# SOME BASIC PROPERTIES OF DESCRIPTIVE LINER SETS 

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#### Abstract

In this paper, some basic properties of descriptive linear sets(Distance set, Difference set, Midpoint set, etc. of linear sets) have been established.


## 1. Introduction

In 1920, H. Steinhaus [10] introduced the concept distance set $D(A)=$ $\{|x-y|: x \in A, y \in A\}$ of a linear set $A$ and he established that distance set of a linear set of positive Lebesgue measure contains an interval of the form $[0, \alpha], \alpha>0$. In the same paper he also established the analogous result by introducing the notions difference set $\Delta(A)=\{x-y: x, y \in A\}$ of linear set $A$ and sum set $A+B=\{x+y: x \in A, y \in B\}$ of two linear sets $A$ and $B$.

Definition 1.1. [8] A set $A$ is said to have the property of Baire if it can be expressed as symmetric difference of an open set and a set of first category.

The category analogue of Steinhaus's result was established by Piccard [9] in the following way:
If both $A$ and $B$ are second category subsets of the real line $\mathbb{R}$, each having the property of Baire, then the set $A+B=\{a+b: a \in A, b \in B\}$ contains an interval. Several authors ([1], [2], [3], [4], [5], [6], [7]) have generalized the results of Steinhaus and Picard in many directions. Throughout this paper we shall denote by $\mathbb{N}, \mathbb{Q}$ and $\mathbb{R}$ respectively the set of natural numbers, the set of rational numbers and the set of real numbers also $\mathbb{Q}^{*}=\mathbb{Q} \backslash\{0\}$. In order to describe the basic properties of descriptive linear sets (Distance set, Difference set, Sum set, Midpoint set, etc. of linear sets), we define the following sets:

Definition 1.2. Let $A$ and $B$ be any two non-empty subsets of the real line $\mathbb{R}$. Let $p, q \in[-1,1] \cap \mathbb{Q}^{*}$. Then we define the following sets;

- $L_{p, q}(A)=\{p x+q y: x, y \in A\}$.
- $L_{p, q}^{*}(A)=\{p x+q y: x \neq y, x, y \in A\}$.
- $L_{p, q}(A, B)=\{p x+q y: x \in A, y \in B\}$.


## 2. Results

Theorem 2.1. If $\left\{A_{n}\right\}$ is monotone decreasing sequence of closed and bounded linear sets, then $L_{p, q}\left(\cap A_{n}\right)=\cap L_{p, q}\left(A_{n}\right)$, for any $p, q \in[-1,1] \cap \mathbb{Q}^{*}$.

Proof. As $\cap A_{n} \subset A_{n}$ for all $n \in \mathbb{N}$ and for any $p, q \in[-1,1] \cap \mathbb{Q}^{*}$,

$$
L_{p, q}\left(\cap A_{n}\right)=\left\{p x+q y: x, y \in \cap A_{n}\right\} \subset\left\{p x+q y: x, y \in A_{n}\right\}=L_{p, q}\left(A_{n}\right)
$$

Thus $L_{p, q}\left(\cap A_{n}\right) \subset \cap L_{p, q}\left(A_{n}\right)$, for any $p, q \in[-1,1] \cap \mathbb{Q}^{*}$. On the other hand, let $d \in \cap L_{p, q}\left(A_{n}\right)$. This implies that $d \in L_{p, q}\left(A_{n}\right)$ for all $n \in \mathbb{N}$. Then there exist two sequences $\left\{x_{n}\right\}$ and $\left\{y_{n}\right\}$ in $A_{n}$ such that $p x_{n}+q y_{n}=d$, for any $p, q \in[-1,1] \cap \mathbb{Q}^{*}$. As $\left\{A_{n}\right\}$ is monotone decreasing, so $A_{n} \subset A_{1}$, hence $\left\{x_{n}\right\} \subset A_{1}$, which is closed and bounded and has a limit point say $x \in A_{1}$. Hence $\left\{x_{n}\right\}$ has a subsequence $\left\{x_{n_{i}}\right\}$ such that $x_{n_{i}} \rightarrow x$ and the corresponding subsequence $\left\{y_{n_{i}}\right\}$ of $y_{n}$ has a limit $y \in A_{1}$. As $\left\{x_{n_{i}}\right\} \subset A_{n}$ and $\left\{y_{n_{i}}\right\} \subset A_{n}$ for each $n$, so $x_{n_{i}} \in A_{n}$ and $y_{n_{i}} \in A_{n}$ and consequently $p x+q y=d \in L_{p, q}\left(\cap A_{n}\right)$, for any $p, q \in[-1,1] \cap \mathbb{Q}^{*}$. Thus $\cap L_{p, q}\left(A_{n}\right) \subset$ $L_{p, q}\left(\cap A_{n}\right)$, for each $p, q \in[-1,1] \cap \mathbb{Q}^{*}$. Hence the result.

Remark 2.2. The above result does not hold when we replace intersection by union. For example:

Example 2.3. Let $A=[0,1]$ and $B=[2,3]$, then $L_{\frac{1}{3}, \frac{2}{3}}(A \cup B)=[0,3]$. But $L_{\frac{1}{3}, \frac{2}{3}}(A)=[0,1]$ and $L_{\frac{1}{3}, \frac{2}{3}}(B)=[2,3]$. So, $L_{\frac{1}{3}, \frac{2}{3}}(A \cup B) \neq L_{\frac{1}{3}, \frac{2}{3}}(A) \cup L_{\frac{1}{3}, \frac{2}{3}}(B)$.

Remark 2.4. The above result does not hold without the sequence of 'closed' sets. For example:

Example 2.5. Let $A_{n}=\left\{1 \pm\left(\frac{1}{2}\right)^{n+t}: t=0,1,2,3, \ldots.\right\}$, for $n \in \mathbb{N}$. Then $\left\{A_{n}\right\}$ is monotone with $\cap A_{n}=\emptyset$ but $1 \in L_{\frac{1}{3}, \frac{2}{3}}\left(A_{n}\right)$ for all $n$. So, $1 \in$ $\cap L_{\frac{1}{3}, \frac{2}{3}}\left(A_{n}\right)$. It follows that $L_{\frac{1}{3}, \frac{2}{3}}\left(\cap A_{n}\right) \neq \cap L_{\frac{1}{3}, \frac{2}{3}}\left(A_{n}\right)$.

Theorem 2.6. If $A$ is an open subset of $\mathbb{R}$, then for any $p, q \in[-1,1] \cap \mathbb{Q}^{*}$, $L_{p, q}(A)$ and $L_{p, q}^{*}(A)$ are open in $\mathbb{R}$.

Proof. Let $l \in L_{p, q}(A)$, for any $p, q \in[-1,1] \cap \mathbb{Q}^{*}$. Then there exist $x, y \in A$ such that $l=p x+q y$. If possible let $I=(l-\epsilon, l+\epsilon), \epsilon>0$, be not contained
in $L_{p, q}(A), p, q \in[-1,1] \cap \mathbb{Q}^{*}$. Then there exists $r \in I$ such that $r \notin L_{p, q}(A)$, $p, q \in[-1,1] \cap \mathbb{Q}^{*}$ and thus $l-\epsilon<r<l+\epsilon \Rightarrow p x+q y-\epsilon<r<p x+q y+\epsilon$. This implies:
Case I: When $q>0, y-\frac{\epsilon}{q}<\frac{r-p x}{q}<y+\frac{\epsilon}{q}$. This shows that $\frac{r-p x}{q} \in I_{y}=$ $\left(y-\frac{\epsilon}{q}, y+\frac{\epsilon}{q}\right) \subset A(y$ being an interior point of $A)$ and hence $p x+q \frac{r-p x}{q}=$ $r \in L_{p, q}(A)$ which is a contradiction.
Case II: When $q<0, y-\frac{\epsilon}{q}>\frac{r-p x}{q}>y+\frac{\epsilon}{q}$. This shows that $\frac{r-p x}{q} \in$ $I_{y}=\left(y+\frac{\epsilon}{q}, y-\frac{\epsilon}{q}\right) \subset A(y$ being an interior point of $A)$ and hence $p x+$ $q \frac{r-p x}{q}=r \in L_{p, q}(A)$ which is a contradiction.
Hence for both the cases $I \subset L_{p, q}(A), p, q \in[-1,1] \cap \mathbb{Q}^{*}$. Since $l$ is arbitrary $L_{p, q}(A)$ is an open set. If we take $x \neq y$ then $L_{p, q}^{*}(A)$ is also an open set.

Theorem 2.7. If $A(\subset \mathbb{R})$ is closed and bounded, then $L_{p, q}(A)$ for any $p, q \in[-1,1] \cap \mathbb{Q}^{*}$, is closed.

Proof. Let $l \in L_{p, q}(A)$ for any $p, q \in[-1,1] \cap \mathbb{Q}^{*}$. Then there is a sequence $\left\{l_{n}\right\} \subset L_{p, q}(A)$ such that $l_{n} \rightarrow l$. Let $\left\{x_{n}\right\}$ and $\left\{y_{n}\right\}$ be two sequences in $A$ such that $p x_{n}+q y_{n}=l_{n}$. Since $A$ is closed and bounded there exists a sub sequence $\left\{x_{n_{i}}\right\}$ of $\left\{x_{n}\right\}$ such that $x_{n_{i}} \rightarrow x$ in $A$ and therefore possesses such sequence $\left\{x_{n_{i_{k}}}\right\}$ for which $x_{n_{i_{k}}} \rightarrow x$. The corresponding subsequence $\left\{y_{n_{i}}\right\}$ also has a limit point $y \in A$ and therefore possesses such sequence $\left\{y_{n_{i_{k}}}\right\}$ for which $\left\{y_{n_{i_{k}}}\right\} \rightarrow y$. Now $p x+q y=p\left(x-x_{n_{i_{k}}}\right)+q\left(y-y_{n_{i_{k}}}\right)+\left(p x_{n_{i_{k}}}+q y_{n_{i_{k}}}\right)$. Taking $k \rightarrow \infty$ we have $p x+q y=l \in L_{p, q}(A)$ for any $p, q \in[-1,1] \cap \mathbb{Q}^{*}$. i.e. $\overline{L_{p, q}(A)} \subset L_{p, q}(A)$ and consequently $L_{p, q}(A)$ for any $p, q \in[-1,1] \cap \mathbb{Q}^{*}$, is closed.

Remark 2.8. $L_{p, q}^{*}(A)$ for any $p, q \in[-1,1] \cap \mathbb{Q}^{*}$, of a closed set $A$ may not be a closed set. For example:
Let $A=[0,1]$, then $L_{p, q}^{*}(A)=(0,1)$.
Theorem 2.9. If $A(\subset \mathbb{R})$ is bounded and perfect, then for any $p, q \in$ $[-1,1] \cap \mathbb{Q}^{*}, L_{p, q}(A)$ is perfect.

Proof. Let $l \in L_{p, q}(A)$ for any $p, q \in[-1,1] \cap \mathbb{Q}^{*}$ and $x, y \in A$ so that $p x+q y=l$. Suppose $l>0$. Let $0<\epsilon<l$ and $I=(l-\epsilon, l+\epsilon)$. Since $A$ is perfect, so the interval $\left(y-\frac{\epsilon}{q}, y+\frac{\epsilon}{q}\right),(q>0)\left[\right.$ or $\left.\left(y+\frac{\epsilon}{q}, y-\frac{\epsilon}{q}\right),(q<0)\right]$ intersects $A$ in atleast one point, say $y_{1}\left(y_{1} \neq y\right)$. Let $l_{1}=p x+q y_{1}$ for any $p, q \in[-1,1] \cap \mathbb{Q}^{*}$. Then $l_{1} \in L_{p, q}(A)$ for any $p, q \in[-1,1] \cap \mathbb{Q}^{*}$ and $l \neq l_{1}$. It is clear that, if $q>0, y-\frac{\epsilon}{q}<y_{1}<y+\frac{\epsilon}{q}$. This implies that $p x+q y-\epsilon<p x+q y_{1}<p x+q y+\epsilon$ i.e. $l-\epsilon<l_{1}<l+\epsilon$. Thus $l_{1} \in I$. In
the similar way we can show that $l_{1} \in I$ when $l<0$. If $l=0$ then for any $y \in A, y+(-1) y=l$. Given $\epsilon>0$, the interval $(y-\epsilon, y+\epsilon)$ intersects $A$ in some point say $y_{1}(\neq y)$. Let $l_{1}=y_{1}+(-1) y$. Then clearly $l_{1} \in I$ and $l \neq l_{1}$. Hence $L_{p, q}(A)=\left(L_{p, q}(A)\right)^{\prime}$ and consequently $L_{p, q}(A)$ is perfect.

Theorem 2.10. If $A(\subset \mathbb{R})$ is of second category, for any $p, q \in[-1,1] \cap \mathbb{Q}^{*}$, both $L_{p, q}(A)$ and $L_{p, q}^{*}(A)$ are of second category.

Proof. If possible let $L_{p, q}^{*}(A)\left(\subset L_{p, q}(A)\right)$, where $p, q \in[-1,1] \cap \mathbb{Q}^{*}$, be of first category. Then the set $\frac{L_{p, q}^{*}(A)-q a}{p}=\left\{\frac{p x+q y-q a}{p}: x, y \in A, x \neq y\right\}$, (where $a \in A$ is fixed) is of first category. Let $x \in A$ and $x \neq a$. Then $p x+q a \in L_{p, q}^{*}(A)$ implies that $x \in \frac{L_{p, q}^{*}(A)-q a}{p}$. Thus $x \in A \backslash\{a\}$ implies that $x \in \frac{L_{p, q}^{*}(A)-q a}{p}$.i.e. $A \backslash\{a\} \subset \frac{L_{p, q}^{*}(A)-q a}{p}$. i.e. $A=(A \backslash\{a\}) \cap\{a\}$ is of first category which is a contradiction. So, $L_{p, q}^{*}(A)$ is of second category and hence the result.

From the results of H. I. Miller[7] we get the following two results.
Theorem 2.11. If $A$ and $B$ are linear sets of positive measure, then $L_{p, q}(A, B)$ for any $p, q \in[-1,1] \cap \mathbb{Q}^{*}$, contains an interval.

Theorem 2.12. If $A$ and $B$ are linear sets of second category and each possesses the property of Baire, then $L_{p, q}(A)$ for any $p, q \in[-1,1] \cap \mathbb{Q}^{*}$, contains an interval.

Theorem 2.13. Let $A$ and $B$ be two liner sets of positive Lebesgue measure and $F$ be a countable dense subset of $\mathbb{R}$. Then for each $d \in F, E=\{x \in$ $\left.A: \exists y \in B, p x+q y=d ; p, q \in[-1,1] \cap \mathbb{Q}^{*}\right\}$ has positive measure.

Proof. Clearly $E$ is Lebesgue measurable. If possible let us assume Lebesgue measure $\mu(E)=0$ and let $H=A \backslash E$, then $\mu(H)=\mu(A)>0$. Then $L_{p, q}(H, B)$, where $p, q \in[-1,1] \cap \mathbb{Q}^{*}$, contains an interval. Hence $F \cap$ $L_{p, q}(H, B) \neq \emptyset$, where $p, q \in[-1,1] \cap \mathbb{Q}^{*}$. Then for $x \in H \subset A$ and $y \in B$, there exists $c \in F$ such that $c=p x+q y$. This shows that $x \in E$, a contradiction. Thus $E$ is of positive measure.

Theorem 2.14. Let $A$ and $B$ be two liner sets of second category each having the property of Baire and $F$ be a countable dense subset of $\mathbb{R}$. Then for any $d \in F, G=\left\{x \in A: \exists y \in B, p x+q y=d ; p, q \in[-1,1] \cap \mathbb{Q}^{*}\right\}$, is of second category.

Proof. If possible let $G=\left\{x \in A: \exists y \in B, p x+q y=d ; p, q \in[-1,1] \cap \mathbb{Q}^{*}\right\}$ for any $d \in F$, be first category. Clearly $A \backslash \cup_{d \in F} G=H$ (say) is a residual set. Hence $F \cap L_{p, q}(H, B) \neq \emptyset$. Then for some $c \in F$, there exist $x \in H \subset A$ and $y \in B$ such that $c=p x+q y ; p, q \in[-1,1] \cap \mathbb{Q}^{*}$ and consequently this $x \in G$ which is a contradiction. Hence $G$ is a second category set.

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