New Lindley-Rayleigh Distribution with Statistical properties and Applications

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Abstract *- In this study, we have launched a new two-parameter probability model called the New Lindley-Rayleigh distribution. The proposed model accommodates unimodal and bathtub, and a broad variety of monotone failure rates. Some statistical and mathematical properties of this distribution are discussed. Four widely used estimation methods are employed to estimate the model parameters namely maximum likelihood estimators (MLE), least-square (LSE) and Cramer-Von-Mises (CVM) methods. By using the maximum likelihood estimate we have constructed the asymptotic confidence interval for the model parameters. The potentiality of the proposed distribution is revealed by using a real dataset, where the proposed distribution provided better fit in comparison with some other lifetime distributions. The importance of the proposed distribution is illustrated by using a real dataset, and found that it provides a better fitting in comparison with other lifetime distributions.*

Keywords - *Lindley G-Family, MLE, LSE, CVE.*

I. INTRODUCTION

The reference [1] has developed a new distribution called the Rayleigh distribution as a special case of the Weibull distribution. The CDF of Rayleigh distribution is

$$F(x) = 1 - e^{-\alpha x^2};\ \alpha > 0, x > 0 \tag{1.1}$$

probability density function (PDF) is

$$f(x) = 2\alpha x e^{-\alpha x^2};\ \alpha > 0, x > 0 \tag{1.2}$$

where α is a scale parameter of the Rayleigh distribution.

The Rayleigh distribution has been widely used in reliability analysis and in applications of several different fields which provide flexibility for modeling real data. The Rayleigh distribution has been used in different formats such as it is used in an application for communication engineering by [2]. The generalized Rayleigh distribution has introduced by [3]. The reference [4] had made a study on Rayleigh distribution and explored that it is applicable for clinical data. The estimation of the parameter of the Rayleigh distribution was performed by [5]. The reference [6] presented the Kumaraswamy generalized Rayleigh distribution for analyzing lifetime data. Marshall–Olkin extended generalized Rayleigh distribution was introduced by [7]. The reference [8] had introduced the Slashed Generalized Rayleigh Distribution which was created as the quotient of two independent random variables, one being a generalized Rayleigh distribution in the numerator and power of the uniform distribution in the denominator.

The reference [9] had introduced the New Lindley-Rayleigh distribution with application to lifetime data. The reference [10] has developed the modified slashed-Rayleigh distribution. They developed it as the quotient of two independent random variables, one being a Rayleigh distribution in the numerator and power of the exponential distribution in the denominator. The reference [11] has introduced a new form of generalized Rayleigh distribution called the Alpha Power generalized Rayleigh (APGR) distribution by following the idea of an extension of the distribution families with the Alpha Power transformation.

Researchers in the last few decades have developed various extensions and a modified form of the Lindley distribution which was developed by [12] in the context of Bayesian statistics, as a counterexample to fiducial statistics. An extensive study on the Lindley distribution was done by [13].

A random variable Y follows Lindley distribution with parameter θ and its probability density function (PDF) is given by

$$f\left(y\right)=\frac{\theta^2}{\theta+1}\left(1+y\right)e^{-\theta y};\ \mathrm{y}>0,\theta>0$$

And its cumulative density function (CDF) is

$$F\left(y\right)=1-\frac{1+\theta+\theta y}{1+\theta}e^{-\theta y};\ \mathrm{y}>0,\theta>0$$

In this article, we put forward the New New Lindley Rayleigh (NL-R) distribution to enhance the capability of the Lindley distribution using the Lindley-G family by inserting only one extra parameter. It is a member of the Lindley-G family introduced by [14]. Here, we have taken the Lindley distribution as a generator and the Rayleigh as a baseline distribution. The motivation of this study is to obtain a more flexible model by adding just one extra parameter to the Rayleigh distribution to achieve a better fit to the real data. We study the properties of the NL-R distribution and explore its applicability.

The contents of this article are organized as follows. The new New Lindley Rayleigh distribution is introduced and various distributional properties are discussed in Section 2. Four widely used estimation methods are employed to estimate the model parameters namely maximum likelihood estimators (MLE), least-square (LSE) and Cramer-Von-Mises (CVM) methods, further, the maximum likelihood estimators are used to construct the asymptotic confidence intervals using the observed information matrix is discussed in Section 3. In Section 4 a real data sets have been taken to investigate the applications and suitability of the proposed distribution. In this section, we present the ML estimators of the parameters and approximate confidence intervals also AIC, BIC, AICC, HQIC are calculated to assess the validity of the NL-R model. Finally, Section 5 ends up with some general concluding remarks.

II. THE NEW LINDLEY RAYLEIGH (NL-R) DISTRIBUTION

The proposed distribution is developed by using Lindley-G family defined by [14] as,

Let X be a random variable that follows the baseline distribution $G\left(x,\xi\right)$ if its cumulative density function (CDF) is given by,

$$F\left(x\right)=\left[G\left(x;\xi\right)\right]^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln G\left(x;\xi\right)\right\},\qquad x,\theta>0 \tag{2.1}$$

and its probability density function (PDF) is,

$$f\left(x\right)=\left(\frac{\theta^2}{1+\theta}\right)\left[G\left(x;\xi\right)\right]^{\theta-1}\left\{1-\ln G\left(x;\xi\right)\right\}g\left(x;\xi\right),\qquad x,\theta>0 \tag{2.2}$$

where $G\left(x,\xi\right)$ and $g\left(x,\xi\right)$ are the CDF and PDF of baseline distribution and ξ is parameter space of baseline distribution. Inserting (1.1) and (1.2) respectively in (2.1) and (2.2) we obtained the CDF and PDF of New Lindley Rayleigh (NL-R) distribution as

$$F\left(x\right)=\left[1-e^{-\alpha x^2}\right]^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left(1-e^{-\alpha x^2}\right)\right\},\qquad x>0,\ \alpha,\theta>0 \tag{2.3}$$

and $$f\left(x\right)=2\alpha\left(\frac{\theta^2}{1+\theta}\right)xe^{-\alpha x^2}\left[1-e^{-\alpha x^2}\right]^{\theta-1}\left\{1-\ln\left(1-e^{-\alpha x^2}\right)\right\},\qquad x>0,\ \alpha,\theta>0 \tag{2.4}$$

where α is the scale parameter and θ is the shape parameter of the NL-R distribution.

Figure 1 demonstrates the graph for PDF and hazard function for NL-R distribution for different values of parameters. From Fig. 1 (left panel), the density function of the NL-R distribution can bear different shapes according to the values of the parameters. Fig. 1 (right panel) demonstrates the increasing, decreasing, decreasing-increasing and constant graph of the hazard function. This proves that NL-R distribution is more flexible than Rayleigh distribution.

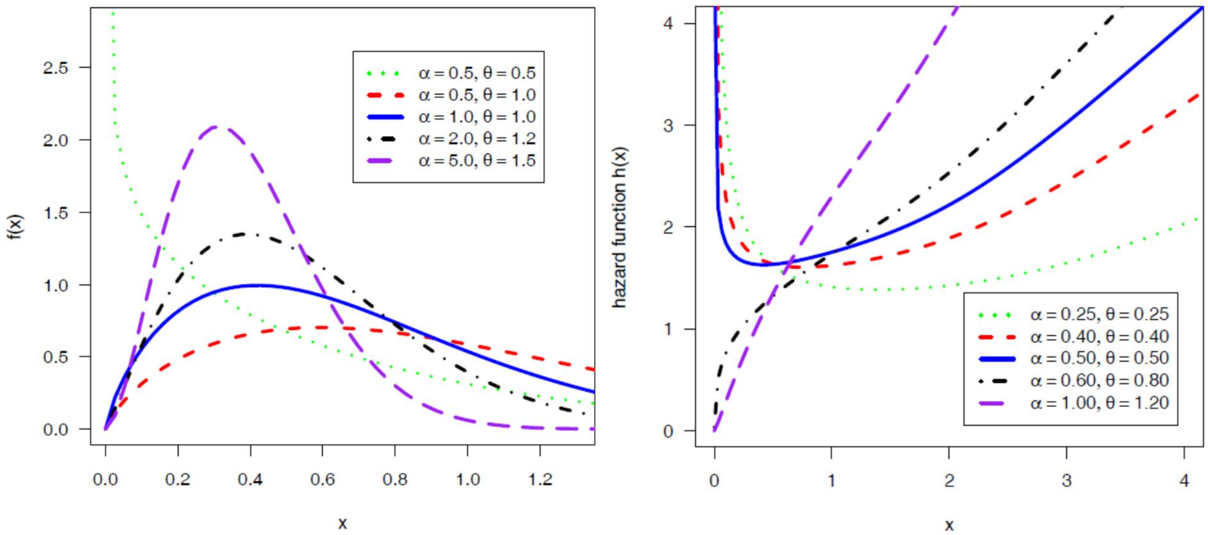


Fig. 1. Graph of PDF (left panel) and hazard function (right panel) for different values of α and θ.

Survival function:

The survival function $R(t)$, which is the probability of an item not failing up to time *t*, is defined by $R(t)=1-F(t)$. The survival /reliability function of a New Lindley-Rayleigh distribution is given by

$$R(t)=1-\left[1-e^{-\alpha t^2}\right]^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left(1-e^{-\alpha t^2}\right)\right\},\qquad t>0,\ \alpha,\theta>0 \tag{2.5}$$

The hazard rate function (HRF)

Let t be survival time of a component or item and the probability that it will not survive for an additional time Δt then, hazard rate function is,

$$h(t)=\frac{f(t)}{R(t)}=\frac{f(t)}{1-F(t)};\ 0<t<\infty$$

where R(t) is a reliability function.

Hence let, $X \sim \text{L-R}(\alpha,\theta)$ then its hazard rate function is

$$h(x)=\frac{2\alpha\left(\frac{\theta^2}{1+\theta}\right)xe^{-\alpha x^2}\left[1-e^{-\alpha x^2}\right]^{\theta-1}\left\{1-\ln\left(1-e^{-\alpha x^2}\right)\right\}}{1-\left[1-e^{-\alpha x^2}\right]^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left(1-e^{-\alpha x^2}\right)\right\}},\quad x>0,\ \alpha,\theta>0 \tag{2.6}$$

Quantile function of NL-R distribution is,

The value of the p^th quantile can be obtained by solving the following equation,

$$Q(p) = F^{-1}(p)$$

And we get quantile function by inverting (2.3) as

$$p - \left[1 - e^{-\alpha x^2}\right]^{\theta} \left\{1 - \left(\frac{\theta}{1+\theta}\right) \ln\left(1 - e^{-\alpha x^2}\right)\right\} = 0, \quad 0 < p < 1 \tag{2.7}$$

For the generation of the random numbers of the NL-R distribution, we suppose simulating values of random variable X with the CDF (2.3). Let U denote a uniform random variable in (0,1), then the simulated values of X can be obtained by

$$u - \left[1 - e^{-\alpha x^2}\right]^{\theta} \left\{1 - \left(\frac{\theta}{1+\theta}\right) \ln\left(1 - e^{-\alpha x^2}\right)\right\} = 0, \quad 0 < u < 1 \tag{2.8}$$

Skewness and Kurtosis:

The Bowley's skewness based on quartiles is,

$S_{kb} = \dfrac{Q_3 + Q_1 - 2Median}{Q_3 - Q_1}$, where and Q_3 *and* Q_1 are the upper quartile and lower quartile respectively.

Coefficient of kurtosis based on octiles given by [15] is

$$K_u(Moors) = \frac{Q(0.875) - Q(0.625) + Q(0.375) - Q(0.125)}{Q(3/4) - Q(1/4)}$$

III. ESTIMATION OF PARAMETERS

A. Maximum Likelihood Estimates

The parameters of the NL-R distribution can be obtained by maximum likelihood (MLE) as follows. Let $x_1, \ldots, x_n$ be a random sample of size *n* from a two-parameter NL-R(α, θ) with PDF (2.4). The likelihood function of the NL-R distribution is given by,

$$L(\alpha, \theta \mid \underline{x}) = 2\alpha\left(\frac{\theta^2}{1+\theta}\right) \prod_{j=1}^{n} x e^{-\alpha x_j^{2}} \left[1 - e^{-\alpha x_j^{2}}\right]^{\theta - 1} \left\{1 - \ln\left(1 - e^{-\alpha x_j^{2}}\right)\right\}, \qquad x > 0, \ \alpha, \theta > 0$$

Hence log-likelihood function is obtained as,

$$l(\alpha, \theta \mid \underline{x}) = 2n \ln\theta - n\ln(1+\theta) + n\ln(2\alpha) + \sum_{j=1}^{n} \ln x_j - \alpha \sum_{j=1}^{n} x_j^{2} + (\theta - 1)\sum_{j=1}^{n} \ln(1 - e^{-\alpha x_j^{2}}) + \sum_{j=1}^{n} \ln\left\{1 - \ln\left(1 - e^{-\alpha x_j^{2}}\right)\right\} \tag{3.1.1}$$

Differentiating (3.1.1) with respect to α and θ we get,

$$\frac{\partial l\left(\alpha,\theta\mid \underline{x}\right)}{\partial\alpha}=\frac{n}{\alpha}-\sum_{j=1}^{n}x_j^2-(\theta-1)\sum_{j=1}^{n}\frac{x_j^2}{1-e^{\alpha x_j^2}}+\sum_{j=1}^{n}\frac{x_j^2}{\left(1-e^{\alpha x_j^2}\right)\left\{1-\ln\left(1-e^{-\alpha x_j^2}\right)\right\}}=0$$

$$\frac{\partial l\left(\alpha,\theta\mid \underline{x}\right)}{\partial\theta}=\frac{2n}{\theta}-\frac{n}{1+\theta}+\sum_{j=1}^{n}\ln\left(1-e^{-\alpha x_j^2}\right)=0$$

By solving these two non-linear equations we get the estimated values of the parameters of the New Lindley Rayleigh distribution. Since is difficult to solve them manually but one can use computer programming to solve them numerically.

Let us denote the parameter space by $\underline{\Omega}=(\alpha,\theta)$ and the corresponding MLE of $\underline{\Omega}$ as $\hat{\underline{\Omega}}=(\hat{\alpha},\hat{\theta})$, then the asymptotic normality results in, $\left(\hat{\underline{\Omega}}-\underline{\Omega}\right)\to N_2\left[0,\left(I\left(\underline{\Omega}\right)\right)^{-1}\right]$ where $I\left(\underline{\Omega}\right)$ is the Fisher's information matrix defined as

$$I\left(\underline{\Omega}\right)=-\begin{bmatrix}E\left(\frac{\partial^2 l}{\partial\alpha^2}\right) & E\left(\frac{\partial^2 l}{\partial\theta\partial\alpha}\right)\\ E\left(\frac{\partial^2 l}{\partial\theta\partial\alpha}\right) & E\left(\frac{\partial^2 l}{\partial\theta^2}\right)\end{bmatrix}$$

For this further diff. (3.1.1) w.r. to parameters α and θ we get,

$$\frac{\partial^2 l\left(\alpha,\theta\mid \underline{x}\right)}{\partial\alpha^2}=-\frac{2n}{\alpha^2}+(\theta-1)\sum_{j=1}^{n}x_j^4\frac{e^{\alpha x_j^2}}{\left(1-e^{\alpha x_j^2}\right)^2}-\sum_{j=1}^{n}\frac{x_j^4\left\{e^{\alpha x_j^2}\ln\left(1-e^{-\alpha x_j^2}\right)-e^{-\alpha x_j^2}+1\right\}}{\left(1-e^{\alpha x_j^2}\right)\left\{1-\ln\left(1-e^{-\alpha x_j^2}\right)\right\}^2}$$

$$\frac{\partial^2 l\left(\alpha,\theta\mid \underline{x}\right)}{\partial\theta^2}=-\frac{2n}{\theta^2}-\frac{n}{\left(1+\theta\right)^2}$$

$$\frac{\partial^2 l\left(\alpha,\theta\mid \underline{x}\right)}{\partial\alpha\partial\theta}=-\sum_{j=1}^{n}\frac{x_j^{\,2}}{1-e^{\alpha x_j^2}}$$

In practice, it is useless that the MLE has asymptotic variance $\left(I\left(\underline{\Omega}\right)\right)^{-1}$ because we don't know $\underline{\Omega}$. Hence we approximate the asymptotic variance by substituting the estimated value of the parameters.

The common procedure is to use the observed Fisher information matrix $O(\hat{\underline{\Omega}})$ as an estimate of the information matrix $I\left(\underline{\Omega}\right)$ given by

$$O(\hat{\underline{\Omega}})=-\begin{pmatrix}\frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\theta}\\ \frac{\partial^2 l}{\partial\alpha\partial\theta} & \frac{\partial^2 l}{\partial\theta^2}\end{pmatrix}_{|(\hat{\alpha},\hat{\theta})}=-H\left(\underline{\Omega}\right)_{|\left(\underline{\delta}=\hat{\underline{\delta}}\right)}$$

Where H is the Hessian matrix.

The Newton-Raphson algorithm to maximize the likelihood produces the observed information matrix. Therefore, the variance-covariance matrix is given by,

$$\left[-H(\underline{\Omega})\Big|_{(\underline{\delta}=\hat{\underline{\delta}})}\right]^{-1}=\begin{pmatrix}\operatorname{var}(\hat{\alpha}) & \operatorname{cov}(\hat{\alpha},\hat{\theta})\\ \operatorname{cov}(\hat{\alpha},\hat{\theta}) & \operatorname{var}(\hat{\theta})\end{pmatrix}$$

Hence from the asymptotic normality of MLEs, approximate 100(1-α) % confidence intervals for α and θ can be constructed as,

$\hat{\alpha}\pm z_{\gamma/2}SE(\hat{\alpha})$ and $\hat{\theta}\pm z_{\gamma/2}SE(\hat{\theta})$ where $z_{\gamma/2}$ is the upper percentile of standard normal variate.

B. Method of Least-Square Estimation (LSE)

The reference [16] has introduced the ordinary least square estimators and weighted least square estimators to estimate the parameters of Beta distributions. In this study, we apply the same technique for the NL-R distribution. The least-square estimators of the unknown parameters α and θ of NL-R distribution can be obtained by minimizing

$$Z(X;\alpha,\theta)=\sum_{i=1}^{n}\left[F(X_i)-\frac{i}{n+1}\right]^2 \tag{3.2.1}$$

with respect to unknown parameters α and θ.

Suppose $F(X_{(i)})$ denotes the cumulative distribution function of the ordered random variables $X_{(1)}<X_{(2)}<\ldots<X_{(n)}$, where $\{X_1,X_2,\ldots,X_n\}$ is a random sample of size n from a CDF F(.). Therefore, the least square estimators of α and θ say $\hat{\alpha}$ and $\hat{\theta}$ respectively, can be obtained by minimizing

$$Z(X;\alpha,\theta)=\sum_{i=1}^{n}\left[\left[1-e^{-\alpha x_i^2}\right]^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left(1-e^{-\alpha x_i^2}\right)\right\}-\frac{i}{n+1}\right]^2 \tag{3.2.2}$$

with respect to α and θ. To obtain the least square estimators, we have to solve the following two nonlinear equations equating to zero,

$$\frac{\partial Z}{\partial\alpha}=\frac{2}{1+\theta}\sum_{i=1}^{n}x_i^2e^{-\alpha x_i^2}\left[1-\theta\ln\left(1-e^{-\alpha x_i^2}\right)\right]\left[\left[1-e^{-\alpha x_i^2}\right]^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left(1-e^{-\alpha x_i^2}\right)\right\}-\frac{i}{n+1}\right]$$

$$\frac{\partial Z}{\partial\theta}=\frac{2}{(1+\theta)^2}\sum_{i=1}^{n}e^{-\alpha x_i^2}\left(1-e^{-\alpha x_i^2}\right)\ln\left(1-e^{-\alpha x_i^2}\right)\left[\left[1-e^{-\alpha x_i^2}\right]^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left(1-e^{-\alpha x_i^2}\right)\right\}-\frac{i}{n+1}\right]$$

C. Method of Cramer-Von-Mises (CVM)

One of the important estimation methods is Cramér-von-Mises type minimum distance estimators, [17] because it provides empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. The CVM estimators of α and θ are obtained by minimizing the function

$$C(\alpha,\theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_{i:n}\mid\alpha,\theta)-\frac{2i-1}{2n}\right]^2 \tag{3.3.1}$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[\left[1-e^{-\alpha x_i^2}\right]^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left(1-e^{-\alpha x_i^2}\right)\right\}-\frac{2i-1}{2n}\right]^2$$

To obtain the CVM estimators, we have to solve the following two nonlinear equations equating to zero,

$$\frac{\partial C}{\partial \alpha}=\frac{2}{1+\theta}\sum_{i=1}^{n}x_i^2 e^{-\alpha x_i^2}\left[1-\theta\ln B(x_i)\right]\left[\left[B(x_i)\right]^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln\left(B(x_i)\right)\right\}-\frac{2i-1}{2n}\right]$$

$$\frac{\partial C}{\partial \theta}=\frac{2}{(1+\theta)^2}\sum_{i=1}^{n}e^{-\alpha x_i^2}B(x_i)\ln B(x_i)\left[\left[B(x_i)\right]^{\theta}\left\{1-\left(\frac{\theta}{1+\theta}\right)\ln B(x_i)\right\}-\frac{2i-1}{2n}\right]$$

here $B(x_i)=1-e^{-\alpha x_i^2}$.

IV. ILLUSTRATION WITH REAL DATA ANALYSIS

For the data analysis, we are using a real data set that was used by, [18]. The data represents thirty successive values of March precipitation (inches) for Minneapolis/St Paul.

0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05

We have computed the maximum likelihood estimates By using the log-likelihood function (3.1.1), directly by using R software [19]. By using the maximum likelihood estimation method for the above data set, we have obtained $\hat{\alpha}$ = 0.21700 and $\hat{\theta}$ = 1.21069 and its corresponding Log-Likelihood value is -38.41925. In Table 1 we have presented the MLE's with their standard errors (SE) and 95% confidence interval for α and θ.

Table 1

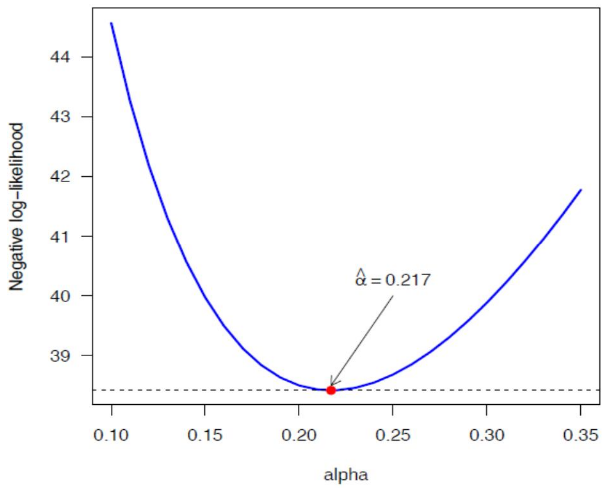
MLE, SE And 95% Confidence Interval

Parameter	MLE	SE	95% ACI
Alpha	0.2170	0.06176	(0.09595, 0.33805)
Theta	1.2107	0.24267	(0.73506, 1.68632)

Hence the Hessian variance-covariance matrix is obtained as,

$$\left[-H\left(\underline{\Omega}\right)\Big|_{\left(\underline{\delta}=\hat{\underline{\delta}}\right)}\right]^{-1}=\begin{pmatrix}\operatorname{var}(\hat{\alpha}) & \operatorname{cov}(\hat{\alpha},\hat{\theta})\\ \operatorname{cov}(\hat{\alpha},\hat{\theta}) & \operatorname{var}(\hat{\theta})\end{pmatrix}=\begin{pmatrix}0.00381 & 0.01072\\ 0.01072 & 0.05889\end{pmatrix}$$

The Profile log-likelihood functions of parameters α and θ are displayed in Fig. 2. It can be explored that the estimated parameters using the MLE method are unique.



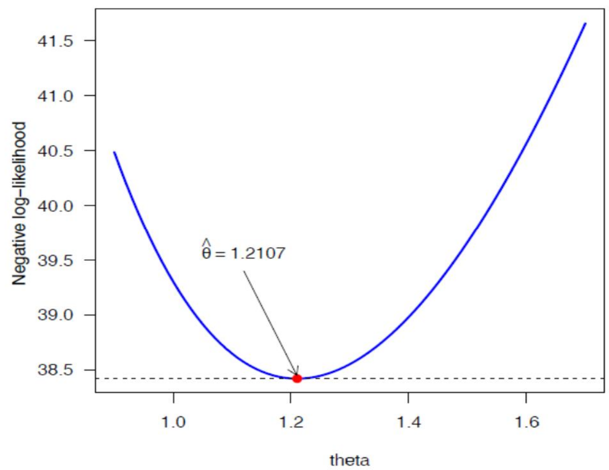


Fig. 2. Profile log-likelihood function of α and θ.

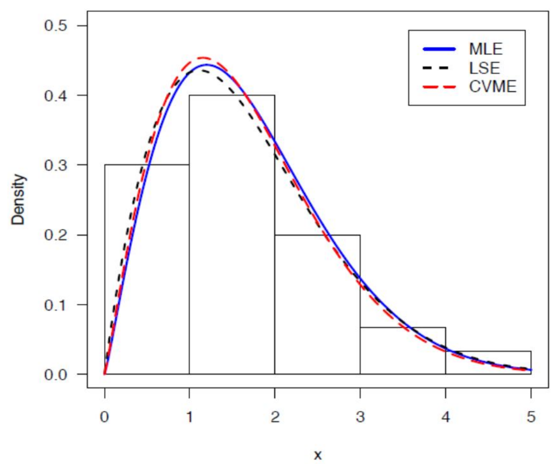


Fig. 3. The plot of fitted density functions of estimation methods MLE, LSE and CVME.

In Table 2 we have presented the estimated parameters, log-likelihood, AIC, BIC and AICC for MLE, LSE and CVM methods.

Table 2
Estimated Parameters, Log-Likelihood, AIC, BIC, AICC and HQIC

Method of Estimation	$\hat{\alpha}$	$\hat{\theta}$	-LL	AIC	BIC	AICC	HQIC
MLE	0.2170	1.2107	38.4192	80.8385	83.6409	81.25229	81.7350
LSE	0.2035	1.1146	38.5087	81.01732	83.81971	81.43111	81.9138
CVE	0.2261	1.1943	38.4596	80.91917	83.72156	81.33296	81.8157

Table 3
The KS, AD and CVM Statistics With P-Value

Method of Estimation	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
MLE	0.0662(0.9994)	0.0206(0.9969)	0.1514(0.9986)
LSE	0.0652(0.9995)	0.0161(0.9994)	0.1445(0.9990)
CVE	0.0586(0.9999)	0.01388(0.9998)	0.1339(0.9995)

In Fig. 4 we have displayed the contour plot of the estimated parameters by MLE and the fitted CDF with empirical distribution function [20].

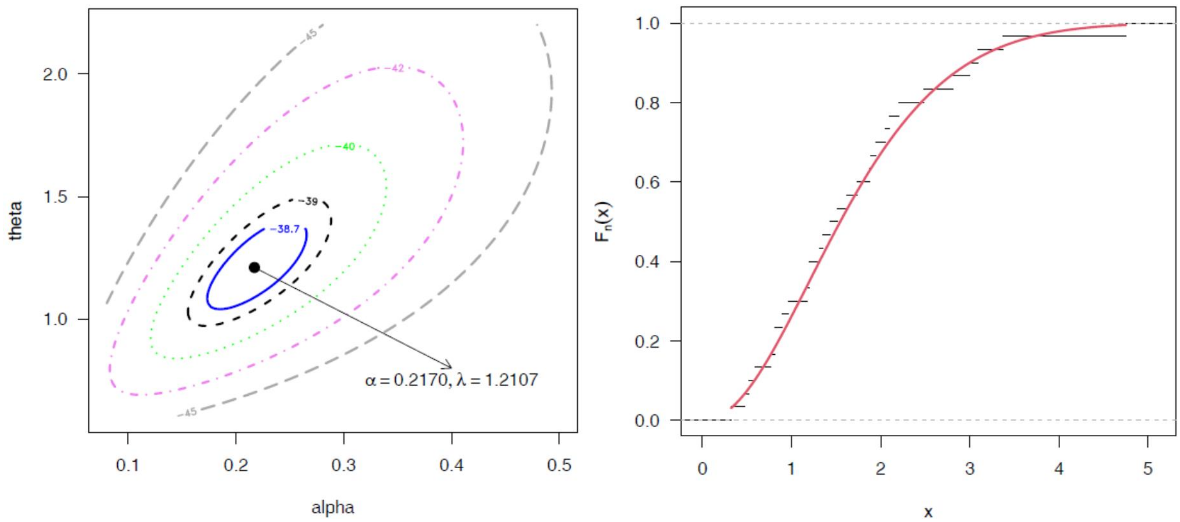


Fig. 4. Contour plot (left panel) and the fitted CDF with empirical distribution function (right panel)

One way to assess how well a particular theoretical model describes a data distribution is to plot the data quantiles against theoretical quantiles. In Fig. 5 we have plotted the P-P and Q-Q plot and verified that the new proposed model fits the data very well.

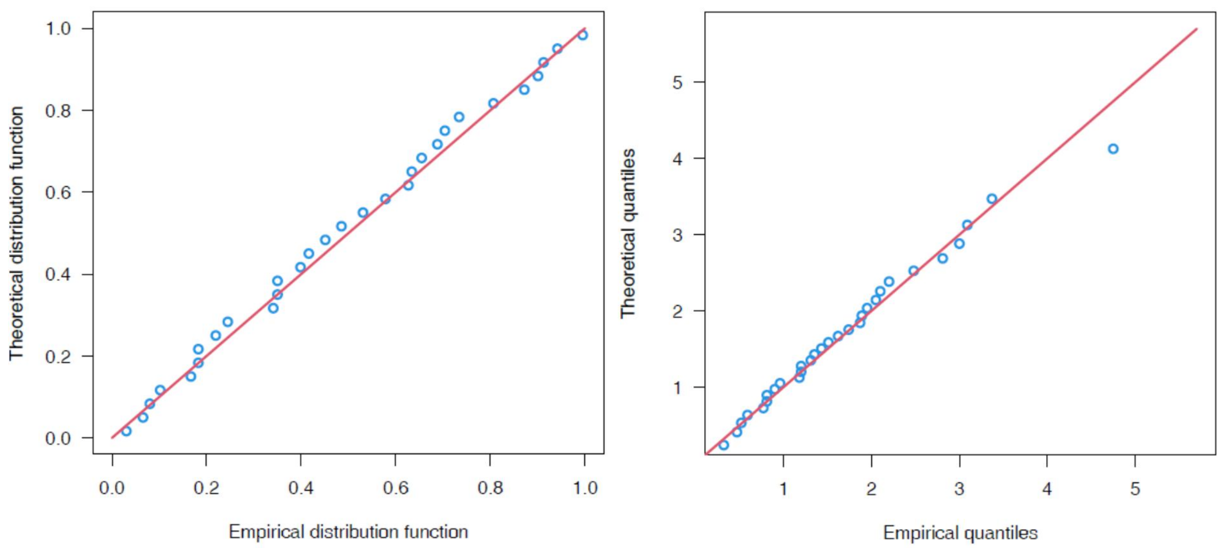


Fig. 5. The graph of the P-P plot (left panel) and Q-Q plot (right panel) of the NL-R distribution

For the illustration purpose we have fitted the following probability distributions models

A. Generalized Rayleigh distribution

The probability density function of Generalized Rayleigh (GR) distribution [21] is

$$f_{GR}\,(x;\,\alpha,\,\lambda) = 2\,\alpha\,\lambda^2\,x\,e^{-(\lambda x)^2}\left\{1-e^{-(\lambda x)^2}\right\}^{\alpha-1} \quad ;(\alpha,\lambda)>0,\, x>0$$

Here α and λ are the shape and scale parameters respectively.

B. Exponential power (EP) distribution

The probability density function Exponential power (EP) distribution [22] is

$$f_{EP}(x) = \alpha\,\lambda^{\alpha}\;x^{\alpha-1}\,e^{(\lambda\,x)^{\alpha}}\,\exp\left\{1-e^{(\lambda\,x)^{\alpha}}\right\}\quad;(\alpha,\lambda)>0,\quad x\ge 0.$$

where α and λ are the shape and scale parameters, respectively.

C. Gompertz distribution (GZ)

The probability density function of Gompertz distribution [23] with parameters α and θ is

$$f_{GZ}\left(x\right)=\theta\;\;e^{\alpha\,x}\;exp\left\{\frac{\theta}{\alpha}\left(1-e^{\alpha\,x}\right)\right\}\quad;x\ge 0,\theta>0,-\infty<\alpha<\infty.$$

D. Exponential Extension (EE) distribution

The density of exponential extension (EE) distribution [24] with parameters α and λ is

$$f_{EE}(x)=\alpha\,\lambda\,(1+\lambda x)^{\alpha-1}\,\exp\left\{1-(1+\lambda x)^{\alpha}\right\}\quad;x\ge 0,\alpha>0,\lambda>0.$$

For the test of goodness of fit and adequacy of the proposed model, Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) are calculated and presented in Table 3.

Table 3
Log-likelihood (LL), AIC, BIC, CAIC and HQIC

Model	-LL	AIC	BIC	CAIC	HQIC
LR	38.4193	80.8385	83.6409	81.2830	81.7350
GR	38.8284	81.6568	84.4592	82.1012	82.5533
EP	40.4769	84.9537	87.7561	85.3675	85.8502
GZ	41.0762	86.1523	88.9547	86.5968	87.0488
EE	41.4221	86.8442	89.6466	87.2580	87.7407

We have displayed the histogram and the fitted probability density functions and the empirical cumulative distribution function with estimated distribution function in Figure 5. For the given data set we have found that the proposed distribution provides a better fit and more reliable results than selected ones.

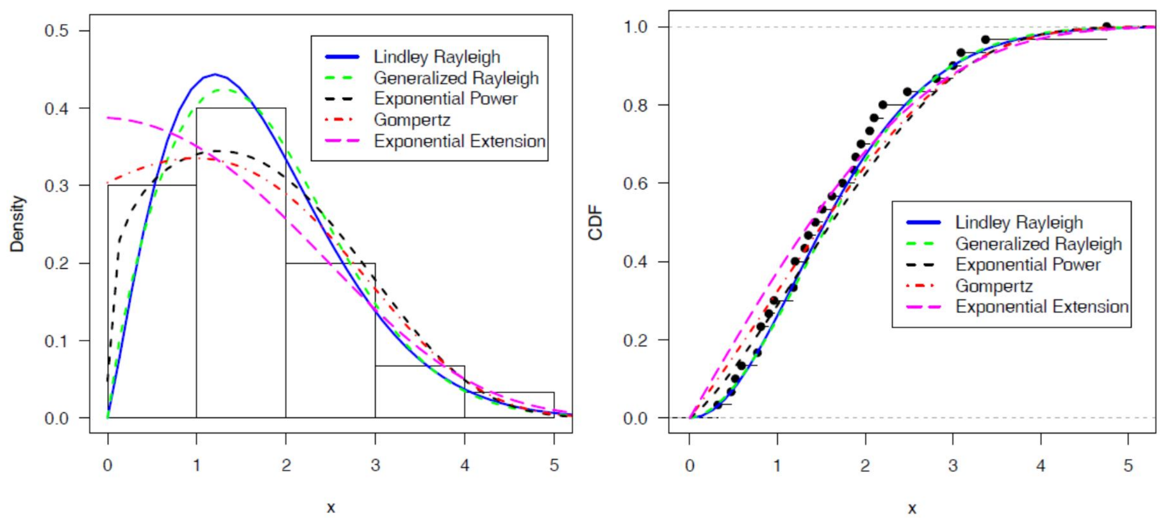


Fig. 6. The Histogram and the PDF of fitted distributions (left panel) and Empirical CDF with estimated CDF (right panel).

We have reported the test statistics and their corresponding p-value of the NL-R distribution and competing models in Table 4. The result shows that the NL-R distribution has the minimum value of the test statistic and higher *p*-value hence we conclude that the NL-R distribution gets quite better fit and more consistent and reliable results from others taken for comparison.

Table 4
The Goodness-of-Fit Statistics and Their Corresponding p-Value

Model	*KS(p-value)*	A^2*(p-value)*	*W(p-value)*
LR	0.0662(0.9994)	0.0206(0.9969)	0.1514(0.9986)
GR	0.0770(0.9942)	0.0341(0.9631)	0.2267(0.9813)
EP	0.1164(0.8108)	0.0738(0.7321)	0.5165(0.7286)
GZ	0.1149(0.8230)	0.0836(0.6749)	0.6440(0.6060)
EE	0.1584(0.4390)	0.1571(0.3702)	1.0251(0.3437)

V. CONCLUSION

We have proposed the new Lindley Rayleigh (NL-R) distribution generated by a new class of Lindley generated distributions. We have derived important properties of the NL-R distribution like hazard rate function, quantile function, and expression for random number generation. We have illustrated the application of NL-R distribution to real data sets used by researchers earlier. We have employed four well-known estimation methods viz. maximum likelihood estimation (MLE), ordinary least square method (LSE), and Cramér-Von-Mises (CVM). By observing the results of these all methods of estimation we conclude that the new Lindley Rayleigh (NL-R) distribution performs better. The importance of the proposed distribution is illustrated by using a real dataset, and found that it provides a better fitting in comparison with other lifetime distributions.

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