{2^k}}\zeta\log(y\Upsilon)^y\Big)dy\Big) + O(\Upsilon^{\epsilon}) \\ &= O\Big(\sum_{p\leq\Upsilon} p^2\sum_{c<p/d}\frac{1}{c}\int_p^{\Upsilon^{2^k}}\frac{y\,\frac{1}{p}\,p^{\epsilon}}{y^2}dy\Big) + O(\Upsilon^{2+\epsilon}) \\ &= O(\Upsilon^{3+\epsilon}). \end{aligned} \tag{4.6}$$

Similarly, we find

$$\sum_{p\leq\Upsilon} p|G_3| = \sum_{p\leq\Upsilon} p|G_4| = O(\Upsilon^{3+\epsilon}). \tag{4.7}$$

Adding the equations (4.5), (4.6), (4.7) we get,

$$\sum_{p\leq\Upsilon} p\left|\sum_{\psi \bmod p}\frac{\psi(ab)}{|L(1,\psi)|^{2k}}\right| = O(\frac{r(ab)}{ab}\Upsilon^{3+\epsilon}). \qquad \blacksquare$$

NOTE: For prime power p^α note that $\mu(1)=1, \mu(p)=-1$ and $\mu(p^i)=0$ if $i>1$, so that we have

$$r(p^\alpha) = \sum_{c_1c_2\cdots=p^\alpha}\mu(c_1)\mu(c_2)\ldots\mu(c_k) = \begin{cases} (-1)^\alpha d_k^\alpha, \text{ if } \alpha\leq k, \\ 0, \text{ otherwise.}\end{cases}$$

and asymptotic formula is

$$\sum_{\substack{c=1\\(c,p)=1}}\frac{r^2(c)}{c^2} = \prod_{p_1}\left[1+\frac{(d_k^1)^2}{p_1^2}+\frac{(d_k^2)^2}{p_1^4}+\ldots\frac{(d_k^k)^2}{p_1^{2k}}\right] = d(k)+O\left(\frac{1}{p^2}\right).$$

Lemma 4.3 *Let $\Upsilon\geq 2$. For an integer k, we have the following identity:*

$$\sum_{p\leq\Upsilon}\sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}}\frac{p(p-1)^2}{|L(1,\psi)|^{2k}} = O(\Upsilon^{5+\epsilon}).$$

Proof: We have to calculate

$$\sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}}\frac{1}{|L(1,\psi)|^{2k}}.$$

As

$$\frac{1}{L^k(1,\psi)} = \sum_{c<\Upsilon^{2^k}}^{\infty}\frac{r(c)\psi(c)}{c}+\int_{\Upsilon^{2^k}}^{\infty}\frac{E(y,\psi)}{y^2}dy, \quad \text{where } E(y,\psi)=\sum_{\Upsilon^{2^k}<c\leq y} r(c)\psi(c).$$

Now

$$\sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}}\frac{1}{|L(1,\psi)|^{2k}} = \sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}}\left(\sum_{c\leq\Upsilon^{2^k}}\frac{r(c)\psi(c)}{c}+\int_{\Upsilon^{2^k}}^{\infty}\frac{E(y,\psi)}{y^2}dy\right)$$
$$\left(\sum_{c\leq\Upsilon^{2^k}}\frac{r(c)\overline{\psi}(c)}{c}+\int_{\Upsilon^{2^k}}^{\infty}\frac{E(y,\overline{\psi})}{y^2}dy\right).$$

So we have,

$$\sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}} \frac{1}{|L(1,\psi)|^{2k}} = \sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}} \left(\sum_{c\leq\Upsilon^{2^k}} \Big(\frac{r(c)\psi(c)}{c}\Big)\Big(\frac{r(c)\overline{\psi}(c)}{c}\Big)\right) + \sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}} \left|\int_{\Upsilon^{2^k}}^{\infty} \frac{E(y,\psi)}{y^2}dy\right|^2$$

$$+ \sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}} \left(\sum_{c\leq\Upsilon^{2^k}} \frac{r(c)\psi(c)}{c}\right) \int_{\Upsilon^{2^k}}^{\infty} \frac{E(y,\overline{\psi})}{y^2}dy$$

$$+ \sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}} \left(\sum_{c\leq\Upsilon^{2^k}} \frac{r(c)\overline{\psi}(c)}{c}\right) \int_{\Upsilon^{2^k}}^{\infty} \frac{E(y,\psi)}{y^2}dy\Bigg)$$

$$= H_1 + H_2 + H_3 + H_4.$$

Here

$$\sum_{p\leq\Upsilon} p(p-1)^2(p-1)H_1 = \sum_{p\leq\Upsilon} p(p-1)^2(p-1) \sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}} \left|\sum_{c\leq\Upsilon^{2^k}} \frac{r(c)\psi(c)}{c}\right|^2 \ll O(\Upsilon^{5+\epsilon}),$$

$$\sum_{p\leq\Upsilon} p(p-1)^2 H_2 = \sum_{p\leq\Upsilon} p(p-1)^2 \sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}} \left(\sum_{c\leq\Upsilon^{2^k}} \frac{r(c)\psi(c)}{c}\right) \int_{\Upsilon^{2^k}}^{\infty} \frac{E(y,\overline{\psi})}{y^2}dy \ll \Upsilon^{4+\epsilon},$$

$$\sum_{p\leq\Upsilon} p(p-1)^2 H_3 = \sum_{p\leq\Upsilon} p(p-1)^2 \sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}} \left(\sum_{c\leq\Upsilon^{2^k}} \frac{r(c)\overline{\psi}(c)}{c}\right) \int_{\Upsilon^{2^k}}^{\infty} \frac{E(y,\psi)}{y^2}dy\Bigg) \ll \Upsilon^{4+\epsilon},$$

$$\sum_{p\leq\Upsilon} p(p-1)^2 H_4 = \sum_{p\leq\Upsilon} p(p-1)^2 \sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}} \left|\int_{\Upsilon^{2^k}}^{\infty} \frac{E(y,\psi)}{y^2}dy\right|^2 \ll \Upsilon^{4+\epsilon}.$$

Therefore,

$$\sum_{p\leq\Upsilon} p(p-1)^2 \sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}} \frac{1}{|L(1,\psi)|^{2k}} = O(\Upsilon^{5+\epsilon}). \qquad \blacksquare$$

5 Main Result:

In this section we establish the hybrid power mean of $2k^{th}$ power inversion of L-functions, trigonometric sums and general quartic Gauss sums for the case $p \equiv 3 \pmod 4$.

Theorem 5.1 *Let p be an prime with $p \le \Upsilon$. Then for any positive integer k, ϑ with $(\vartheta, p) = 1$ and $|\vartheta| \le \Upsilon$, we have*

$$\sum_{p\le\Upsilon}\ \sum_{\psi \bmod p} \left|\sum_{a=1}^{p-1}\ \sum_{b=1}^{p-1}\psi(ab)e\left(\frac{f(a,b)}{p}\right)\right|^2 |G(\vartheta,4,\psi;p)|^2 \frac{1}{|L(1,\psi)|^{2k}} \ = O\left(\Upsilon^{5+\epsilon}\right),$$

where $f(a,b)$ is a l^{th}-degree polynomial, Υ is any positive number with $\Upsilon \ge 2$, ψ is a Dirichlet character (mod p)*, ϵ is any fixed positive number.*

Proof: For any non-principal character $\psi \pmod p$, $|L(1,\psi)|^2 = |L(1,\overline{\psi})|^2$ and if ψ an even character (mod p) then $\overline{\psi}$ is also an even character (mod p). Using these properties we get,

$$\sum_{p\le\Upsilon}\ \sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}} \left|\sum_{a=1}^{p-1}\ \sum_{b=1}^{p-1}\psi(ab)e\left(\frac{f(a,b)}{p}\right)\right|^2 |G(\vartheta,4,\psi;p)|^2 \frac{1}{|L(1,\psi)|^{2k}}$$

$$= \sum_{p\le\Upsilon}\ \sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}} \left|\sum_{a=1}^{p-1}\ \sum_{b=1}^{p-1}\overline{\psi}(ab)e\left(\frac{f(a,b)}{p}\right)\right|^2 |G(\vartheta,4,\overline{\psi};p)|^2 \frac{1}{(|L(1,\overline{\psi})|^2)^k}$$

$$= \sum_{p\le\Upsilon}\ \sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}} \left|\sum_{a=1}^{p-1}\ \sum_{b=1}^{p-1}\psi(ab)e\left(\frac{f(a,b)}{p}\right)\right|^2 |G(\vartheta,4,\overline{\psi};p)|^2 \frac{1}{|L(1,\psi)|^{2k}}.$$

So,

$$\frac{1}{2}\sum_{p\le\Upsilon}\ \sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}} \left|\sum_{a=1}^{p-1}\ \sum_{b=1}^{p-1}\psi(ab)e\left(\frac{f(a,b)}{p}\right)\right|^2 \left(|G(\vartheta,4,\psi;p)|^2 + |G(\vartheta,4,\overline{\psi};p)|^2\right) \frac{1}{|L(1,\psi)|^{2k}}$$

$$= \frac{1}{2}\sum_{p\le\Upsilon}\ \sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}} \left|\sum_{a=1}^{p-1}\ \sum_{b=1}^{p-1}\psi(ab)e\left(\frac{f(a,b)}{p}\right)\right|^2 (4p) \frac{1}{|L(1,\psi)|^{2k}}$$

$$= 2\sum_{p\le\Upsilon}p\ \sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}} \Bigg[(p-1)^2 + \sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\psi(ab)e\left(\frac{g(a,b,u,v)}{p}\right)$$

$$+ \sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\sum_{b=2}^{p-1}\psi(b)e\left(\frac{g(1,b,u,v)}{p}\right) + \sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\sum_{a=2}^{p-1}\psi(a)e\left(\frac{g(a,1,u,v)}{p}\right)\Bigg] \frac{1}{|L(1,\psi)|^{2k}}$$

$$= 2\sum_{p\le\Upsilon}\ \sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}} \frac{p(p-1)^2}{|L(1,\psi)|^{2k}} + 2\sum_{p\le\Upsilon}p\ \sum_{\substack{\psi\neq\psi_0\\ \psi(-1)=1}} \sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\psi(ab)e\left(\frac{g(a,b,u,v)}{p}\right) \frac{1}{|L(1,\psi)|^{2k}}$$

$$+2\sum_{p\le\Upsilon}p\sum_{\substack{\psi\ne\psi_0\\ \psi(-1)=1}}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\sum_{b=2}^{p-1}\psi(b)e\Big(\frac{g(1,b,u,v)}{p}\Big)\frac{1}{|L(1,\psi)|^{2k}}+2\sum_{p\le\Upsilon}p\sum_{\substack{\psi\ne\psi_0\\ \psi(-1)=1}}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\sum_{a=2}^{p-1}\psi(a)e\Big(\frac{g(a,1,u,v)}{p}\Big)\frac{1}{|L(1,\psi)|^{2k}}$$

$$=2\sum_{p\le\Upsilon}\sum_{\substack{\psi\ne\psi_0\\ \psi(-1)=1}}\frac{p(p-1)^2}{|L(1,\psi)|^{2k}}+2\sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\psi(ab)e\left(\frac{g(a,b,u,v)}{p}\right)\sum_{p\le\Upsilon}p\sum_{\psi\ne\psi_0}\psi(ab)\frac{1}{|L(1,\psi)|^{2k}}$$

$$+2\sum_{a=2}^{p-1}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\psi(a)e\left(\frac{g(a,1,u,v)}{p}\right)\sum_{p\le\Upsilon}p\sum_{\psi\ne\psi_0}\psi(a)\frac{1}{|L(1,\psi)|^{2k}}$$

$$+2\sum_{b=2}^{p-1}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\psi(b)e\left(\frac{g(1,b,u,v)}{p}\right)\sum_{p\le\Upsilon}p\sum_{\psi\ne\psi_0}\psi(b)\frac{1}{|L(1,\psi)|^{2k}}$$

$$=2\sum_{p\le\Upsilon}\sum_{\substack{\psi\ne\psi_0\\ \psi(-1)=1}}\frac{p(p-1)^2}{|L(1,\psi)|^{2k}}+2\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}\psi(ab)e\left(\frac{g(a,b,u,v)}{p}\right)\sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\sum_{p\le\Upsilon}p\sum_{\psi\ne\psi_0}\psi(ab)\frac{1}{|L(1,\psi)|^{2k}}$$

$$+2\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\left(\frac{g(a,1,u,v)}{p}\right)\sum_{p\le\Upsilon}p\sum_{a=2}^{p-1}\sum_{\psi\ne\psi_0}\psi(a)\frac{1}{|L(1,\psi)|^{2k}}+2\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\left(\frac{g(1,b,u,v)}{p}\right)\sum_{p\le\Upsilon}p\sum_{b=2}^{p-1}\sum_{\psi\ne\psi_0}\psi(b)\frac{1}{|L(1,\psi)|^{2k}},$$

where $g(a,b,u,v)=f(au,bv)-f(u,v)=\sum_{j=0}^{l}a_j(a^jb^j-1)u^jv^j$, $t_j=a_j(a^jb^j-1)$, $j=0,1,2,3,...,l$. $\sum_{a=2}^{p-1}{}^{*}\sum_{b=2}^{p-1}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}$ and $\sum_{a=2}^{p-1}{}^{**}\sum_{b=2}^{p-1}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}$ means $p\nmid(t_0,t_1,t_2,...t_l)$ and $p\mid(t_0,t_1,t_2,...t_l)$ respectively.

case : 1 when $p\nmid(t_0,t_1,t_2,...t_l)$, we have

$$=2\Bigg|\sum_{a=2}^{p-1^*}\sum_{b=2}^{p-1}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\left(\frac{g(a,b,u,v)}{p}\right)\sum_{p\le\Upsilon}p\sum_{\psi\ne\psi_0}\psi(ab)\frac{1}{|L(1,\psi)|^{2k}}$$
$$+\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\left(\frac{g(a,1,u,v)}{p}\right)\sum_{p\le\Upsilon}p\sum_{a=2}^{p-1}\sum\nolimits_{\psi\ne\psi_0}\psi(a)\frac{1}{|L(1,\psi)|^{2k}}$$
$$+\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\left(\frac{g(1,b,u,v)}{p}\right)\sum_{p\le\Upsilon}p\sum_{b=2}^{p-1}\sum\nolimits_{\psi\ne\psi_0}\psi(b)\frac{1}{|L(1,\psi)|^{2k}}\Bigg|,$$

$$\ll 2\sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\left(\frac{g(a,b,u,v)}{p}\right)\sum_{p\le\Upsilon}p\left|\sum_{\psi\ne\psi_0}\psi(ab)\frac{1}{|L(1,\psi)|^{2k}}\right|$$

$$+2\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\Big(\frac{g(a,1,u,v)}{p}\Big)\sum_{p\leq\Upsilon}p\sum_{a=2}^{p-1}\Big|\sum\nolimits_{\psi\neq\psi_0}\psi(a)\frac{1}{|L(1,\psi)|^{2k}}\Big|$$

$$+2\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\Big(\frac{g(1,b,u,v)}{p}\Big)\sum_{p\leq\Upsilon}p\sum_{b=2}^{p-1}\Big|\sum\nolimits_{\psi\neq\psi_0}\psi(b)\frac{1}{|L(1,\psi)|^{2k}}\Big|$$

$$\ll O\Big(2\sum_{a=2}^{p-1}\sum_{b=2}^{p-1}p^{2-\frac{2}{l}}\frac{r(ab)}{ab}\Upsilon^{3+\epsilon}\Big)+O\Big(2p^{2-\frac{2}{l}}\ \ \Upsilon^{3+\epsilon}\Big)$$

$$\ll O(\Upsilon^{5+\epsilon}),$$

where $r(ab)\ll p^{\epsilon}$.

case : 2 when $p\mid(t_0,t_1,t_2,...t_l)$,

$$=2\Big|\sum_{a=2}^{p-1^{**}}\sum_{b=2}^{p-1}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\Big(\frac{g(a,b,u,v)}{p}\Big)\sum_{p\leq\Upsilon}p\sum_{\psi\neq\psi_0}\psi(ab)\frac{1}{|L(1,\psi)|^{2k}}$$

$$+\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\Big(\frac{g(a,1,u,v)}{p}\Big)\sum_{p\leq\Upsilon}p\sum_{a=2}^{p-1}\sum\nolimits_{\psi\neq\psi_0}\psi(a)\frac{1}{|L(1,\psi)|^{2k}}$$

$$+\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\Big(\frac{g(1,b,u,v)}{p}\Big)\sum_{p\leq\Upsilon}p\sum_{b=2}^{p-1}\sum\nolimits_{\psi\neq\psi_0}\psi(b)\frac{1}{|L(1,\psi)|^{2k}}\Big|,$$

$$\ll 2\sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\Big(\frac{g(a,b,u,v)}{p}\Big)\Big|\sum_{p\leq\Upsilon}p\sum_{\psi\neq\psi_0}\psi(ab)\frac{1}{|L(1,\psi)|^{2k}}\Big|$$

$$+2\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\Big(\frac{g(a,1,u,v)}{p}\Big)\Big|\sum_{p\leq\Upsilon}p\sum_{a=2}^{p-1}\sum\nolimits_{\psi\neq\psi_0}\psi(a)\frac{1}{|L(1,\psi)|^{2k}}\Big|$$

$$+2\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\Big(\frac{g(1,b,u,v)}{p}\Big)\Big|\sum_{p\leq\Upsilon}p\sum_{b=2}^{p-1}\sum\nolimits_{\psi\neq\psi_0}\psi(b)\frac{1}{|L(1,\psi)|^{2k}}\Big|$$

$$\ll O\Big(2(p-1)^2\sum_{a=2}^{p-1}\sum_{b=2}^{p-1}\frac{r(ab)}{ab}\Upsilon^{3+\epsilon}+2(p-1)^2\sum_{b=2}^{p-1}\Upsilon^{3+\epsilon}+2(p-1)^2\sum_{a=2}^{p-1}\Upsilon^{3+\epsilon}\Big)$$

$$\ll O(\Upsilon^{5+\epsilon}).$$

Combining **case:1**, **case:2** and Lemma 4.3,

$$\sum_{p\le \Upsilon}\sum_{\psi \bmod p}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\psi(ab)e\Big(\frac{f(a,b)}{p}\Big)\right|^2|G(\vartheta,4,\psi;p)|^2\frac{1}{|L(1,\psi)|^{2k}} \le O\Big(2\sum_{p\le \Upsilon}\sum_{\substack{\psi\ne\psi_0\\ \psi(-1)=1}}\frac{p(p-1)^2}{|L(1,\psi)|^{2k}}\Big)$$

$$+O\Bigg(2\Bigg|\sum_{a=2}^{p-1^*}\sum_{b=2}^{p-1}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\Big(\frac{g(a,b,u,v)}{p}\Big)\sum_{p\le\Upsilon}p\sum_{\psi\ne\psi_0}\psi(ab)\frac{1}{|L(1,\psi)|^{2k}}$$

$$+\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\Big(\frac{g(a,1,u,v)}{p}\Big)\sum_{p\le\Upsilon}p\sum_{a=2}^{p-1}\sum_{\psi\ne\psi_0}\psi(a)\frac{1}{|L(1,\psi)|^{2k}}$$

$$+\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\Big(\frac{g(1,b,u,v)}{p}\Big)\sum_{p\le\Upsilon}p\sum_{b=2}^{p-1}\sum_{\psi\ne\psi_0}\psi(b)\frac{1}{|L(1,\psi)|^{2k}}\Bigg|\Bigg)$$

$$+O\Bigg(2\Bigg|\sum_{a=2}^{p-1^{**}}\sum_{b=2}^{p-1}\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\Big(\frac{g(a,b,u,v)}{p}\Big)\sum_{p\le\Upsilon}p\sum_{\psi\ne\psi_0}\psi(ab)\frac{1}{|L(1,\psi)|^{2k}}$$

$$+\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\Big(\frac{g(a,1,u,v)}{p}\Big)\sum_{p\le\Upsilon}p\sum_{a=2}^{p-1}\sum_{\psi\ne\psi_0}\psi(a)\frac{1}{|L(1,\psi)|^{2k}}$$

$$+\sum_{u=1}^{p-1}\sum_{v=1}^{p-1}e\Big(\frac{g(1,b,u,v)}{p}\Big)\sum_{p\le\Upsilon}p\sum_{b=2}^{p-1}\sum_{\psi\ne\psi_0}\psi(b)\frac{1}{|L(1,\psi)|^{2k}}\Bigg|\Bigg)$$

$$\ll O\Big(\Upsilon^{5+\epsilon}\Big).$$

Acknowledgment

The authors would like to thank Central University of Jharkhand for support.

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