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SOME BASIC PROPERTIES OF DESCRIPTIVE LINER SETS

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Abstract. In this paper, some basic properties of descriptive linear sets(Distance set, Difference set, Midpoint set, etc. of linear sets) have been established.

1. Introduction

In 1920, H. Steinhaus [10] introduced the concept distance set $D(A) = \{|x-y| : x \in A, y \in A\}$ of a linear set A and he established that distance set of a linear set of positive Lebesgue measure contains an interval of the form $[0, \alpha], \alpha > 0$. In the same paper he also established the analogous result by introducing the notions difference set $\Delta(A) = \{x - y : x, y \in A\}$ of linear set A and sum set $A + B = \{x + y : x \in A, y \in B\}$ of two linear sets A and B.

Definition 1.1. [8] A set A is said to have the property of Baire if it can be expressed as symmetric difference of an open set and a set of first category.

The category analogue of Steinhaus's result was established by Piccard [9] in the following way:
If both A and B are second category subsets of the real line $\mathbb{R}$, each having the property of Baire, then the set $A + B = \{a + b : a \in A, b \in B\}$ contains an interval. Several authors ([1], [2], [3], [4], [5], [6], [7]) have generalized the results of Steinhaus and Picard in many directions. Throughout this paper we shall denote by $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ respectively the set of natural numbers, the set of rational numbers and the set of real numbers also $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$. In order to describe the basic properties of descriptive linear sets (Distance set, Difference set, Sum set, Midpoint set, etc. of linear sets), we define the following sets:

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Definition 1.2. Let A and B be any two non-empty subsets of the real line $\mathbb{R}$. Let $p, q \in [-1, 1] \cap \mathbb{Q}^*$. Then we define the following sets;

- $L_{p,q}(A) = \{px + qy : x, y \in A\}$.
- $L^*_{p,q}(A) = \{px + qy : x \neq y, x, y \in A\}$.
- $L_{p,q}(A, B) = \{px + qy : x \in A, y \in B\}$.

2. Results

Theorem 2.1. *If $\{A_n\}$ is monotone decreasing sequence of closed and bounded linear sets, then $L_{p,q}(\cap A_n) = \cap L_{p,q}(A_n)$, for any $p, q \in [-1, 1] \cap \mathbb{Q}^*$.*

Proof. As $\cap A_n \subset A_n$ for all $n \in \mathbb{N}$ and for any $p, q \in [-1, 1] \cap \mathbb{Q}^*$,

$$L_{p,q}(\cap A_n) = \{px + qy : x, y \in \cap A_n\} \subset \{px + qy : x, y \in A_n\} = L_{p,q}(A_n).$$

Thus $L_{p,q}(\cap A_n) \subset \cap L_{p,q}(A_n)$, for any $p, q \in [-1, 1] \cap \mathbb{Q}^*$. On the other hand, let $d \in \cap L_{p,q}(A_n)$. This implies that $d \in L_{p,q}(A_n)$ for all $n \in \mathbb{N}$. Then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in A_n such that $px_n + qy_n = d$, for any $p, q \in [-1, 1] \cap \mathbb{Q}^*$. As $\{A_n\}$ is monotone decreasing, so $A_n \subset A_1$, hence $\{x_n\} \subset A_1$, which is closed and bounded and has a limit point say $x \in A_1$. Hence $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to x$ and the corresponding subsequence $\{y_{n_i}\}$ of y_n has a limit $y \in A_1$. As $\{x_{n_i}\} \subset A_n$ and $\{y_{n_i}\} \subset A_n$ for each n, so $x_{n_i} \in A_n$ and $y_{n_i} \in A_n$ and consequently $px + qy = d \in L_{p,q}(\cap A_n)$, for any $p, q \in [-1, 1] \cap \mathbb{Q}^*$. Thus $\cap L_{p,q}(A_n) \subset L_{p,q}(\cap A_n)$, for each $p, q \in [-1, 1] \cap \mathbb{Q}^*$. Hence the result. □

Remark 2.2. The above result does not hold when we replace intersection by union. For example:

Example 2.3. Let $A = [0, 1]$ and $B = [2, 3]$, then $L_{\frac{1}{3},\frac{2}{3}}(A \cup B) = [0, 3]$. But $L_{\frac{1}{3},\frac{2}{3}}(A) = [0, 1]$ and $L_{\frac{1}{3},\frac{2}{3}}(B) = [2, 3]$. So, $L_{\frac{1}{3},\frac{2}{3}}(A \cup B) \neq L_{\frac{1}{3},\frac{2}{3}}(A) \cup L_{\frac{1}{3},\frac{2}{3}}(B)$.

Remark 2.4. The above result does not hold without the sequence of 'closed' sets. For example:

Example 2.5. Let $A_n = \{1 \pm (\frac{1}{2})^{n+t} : t = 0, 1, 2, 3, ...\}$, for $n \in \mathbb{N}$. Then $\{A_n\}$ is monotone with $\cap A_n = \emptyset$ but $1 \in L_{\frac{1}{3},\frac{2}{3}}(A_n)$ for all n. So, $1 \in \cap L_{\frac{1}{3},\frac{2}{3}}(A_n)$. It follows that $L_{\frac{1}{3},\frac{2}{3}}(\cap A_n) \neq \cap L_{\frac{1}{3},\frac{2}{3}}(A_n)$.

Theorem 2.6. *If A is an open subset of $\mathbb{R}$, then for any $p, q \in [-1, 1] \cap \mathbb{Q}^*$, $L_{p,q}(A)$ and $L^*_{p,q}(A)$ are open in $\mathbb{R}$.*

Proof. Let $l \in L_{p,q}(A)$, for any $p, q \in [-1, 1] \cap \mathbb{Q}^*$. Then there exist $x, y \in A$ such that $l = px + qy$. If possible let $I = (l - \epsilon, l + \epsilon)$, $\epsilon > 0$, be not contained in $L_{p,q}(A)$, $p,q \in [-1,1]\cap\mathbb{Q}^*$. Then there exists $r \in I$ such that $r \notin L_{p,q}(A)$, $p,q \in [-1,1]\cap\mathbb{Q}^*$ and thus $l-\epsilon < r < l+\epsilon \Rightarrow px+qy-\epsilon < r < px+qy+\epsilon$. This implies:

Case I: When $q > 0$, $y-\frac{\epsilon}{q} < \dfrac{r-px}{q} < y+\frac{\epsilon}{q}$. This shows that $\dfrac{r-px}{q} \in I_y = (y-\frac{\epsilon}{q}, y+\frac{\epsilon}{q}) \subset A$ (y being an interior point of A) and hence $px+q\dfrac{r-px}{q} = r \in L_{p,q}(A)$ which is a contradiction.

Case II: When $q < 0$, $y-\frac{\epsilon}{q} > \dfrac{r-px}{q} > y+\frac{\epsilon}{q}$. This shows that $\dfrac{r-px}{q} \in I_y = (y+\frac{\epsilon}{q}, y-\frac{\epsilon}{q}) \subset A$ (y being an interior point of A) and hence $px+q\dfrac{r-px}{q} = r \in L_{p,q}(A)$ which is a contradiction.

Hence for both the cases $I \subset L_{p,q}(A)$, $p,q \in [-1,1]\cap\mathbb{Q}^*$. Since l is arbitrary $L_{p,q}(A)$ is an open set. If we take $x \neq y$ then $L^*_{p,q}(A)$ is also an open set. □

Theorem 2.7. *If $A(\subset \mathbb{R})$ is closed and bounded, then $L_{p,q}(A)$ for any $p,q \in [-1,1]\cap\mathbb{Q}^*$, is closed.*

Proof. Let $l \in \overline{L_{p,q}(A)}$ for any $p,q \in [-1,1]\cap\mathbb{Q}^*$. Then there is a sequence $\{l_n\} \subset L_{p,q}(A)$ such that $l_n \to l$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in A such that $px_n + qy_n = l_n$. Since A is closed and bounded there exists a sub sequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x$ in A and therefore possesses such sequence $\{x_{n_{i_k}}\}$ for which $x_{n_{i_k}} \to x$. The corresponding subsequence $\{y_{n_i}\}$ also has a limit point $y \in A$ and therefore possesses such sequence $\{y_{n_{i_k}}\}$ for which $\{y_{n_{i_k}}\} \to y$. Now $px+qy = p(x-x_{n_{i_k}})+q(y-y_{n_{i_k}})+(px_{n_{i_k}}+qy_{n_{i_k}})$. Taking $k \to \infty$ we have $px+qy = l \in L_{p,q}(A)$ for any $p,q \in [-1,1]\cap\mathbb{Q}^*$. i.e. $\overline{L_{p,q}(A)} \subset L_{p,q}(A)$ and consequently $L_{p,q}(A)$ for any $p,q \in [-1,1]\cap\mathbb{Q}^*$, is closed. □

Remark 2.8. $L^*_{p,q}(A)$ for any $p,q \in [-1,1]\cap\mathbb{Q}^*$, of a closed set A may not be a closed set. For example:
Let $A = [0,1]$, then $L^*_{p,q}(A) = (0,1)$.

Theorem 2.9. *If $A(\subset \mathbb{R})$ is bounded and perfect, then for any $p,q \in [-1,1]\cap\mathbb{Q}^*$, $L_{p,q}(A)$ is perfect.*

Proof. Let $l \in L_{p,q}(A)$ for any $p,q \in [-1,1]\cap\mathbb{Q}^*$ and $x,y \in A$ so that $px+qy = l$. Suppose $l > 0$. Let $0 < \epsilon < l$ and $I = (l-\epsilon, l+\epsilon)$. Since A is perfect, so the interval $(y-\frac{\epsilon}{q}, y+\frac{\epsilon}{q}), (q>0)$ [or $(y+\frac{\epsilon}{q}, y-\frac{\epsilon}{q}), (q<0)$] intersects A in atleast one point, say $y_1 (y_1 \neq y)$. Let $l_1 = px+qy_1$ for any $p,q \in [-1,1]\cap\mathbb{Q}^*$. Then $l_1 \in L_{p,q}(A)$ for any $p,q \in [-1,1]\cap\mathbb{Q}^*$ and $l \neq l_1$. It is clear that, if $q > 0$, $y-\frac{\epsilon}{q} < y_1 < y+\frac{\epsilon}{q}$. This implies that $px+qy-\epsilon < px+qy_1 < px+qy+\epsilon$ i.e. $l-\epsilon < l_1 < l+\epsilon$. Thus $l_1 \in I$. In the similar way we can show that $l_1 \in I$ when $l < 0$. If $l = 0$ then for any $y \in A$, $y + (-1)y = l$. Given $\epsilon > 0$, the interval $(y - \epsilon, y + \epsilon)$ intersects A in some point say $y_1(\neq y)$. Let $l_1 = y_1 + (-1)y$. Then clearly $l_1 \in I$ and $l \neq l_1$. Hence $L_{p,q}(A) = (L_{p,q}(A))'$ and consequently $L_{p,q}(A)$ is perfect. □

Theorem 2.10. *If $A(\subset \mathbb{R})$ is of second category, for any $p, q \in [-1, 1] \cap \mathbb{Q}^*$, both $L_{p,q}(A)$ and $L^*_{p,q}(A)$ are of second category.*

Proof. If possible let $L^*_{p,q}(A)(\subset L_{p,q}(A))$, where $p, q \in [-1, 1] \cap \mathbb{Q}^*$, be of first category. Then the set $\frac{L^*_{p,q}(A) - qa}{p} = \{\frac{px + qy - qa}{p} : x, y \in A, x \neq y\}$, (where $a \in A$ is fixed) is of first category. Let $x \in A$ and $x \neq a$. Then $px + qa \in L^*_{p,q}(A)$ implies that $x \in \frac{L^*_{p,q}(A) - qa}{p}$. Thus $x \in A \setminus \{a\}$ implies that $x \in \frac{L^*_{p,q}(A) - qa}{p}$.i.e. $A \setminus \{a\} \subset \frac{L^*_{p,q}(A) - qa}{p}$. i.e. $A = (A \setminus \{a\}) \cap \{a\}$ is of first category which is a contradiction. So, $L^*_{p,q}(A)$ is of second category and hence the result. □

From the results of H. I. Miller[7] we get the following two results.

Theorem 2.11. *If A and B are linear sets of positive measure, then $L_{p,q}(A, B)$ for any $p, q \in [-1, 1] \cap \mathbb{Q}^*$, contains an interval.*

Theorem 2.12. *If A and B are linear sets of second category and each possesses the property of Baire, then $L_{p,q}(A)$ for any $p, q \in [-1, 1] \cap \mathbb{Q}^*$, contains an interval.*

Theorem 2.13. *Let A and B be two liner sets of positive Lebesgue measure and F be a countable dense subset of $\mathbb{R}$. Then for each $d \in F$, $E = \{x \in A : \exists y \in B, px + qy = d; p, q \in [-1, 1] \cap \mathbb{Q}^*\}$ has positive measure.*

Proof. Clearly E is Lebesgue measurable. If possible let us assume Lebesgue measure $\mu(E) = 0$ and let $H = A \setminus E$, then $\mu(H) = \mu(A) > 0$. Then $L_{p,q}(H, B)$, where $p, q \in [-1, 1] \cap \mathbb{Q}^*$, contains an interval. Hence $F \cap L_{p,q}(H, B) \neq \emptyset$, where $p, q \in [-1, 1] \cap \mathbb{Q}^*$. Then for $x \in H \subset A$ and $y \in B$, there exists $c \in F$ such that $c = px + qy$. This shows that $x \in E$, a contradiction. Thus E is of positive measure. □

Theorem 2.14. *Let A and B be two liner sets of second category each having the property of Baire and F be a countable dense subset of $\mathbb{R}$. Then for any $d \in F$, $G = \{x \in A : \exists y \in B, px + qy = d; p, q \in [-1, 1] \cap \mathbb{Q}^*\}$, is of second category.*

Proof. If possible let $G = \{x \in A : \exists y \in B, px + qy = d; p, q \in [-1, 1] \cap \mathbb{Q}^*\}$ for any $d \in F$, be first category. Clearly $A \setminus \cup_{d \in F} G = H$ (say) is a residual set. Hence $F \cap L_{p,q}(H, B) \neq \emptyset$. Then for some $c \in F$, there exist $x \in H \subset A$ and $y \in B$ such that $c = px + qy; p, q \in [-1, 1] \cap \mathbb{Q}^*$ and consequently this $x \in G$ which is a contradiction. Hence G is a second category set. □

References

[1] J.Ceder and D.K.Ganguly, *On projection of big planar sets*, Real Anal. Exch.,9 (1983 -84), 206 –214.

[2] Miljenko Crnjac , Boris Guljas and Harry I.Miller, *On some questions of Ger, Grubb and Kraljević*, Acta Math. Hung., 57(3-4) (1991), 253–257.

[3] P.Erdos and J.C.Oxtoby, *Partitions of the plane into sets having positive measure in every non-null measurable product set*, Trans. Amer. Math. Soc., 79, (1955), 99–102.

[4] Z.Kominek, *Some generalizations of the theorem of S. Piccard*, Prace Nauk. Univ. Slqski Katowice, Prace Mat., 4 (1973), 31–33.

[5] M.E.Kuczma and M.Kuczma, *An elementary proof and an extension of a theorem of Steinhaus*, Glasnik Mat. Ser. III, 6(26),(1971) 11–18.

[6] H.I.Miller, *Generalization of a classical theorem of measure theory*, Radovi Akademije Nauka i Umjetnosti BiH, XLV (1973), 45–48.

[7] H.I.Miller, *Relationships between various gauges of the size of sets of real numbers*, Glasnik Mat. Ser. III, 9(29),(1974), 59–64.

[8] J.C.Oxtoby, *Measure and Category*, Second Edition, Springer - Verlag, NY, Heidelberg Berlin, (1980).

[9] S.Piccard, *Sur les ensembles de distances des points d' un espace Euclidean*, Neuchatel, (1933).

[10] H.Steinhaus, *Sur les distance des points des ensembles de measure positive*, Fund - Math, I, (1920), 93–104.

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