Roll of $\gamma_{\delta\widehat{\Omega}}$-open sets in Digital Topology

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Abstract — *This work is based on quasi-operation in a topological space. Quasi-operation has been extended to the class of $\widehat{\Omega}$-open sets. The new class of $\gamma_{\delta\widehat{\Omega}}$-open sets has been introduced by linking the set of all δ-open sets with quasi-operation on $\widehat{\Omega}$-open sets. Also two kinds of closures such as $\gamma(\delta\widehat{\Omega})$-closure and $\gamma_{\delta\widehat{\Omega}}$-closure have been studied and derived their elementary properties. Moreover, γ_δ-regular and γ_δ-open quasi-operation have been investigated. Also $\gamma(\delta\widehat{\Omega})$-regular space has been defined and a few basic results on it have been derived.*

Keywords — *$\gamma_{\delta\widehat{\Omega}}$-open set, $\gamma_{\delta\widehat{\Omega}}$-closure, $\gamma_{\delta\widehat{\Omega}}$-kernel, $\gamma(\delta\widehat{\Omega})$-interior, $\gamma(\delta\widehat{\Omega})$-closure, γ_δ-regular quasi-operation, γ_δ-open quasi-operation and $\gamma(\delta\widehat{\Omega})$-regular space..*

I. INTRODUCTION

In 1979, Kasahara[1] introduced the concept of operation compact spaces. Following him, Ogata[4] studied the notion of operation on open sets and investigated some related topological properties of the associated family of all the operation-open sets with a given topology and a given operation in 1991. In 2012, the class of $\widehat{\Omega}$-Closed sets has been introduced by Lellis Thivagar et al. [2]. Recently, operation on the class of $\widehat{\Omega}$-open sets has been introduced and studied by us[3]. In this paper, an attempt has been made to introduce the concept of quasi-operation on the class of $\widehat{\Omega}$-open sets in a space X. Moreover, for a given quasi-operation γ on $\widehat{\Omega}O(X)$, the new class of sets known as $\gamma_{\delta\widehat{\Omega}}$-open sets, corresponding $\gamma(\delta\widehat{\Omega})$-closure and $\gamma(\delta\widehat{\Omega})$-interior have been defined and investigated by giving suitable examples in digital topology. Some basic properties with respect to $\gamma_{\delta\widehat{\Omega}}$-closure and $\gamma(\delta\widehat{\Omega})$-closure have been failed in quasi-operation. This gives birth to three notions such as γ_δ-regular quasi-operation, γ_δ-open quasi-operation and $\gamma(\delta\widehat{\Omega})$-regular space in which those properties hold.

II. PRELIMINARIES

In this section, some definitions and results that are used in this paper have been dealt. Always X or (X,τ) denotes a topological space on which no separation axioms assumed, unless otherwise stated. For any subset A of X, the closure (res.interior, kernel) of A is denoted by $cl(A)(res.int(A), \ker(A))$.

Definition 2.1 *([6]) A subset A of (X,τ) is said to be **δ-open set** in (X,τ) if for each point $x\epsilon A$, there exists an open set U in (X,τ) such that $x\epsilon U$ and $int(cl(U)) \subseteq A$. A subset E is said to be δ-closed in (X,τ) if X\E is δ-open in (X,τ).*

Definition 2.2 *([5]) A subset A of a space (X,τ) is called a **regular open set** if $A = int(cl(A))$.*

Definition 2.3*([2], Definition 3.1) Let (X,τ) be a topological space. A is said to be **$\widehat{\Omega}$-closed set** if $\delta cl(A) \subseteq U$ when $A \subseteq U$, where U is a semi-open subset of X. The complement of $\widehat{\Omega}$-closed set is an $\widehat{\Omega}$-open set. The family of all $\widehat{\Omega}$-closed sets in a space (X,τ) is denoted by $\tau_{\widehat{\Omega}}$. Also $\widehat{\Omega}O(X,\tau)$ or $\widehat{\Omega}O(X)$ (resp. $\widehat{\Omega}C(X,\tau)$ or $\widehat{\Omega}C(X)$) denotes the set of all $\widehat{\Omega}$-open sets (resp. $\widehat{\Omega}$-closed sets) on the space X.*

Definition 2.4*([3], Definition 3.1) A function $\gamma : \widehat{\Omega}O(X,\tau) \to P(X)$ is called an **operation on $\widehat{\Omega}O(X)$**, if $U \subseteq \gamma(U)$ for every set $U\epsilon\widehat{\Omega}O(X,\tau)$. For any operation γ, $\gamma(X) = X$, and $\gamma(\emptyset) = \emptyset$.*

Proposition 2.5 *([6]) Every regular open set is a δ-open set.*

Proposition 2.6 *([2], Theorem 3.2) Every δ-closed set is a $\widehat{\Omega}$-closed set in (X,τ) .*

Proposition 2.7 *([3], Theorem 5.3 vii.) Let A be any subset of a topological space (X,τ). Then, $\widehat{\Omega}cl(A) \subseteq \delta cl(A)$.*

Notation 2.8. *i)* $\delta O(X,x)$ *denotes the set of all* δ*-open sets containing x in a space* (X,τ)*.*
ii) $\widehat{\Omega}O(X,x)$ *denotes the set of all* $\widehat{\Omega}$*-open sets containing x in a space* (X,τ)*.*
iii)$\widehat{\Omega}O(\mathbb{Z})$ *or* $\widehat{\Omega}O(\mathbb{Z},k)$ *denotes the set of all* $\widehat{\Omega}$*-open sets in* $(\mathbb{Z},k)$*, digital topological space.*

III. QUASI -OPERATION AND $\gamma_{\delta\widehat{\Omega}}$–OPEN SETS

Definition 3.1. *Let* X *be a topological space. A function* $\gamma : \widehat{\Omega}O(X,\tau) \to P(X)$ *is said to be a* ***quasi-operation on*** $\widehat{\Omega}O(X)$ *if there exists a* $\widehat{\Omega}$*-open subset* U *of* X *such that* $\gamma(U) \neq \emptyset$*.*

Example 3.2. *Let* $\mathbb{Z}$ *be the set of all integers and* $(\mathbb{Z},k)$ *be a digital line(Khalimsky line) where* k *is a digital topology with subbase* $\{2m-1, 2m, 2m+1 / m \epsilon \mathbb{Z}\}$*. Define* $\gamma : \widehat{\Omega}O(\mathbb{Z},k) \to P(\mathbb{Z})$ *by*
$\gamma(U) = \delta cl(\delta int(U)) \cap \{\cap\, cl(\{x\}) / x \in U\}$ *for all* $U \in \widehat{\Omega}O(\mathbb{Z},k)$*. Here,*
$\gamma(\{2m+1\}) = \{2m, 2m+1, 2m+2\} \neq \emptyset$ *for a* $\{2m+1\} \in \widehat{\Omega}O(\mathbb{Z},k)$*. Then* γ *is a quasi-operation on* $\widehat{\Omega}O(\mathbb{Z})$*.*

Proposition 3.3. *Let* X *be a non-empty set. Every operation on* $\widehat{\Omega}O(X)$ *is a quasi-operation on* $\widehat{\Omega}O(X)$ *in a space* X*. But, the converse does not always hold.*
Proof. Let $\gamma : \widehat{\Omega}O(X) \to P(X)$ *be any quasi-operation on* $\widehat{\Omega}O(X)$*. As* $\gamma(X) = X \neq \emptyset$*,* γ *is a quasi-operation on* $\widehat{\Omega}O(X)$*.*

Example 3.4. *Define* $\gamma : \widehat{\Omega}O(\mathbb{Z},k) \to P(\mathbb{Z})$ *by* $\gamma(U) = \delta int(\delta cl(U)) \cap \{2n / n \in \mathbb{Z}\}$ *for all* $U \in \widehat{\Omega}O(\mathbb{Z},k)$*. Consider an* $\widehat{\Omega}$*-open subset* $U = \{2m+1\}$ *of* $\mathbb{Z}$*. Here,* $\gamma(U) = \{2m, 2m+2\} \neq \emptyset$ *and* $U \nsubseteq \gamma(U)$*. So,* γ *is a quasi-operation but not an operation on* $\widehat{\Omega}O(\mathbb{Z})$*.*

Definition 3.5. *Let* $\gamma : \widehat{\Omega}O(X) \to P(X)$ *be a quasi-operation on* $\widehat{\Omega}O(X)$*. A non-empty subset* U *of* X *is called a* $\boldsymbol{\gamma_{\delta\widehat{\Omega}}}$***-open set*** *if for every* $x \in U$*, there exists a* δ*-open set* V *containing* x *such that* $\gamma(V) \subseteq U$*. Assume that* $\emptyset$ *is always a* $\gamma_{\delta\widehat{\Omega}}$*-open subset of* X*.* $\gamma_{\delta\widehat{\Omega}}O(X)$ *denotes the set of all* $\gamma_{\delta\widehat{\Omega}}$*-open subsets of* X *and* $\gamma_{\delta\widehat{\Omega}}O(X,x)$ *denotes the set of all* $\gamma_{\delta\widehat{\Omega}}$*-open subset containing the point* x *of* X*. The complement of a* $\gamma_{\delta\widehat{\Omega}}$*-open set is a* $\gamma_{\delta\widehat{\Omega}}$*-closed set in* X*.* $\gamma_{\delta\widehat{\Omega}}C(X)$ *denotes the set of all* $\gamma_{\delta\widehat{\Omega}}$*-closed subsets of* X*. From the definition,* $\emptyset, X \in \gamma_{\delta\widehat{\Omega}}O(X)$*;* $\emptyset, X \in \gamma_{\delta\widehat{\Omega}}C(X)$*.*

Example 3.6. *Define* $\gamma : \widehat{\Omega}O(\mathbb{Z},k) \to P(\mathbb{Z})$ *by* $\gamma(U) = \delta int(\delta cl(U))$ *for every* $U \in \widehat{\Omega}O(\mathbb{Z},k)$*. Here, an* $\widehat{\Omega}$*-open subset* $A = \{2m+3\}$ *of* $\mathbb{Z}$ *is a* $\gamma_{\delta\widehat{\Omega}}$*-open subset of* $(\mathbb{Z},k)$*.*

Definition 3.7. *Let* γ *be a quasi-operation on* $\widehat{\Omega}O(X)$*. For any subset* A *of a space* X*,* $\boldsymbol{\gamma_{\delta\widehat{\Omega}}}$***-closure of*** $\boldsymbol{A}$ *is denoted by* $\gamma_{\delta\widehat{\Omega}}cl(A)$ *and defined by* $\gamma_{\delta\widehat{\Omega}}cl(A) = \cap\{F / A \subseteq F; X \backslash F \in \gamma_{\delta\widehat{\Omega}}O(X)\}$*.* $\gamma_{\delta\widehat{\Omega}}$*-closure is well defined as* X *is a* $\gamma_{\delta\widehat{\Omega}}$*-closed set.*

Definition 3.8. *Let* γ *be a quasi-operation on* $\widehat{\Omega}O(X)$*. For any subset* A *of a space* X*,* $\boldsymbol{\gamma_{\delta\widehat{\Omega}}}$***-kernel of*** $\boldsymbol{A}$ *is denoted by* $\gamma_{\delta\widehat{\Omega}}\ker(A)$ *and defined by* $\gamma_{\delta\widehat{\Omega}}\ker(A) = \cap\{U / A \subseteq U; U \in \gamma_{\delta\widehat{\Omega}}O(X)\}$*. Since* X *is a* $\gamma_{\delta\widehat{\Omega}}$*-open set,* $\gamma_{\delta\widehat{\Omega}}$*-kernel is well defined.*

Proposition 3.9. *Let* $\gamma : \widehat{\Omega}O(X) \to P(X)$ *be a quasi-operation on* $\widehat{\Omega}O(X)$*. Then,*
$i).\ \gamma_{\delta\widehat{\Omega}}cl(\emptyset) = \emptyset.\ ii).\ \gamma_{\delta\widehat{\Omega}}cl(X) = X.\ iii).\ \gamma_{\delta\widehat{\Omega}}ker(\emptyset) = \emptyset.\ iv).\ \gamma_{\delta\widehat{\Omega}}ker(X) = X.$

Proposition 3.10. *If* $\gamma : \widehat{\Omega}O(X) \to P(X)$ *is a quasi-operation on* $\widehat{\Omega}O(X)$*, then the following statements hold for any two subsets* A *and* B *of* X*.*
i) $A \subseteq \gamma_{\delta\widehat{\Omega}}cl(A)$*.*
ii) $If\ A \subseteq B, then\ \gamma_{\delta\widehat{\Omega}}cl(A) \subseteq \gamma_{\delta\widehat{\Omega}}cl(B)$*.*
iii) $A \subseteq \gamma_{\delta\widehat{\Omega}}ker(A)$*.*
iv) $If\ A \subseteq B, then\ \gamma_{\delta\widehat{\Omega}}ker(A) \subseteq \gamma_{\delta\widehat{\Omega}}ker(B)$*.*
Proof. i) Suppose $x \notin \gamma_{\delta\widehat{\Omega}}cl(A)$*. By the definition of* $\gamma_{\delta\widehat{\Omega}}$*-closure, there exists a subset* F *containing* A *such that* $x \notin F$ *and* F^c *is a* $\gamma_{\delta\widehat{\Omega}}$*-open subset containing* x*. Now,* $x \in F^c \subseteq A^c$*. Therefore,* $x \notin A$*. Hence (i) holds.*
ii) Let $x \in \gamma_{\delta\widehat{\Omega}}cl(A)$ *be arbitrary and* F *be any subset of* X *such that* $B \subseteq F$ *and* $X \backslash F$ *is a* $\gamma_{\delta\widehat{\Omega}}$*-open set in* X*. As* $x \in \gamma_{\delta\widehat{\Omega}}cl(A)$*,* $x \in F$*. Then,* $x \in \cap\{F / B \subseteq F; X \backslash F \in \gamma_{\delta\widehat{\Omega}}O(X)\} = \gamma_{\delta\widehat{\Omega}}cl(B)$*.*
iii) If $x \notin \gamma_{\delta\widehat{\Omega}}ker(A) = \cap\{U / A \subseteq U; U \in \gamma_{\delta\widehat{\Omega}}O(X)\}$*, then* $x \notin U$ *for some* $\gamma_{\delta\widehat{\Omega}}$*-open set* U *containing* A*. Therefore,* $x \notin A$*.*

iv) Let $x \in \gamma_{\delta\hat{\Omega}} ker(A)$ *and* U *be any* $\gamma_{\delta\hat{\Omega}}$*-open set such that* $B \subseteq U$. *Since* $A \subseteq B$, $A \subseteq U$. *By the definition of* $\gamma_{\delta\hat{\Omega}}$*-kernel,* $x \in U$. *So,* $x \in \cap\{U/B \subseteq U; U \in \gamma_{\delta\hat{\Omega}} O(X)\}$. *Therefore,* $x \in \gamma_{\delta\hat{\Omega}} ker(B)$.

Proposition 3.11. *Let* $\gamma : \hat{\Omega}O(X) \to P(X)$ *be a quasi-operation on* $\hat{\Omega}O(X)$. *Then, the following two statements hold for any subset* A *of* X.
i) $\gamma_{\delta\hat{\Omega}} cl(A) = \{x \in X/U \cap A \neq \emptyset \,\forall\, U \in \gamma_{\delta\hat{\Omega}} O(X,x)\}$.
ii) $\gamma_{\delta\hat{\Omega}} ker(A) = \{x \in X/F \cap A \neq \emptyset \,\forall\, F \in \gamma_{\delta\hat{\Omega}} C(X,x)\}$.

Proof. i) Let $y \notin \{x \in X/\, U \in \gamma_{\delta\hat{\Omega}} O(X,x) \text{ such that } U \cap A \neq \emptyset\}$. *Then, there exists a* $\gamma_{\delta\hat{\Omega}}$*-open set* U *containing* y *such that* $U \cap A = \emptyset$. *Put* $F = U^c$. *Now,* $y \notin U^c = F$ *such that* $A \subseteq F$. *Then,*
$y \notin \cap\{F/A \subseteq F; X\backslash F \in \gamma_{\delta\hat{\Omega}} O(X)\} = \gamma_{\delta\hat{\Omega}} cl(A)$.*Therefore,*
$\gamma_{\delta\hat{\Omega}} cl(A) \subseteq \{x \in X/\, U \in \gamma_{\delta\hat{\Omega}} O(X,x) \text{ such that } U \cap A \neq \emptyset\}$.
On the other hand, assume that $y \notin \gamma_{\delta\hat{\Omega}} cl(A)$. *Then,* $y \notin F$ *for some subset* F *of* X *such that* $A \subseteq F$ *and* $X\backslash F$ *is an* $\gamma_{\delta\hat{\Omega}}$*-open set in* X. *Put* $U = X\backslash F$. *Then,* U *is a* $\gamma_{\delta\hat{\Omega}}$*-open set containing* y *such that* $U \subseteq A^c$ *or* $U \cap A = \emptyset$. *Then,* $y \notin \{x \in X/U \in \gamma_{\delta\hat{\Omega}} O(X,x) \text{ such that } U \cap A \neq \emptyset\}$. *Therefore,*
$\{x \in X/U \in \gamma_{\delta\hat{\Omega}} O(X,x) \text{ such that } U \cap A \neq \emptyset\} \subseteq \gamma_{\delta\hat{\Omega}} cl(A)$. *Hence*
$\gamma_{\delta\hat{\Omega}} cl(A) = \{x \in X/\, U \in \gamma_{\delta\hat{\Omega}} O(X,x) \text{ such that } U \cap A \neq \emptyset\}$.
ii) Let $y \notin \{x \in X/F \in \gamma_{\delta\hat{\Omega}} C(X,x) \text{ such that } F \cap A \neq \emptyset\}$. *Then, there exists a* $\gamma_{\delta\hat{\Omega}}$*-closed set* F *containing* y *such that* $F \cap A = \emptyset$. *Here,* $A \subseteq F^c$, *put* $F^c = U$. *Now,* $y \notin F^c = U$ *such that* $A \subseteq U$. *Then,*

$y \notin \cap\{U/A \subseteq U \text{ such that } U \in \gamma_{\delta\hat{\Omega}} O(X)\} = \gamma_{\delta\hat{\Omega}} ker(A)$.*Therefore,*
$\gamma_{\delta\hat{\Omega}} ker(A) \subseteq \{x \in X/F \in \gamma_{\delta\hat{\Omega}} C(X,x) \text{ such that } F \cap A \neq \emptyset\}$.
On the other hand, assume that $y \notin \gamma_{\delta\hat{\Omega}} ker(A)$. *Then,* $y \notin U$ *for some* $\gamma_{\delta\hat{\Omega}}$*-open set* U *of* X *such that* $A \subseteq U$. *Put* $F = X\backslash U$. *Now,* F *is a* $\gamma_{\delta\hat{\Omega}}$*-closed set containing* y *such that* $F \subseteq A^c$ *or* $F \cap A = \emptyset$. *Then,*
$y \notin \{x \in X/F \in \gamma_{\delta\hat{\Omega}} C(X,x) \text{ such that } F \cap A \neq \emptyset\}$.*Therefore,*
$\{x \in X/F \in \gamma_{\delta\hat{\Omega}} C(X,x) \text{ such that } F \cap A \neq \emptyset\} \subseteq \gamma_{\delta\hat{\Omega}} ker(A)$. *Hence*
$\gamma_{\delta\hat{\Omega}} ker(A) = \{x \in X/F \in \gamma_{\delta\hat{\Omega}} C(X,x) \text{ such that } F \cap A \neq \emptyset\}$.

IV. $\gamma(\delta\hat{\Omega})$-CLOSURE AND $\gamma(\delta\hat{\Omega})$-INTERIOR

Definition 4.1. *Let* $\gamma : \hat{\Omega}O(X) \to P(X)$ *be a quasi-operation on* $\hat{\Omega}O(X)$ *and* A *be any non-empty subset of* X. ***$\gamma(\delta\hat{\Omega})$-closure of A*** *is denoted by* $\gamma(\delta\hat{\Omega})cl(A)$ *and defined by* $\gamma(\delta\hat{\Omega})cl(A) = \{x \in X/\gamma(U) \cap A \neq \emptyset \text{ for every } U \in \delta O(X,x)\}$. *Always* $\gamma(\delta\hat{\Omega})cl(\emptyset) = \emptyset$.

Example 4.2. *Let* $\gamma : \hat{\Omega}O(\mathbb{Z},k) \to P(\mathbb{Z})$ *be defined by* $\gamma(U) = \delta int(\delta cl(U))$ *for all* $U \in \hat{\Omega}O(\mathbb{Z},k)$. *For the non-empty* $\hat{\Omega}$*-open subset* $\{2m, 2m+1\}$ *of* $\mathbb{Z}$, $\gamma(\delta\hat{\Omega})cl(\{2m, 2m+1\}) = \{2m, 2m+1, 2m+2\}$.

Definition 4.3. *Let* $\gamma : \hat{\Omega}O(X) \to P(X)$ *be a quasi-operation on* $\hat{\Omega}O(X)$ *and* A *be any non-empty subset of* X. ***$\gamma(\delta\hat{\Omega})$-interior of A*** *is denoted by* $\gamma(\delta\hat{\Omega})int(A)$ *and defined by* $\gamma(\delta\hat{\Omega})int(A) = \{x \in X/\gamma(U) \subseteq A \text{ for some } U \in \delta O(X,x)\}$. *Always* $\gamma(\delta\hat{\Omega})int(X) = X$.

Example 4.4. *Let* $\gamma : \hat{\Omega}O(\mathbb{Z},k) \to P(\mathbb{Z})$ *be defined by* $\gamma(U) = \delta cl(\delta int(U))$ *for all* $U \in \hat{\Omega}O(\mathbb{Z},k)$. *For the non-empty* $\hat{\Omega}$*-open subset* $\{2m, 2m+1, 2m+2\}$ *of* $\mathbb{Z}$, $\gamma(\delta\hat{\Omega})int(\{2m, 2m+1, 2m+2\}) = \{2m+1\}$.

Definition 4.5. *Let* γ *be a quasi-operation on* $\hat{\Omega}O(X)$. *A subset* F *of a space* X *is a* ***$\gamma(\delta\hat{\Omega})$-closed set*** *in* X *if* $\gamma(\delta\hat{\Omega})cl(F) = F$.

Example 4.6. *Let* $\gamma : \hat{\Omega}O(\mathbb{Z},k) \to P(\mathbb{Z})$ *be defined by* $\gamma(U) = \delta int(\delta cl(U))$ *for all* $U \in \hat{\Omega}O(\mathbb{Z},k)$. *For the non-empty* $\hat{\Omega}$*-open subset* $A = \{2m, 2m+1, 2m+2\}$ *of* $\mathbb{Z}$, $\gamma(\delta\hat{\Omega})cl(A) = \{2m, 2m+1, 2m+2\}) = A$. *Thus* $A = \{2m, 2m+1, 2m+2\}$ *is* $\gamma(\delta\hat{\Omega})$*-closed in* $(\mathbb{Z},k)$.

Remark 4.7. *For a given quasi-operation* γ *on* $\hat{\Omega}O(X)$, *the basic properties* $cl(X) = X$ *and* $int(\emptyset) = \emptyset$ *can not be extended to the* $\gamma(\delta\hat{\Omega})$*-closure and* $\gamma(\delta\hat{\Omega})$*-interior.*

Example 4.8. *i) Define* $\gamma : \hat{\Omega}O(\mathbb{Z},k) \to P(\mathbb{Z})$ *by* $\gamma(U) = \delta cl(\delta int(U)) \cap \{\cap\, cl(\{x\})/x \in U\}$ *for every* $U \in \hat{\Omega}O(\mathbb{Z},k)$. *For a* δ*-open set* $U = \{2m+1, 2m+3, 2m+5\}$ *containing the point* $2m+3$, $\gamma(\{2m+1, 2m+3, 2m+5\}) = \emptyset$. *Then,* $\gamma(U) \cap \mathbb{Z} = \emptyset \cap \mathbb{Z} = \emptyset$. *By the definition of* $\gamma(\delta\hat{\Omega})$*-closure,* $2m+3 \notin \gamma(\delta\hat{\Omega})cl(\mathbb{Z})$, *where as* $2m+3 \in \mathbb{Z}$. *Hence* $\gamma(\delta\hat{\Omega})cl(\mathbb{Z}) \neq \mathbb{Z}$.

ii) Define $\gamma : \hat{\Omega}O(\mathbb{Z},k) \to P(\mathbb{Z})$ *by* $\gamma(U) = \delta int(\delta cl(U)) \cap \{2n/n \in \mathbb{Z}\}$ *for every* $U \in \hat{\Omega}O(\mathbb{Z},k)$. *For the point* $2m-1$, *there exists a* δ*–open subset* $U = \{2m-1\}$ *containing* $2m-1$ *such that* $\gamma(\{2m-1\}) \subseteq \emptyset$. *By the definition of* $\gamma(\delta\hat{\Omega})$*-interior,* $2m-1 \in \gamma(\delta\hat{\Omega})int(\emptyset)$. *Therefore,* $\gamma(\delta\hat{\Omega})int(\emptyset) \neq \emptyset$.

V. γ_δ-REGULAR QUASI-OPERATION AND γ_δ-OPEN QUASI-OPERATION

Definition 4.9. *A quasi-operation* $\gamma : \hat{\Omega}O(X) \to P(X)$ *is said to be a* $\boldsymbol{\gamma_\delta}$ ***-regular quasi-operation*** *if for each* $x \in X$ *and for every pair* $U, V \in \delta O(X,x)$ *there exists* $W \in \delta O(X,x)$ *such that* $\gamma(W) \subseteq \gamma(U) \cap \gamma(V)$.

Example 4.10. *Let* $\gamma : \hat{\Omega}O(\mathbb{Z},k) \to P(\mathbb{Z})$ *be defined by* $\gamma(U) = \delta int(\delta cl(U)) \cap \{2n+1/n \in \mathbb{Z}\}$ *for every* $U \in \hat{\Omega}O(\mathbb{Z},k)$.
For an even integer 2n, *consider* $U = \{2n-1, 2n, 2n+1\} = V \in \delta O(\mathbb{Z}, 2n)$. *Then, there exists* $W = U \in \delta O(\mathbb{Z}, 2n)$ *such that* $\gamma(W) = \gamma(U) \cap \gamma(V) = \{2n-1, 2n+1\}$.
For an odd integer 2n+1, *consider* $U = \{2n+1, 2n+2, 2n+3\} \in \delta O(\mathbb{Z}, 2n+1)$, *and* $V = \{2n+1, 2n+3\} \in \delta O(\mathbb{Z}, 2n+1)$. *Then, there exists* $W = \{2n+1\} \in \delta O(\mathbb{Z}, 2n+1)$ *such that* $\gamma(W) = \{2n+1\} \subseteq \{2n+1, 2n+3\} = \gamma(U) \cap \gamma(V)$. *Therefore,* γ *is a* γ_δ*-regular quasi-operation.*

Definition 4.11. *A quasi-operation* $\gamma : \hat{\Omega}O(X) \to P(X)$ *is said to be a* $\boldsymbol{\gamma_\delta}$ ***-open quasi-operation*** *if for each point* $x \in X$ *and* $U \in \delta O(X,x)$ *there exists an* γ_δ *–open set* W *containing* x *such that* $W \subseteq \gamma(U)$.

Example 4.12. *Let* $\gamma : \hat{\Omega}O(\mathbb{Z},k) \to P(\mathbb{Z})$ *be defined by* $\gamma(U) = \delta int(\delta cl(U))$ *for all* $U \in \hat{\Omega}O(\mathbb{Z},k)$.
For an even integer 2n, *consider* $U = \{2n-1, 2n, 2n+1\} \in \delta O(\mathbb{Z}, 2n)$. *Then, there exists* $W = \{2n-1, 2n, 2n+1\} \in \gamma_{\delta\hat{\Omega}}O(\mathbb{Z}, 2n)$ *such that* $W = \gamma(U) = \{2n-1, 2n, 2n+1\}$.
For an odd integer 2n+1, *consider* $U = \{2n+1\} \in \delta O(\mathbb{Z}, 2n+1)$. *Then, there exists* $W = \{2n+1\} \in \gamma_{\delta\hat{\Omega}}O(\mathbb{Z}, 2n+1)$ *such that* $W = \gamma(U) = \{2n+1\}$. *Therefore,* γ *is a* γ_δ*-open quasi-operation.*
Elementary properties of $\gamma(\delta\hat{\Omega})$*-closure*

Proposition 4.13. *Let* $\gamma : \hat{\Omega}O(X) \to P(X)$ *be a quasi-operation on* $\hat{\Omega}O(X)$ *in a space* X. *Then, the following statements hold for any subsets* A *and* B *of* X.
i) $X \backslash \gamma(\delta\hat{\Omega})cl(A) = \gamma(\delta\hat{\Omega})int(X \backslash A)$.
ii) If $A \subseteq B$, *then* $\gamma(\delta\hat{\Omega})cl(A) \subseteq \gamma(\delta\hat{\Omega})cl(B)$.
iii) Arbitrary union of $\gamma(\delta\hat{\Omega})$*–open sets in* X *is a* $\gamma(\delta\hat{\Omega})$*–open set in* X.
Proof. i) Let $x \in X \backslash \gamma(\delta\hat{\Omega})cl(A)$ *be arbitrary. Then, there exists* $U \in \delta O(X,x)$ *such that* $\gamma(U) \cap A = \emptyset$. *That is,* $\gamma(U) \subseteq A^c$ *for some* $U \in \delta O(X,x)$. *Therefore,* $x \in \gamma(\delta\hat{\Omega})int(X \backslash A)$.
ii) Let $x \in \gamma(\delta\hat{\Omega})cl(A)$ *and* $U \in \delta O(X,x)$ *be arbitrary. Then,* $\gamma(U) \cap A \neq \emptyset$. *By the hypothesis,* $\gamma(U) \cap B \neq \emptyset$ *for every* $U \in \delta O(X,x)$ *and hence* $x \in \gamma(\delta\hat{\Omega})cl(B)$.*Therefore,* $\gamma(\delta\hat{\Omega})cl(A) \subseteq \gamma(\delta\hat{\Omega})cl(B)$.
iii) Let $\{U_\alpha / \alpha \epsilon J\}$ *be any family of* $\gamma_{\delta\hat{\Omega}}$*-open sets in* X. *Let* $U = \cup_{\alpha \epsilon J} U_\alpha$ *and* $x \epsilon U$ *be arbitrary. Then,* $x \epsilon U_\alpha$ *for some* $\alpha \epsilon J$. *Since each* U_α *is a* $\gamma_{\delta\hat{\Omega}}$*-open subset of* X, *there exists* $V \in \delta O(X,x)$ *such that* $\gamma(V) \subseteq U_\alpha \subseteq U$. *Then,* U *is a* $\gamma_{\delta\hat{\Omega}}$*-open subset of* X.

Proposition 4.14. *Let* $\gamma : \hat{\Omega}O(X) \to P(X)$ *be a quasi-operation on* $\hat{\Omega}O(X)$ *in a space* X. *Then,* $\gamma(\delta\hat{\Omega})cl(A) \cup \gamma(\delta\hat{\Omega})cl(B) \subseteq \gamma(\delta\hat{\Omega})cl(A \cup B)$ *for any subsets* A *and* B *of* X.
Proof. Always $A \subseteq A \cup B$ *and* $B \subseteq A \cup B$. *By the Proposition 4.13. ii),* $\gamma(\delta\hat{\Omega})cl(A) \subseteq \gamma(\delta\hat{\Omega})cl(A \cup B)$*and* $\gamma(\delta\hat{\Omega})cl(B) \subseteq \gamma(\delta\hat{\Omega})cl(A \cup B)$ *and hence* $\gamma(\delta\hat{\Omega})cl(A) \cup \gamma(\delta\hat{\Omega})cl(B) \subseteq \gamma(\delta\hat{\Omega})cl(A \cup B)$.

Example 4.15. $\gamma(\delta\hat{\Omega})cl(A \cup B) \subseteq \gamma(\delta\hat{\Omega})cl(A) \cup \gamma(\delta\hat{\Omega})cl(B)$ *fails to hold in a quasi-operation* $\gamma : \hat{\Omega}O(X) \to P(X)$. *For an example, consider* $\mathbb{Z}$, *the set of all integers equipped with* κ, *the digital topology.*
Define $\gamma : \hat{\Omega}O(\mathbb{Z},k) \to P(\mathbb{Z})$ *by*

$$\gamma(U) = \begin{cases} \delta cl(\delta int(U)) \cap 2\mathbb{Z} & \text{if } U = \{2n+1\} \text{ where } x \in \mathbb{Z} \\ \delta cl(\delta int(U)) \cap 2\mathbb{Z}+1 & \text{otherwise} \end{cases}$$

for all $U \in \hat{\Omega}O(\mathbb{Z},k)$. *If* $A = \{2m+1\}$ *and* $B = \{2m\}$, *then* $\gamma(\delta\hat{\Omega})cl(A) = \emptyset = \gamma(\delta\hat{\Omega})cl(B)$. *But* $\gamma(\delta\hat{\Omega})cl(A \cup B) = \gamma(\delta\hat{\Omega})cl(\{2m, 2m+1\}) = \{2m+1\}$. *Therefore,* $\gamma(\delta\hat{\Omega})cl(A \cup B) \nsubseteq \gamma(\delta\hat{\Omega})cl(A) \cup \gamma(\delta\hat{\Omega})cl(B)$.

Proposition 4.16. *Let* $\gamma : \widehat{\Omega}O(X) \to P(X)$ *be a quasi-operation on* $\widehat{\Omega}O(X)$ *in a space* X. *If* γ *is a* γ_{δ}*-regular quasi-operation, then* $\gamma(\delta\widehat{\Omega})cl(A) \cup \gamma(\delta\widehat{\Omega})cl(B) = \gamma(\delta\widehat{\Omega})cl(A \cup B)$ *for any subsets* A *and* B *of* X.

Proof. *Assume that* $x \notin \gamma(\delta\widehat{\Omega})cl(A) \cup \gamma(\delta\widehat{\Omega})cl(B)$. *That is,* $x \notin \gamma(\delta\widehat{\Omega})cl(A)$ *and* $x \notin \gamma(\delta\widehat{\Omega})cl(B)$. *By the definition of* $\gamma(\delta\widehat{\Omega})$*-closure, there exists* $U, V \in \delta O(X, x)$ *such that* $\gamma(U) \cap A = \emptyset$ *and* $\gamma(V) \cap B = \emptyset$. *Since* γ *is a* γ_{δ}*-regular quasi-operation, there exists* $W \in \delta O(X, x)$ *such that* $\gamma(W) \subseteq \gamma(U) \cap \gamma(V)$.

$\gamma(W) \cap (A \cup B) = (\gamma(U) \cap \gamma(V)) \cap (A \cup B) = (\gamma(U) \cap \gamma(V) \cap A) \cup (\gamma(U) \cap \gamma(V) \cap B) \subseteq (\gamma(U) \cap A) \cup (\gamma(U) \cap B) = \emptyset$

So, $x \notin \gamma(\delta\widehat{\Omega})cl(A \cup B)$. *Therefore,* $\gamma(\delta\widehat{\Omega})cl(A \cup B) \subseteq \gamma(\delta\widehat{\Omega})cl(A) \cup \gamma(\delta\widehat{\Omega})cl(B)$. *Hence* $\gamma(\delta\widehat{\Omega})cl(A) \cup \gamma(\delta\widehat{\Omega})cl(B) = \gamma(\delta\widehat{\Omega})cl(A \cup B)$.

Example 4.17. $(\gamma(\delta\widehat{\Omega})cl(A))^c$ *is not always* $\gamma_{\delta\widehat{\Omega}}$*-open in a quasi-operation* $\gamma : \widehat{\Omega}O(X) \to P(X)$. *Define* $\gamma : \widehat{\Omega}O(\mathbb{Z}, k) \to P(\mathbb{Z})$ *by* $\gamma(U) = \delta cl(\delta int(U))$ *for all* $U \in \widehat{\Omega}O(\mathbb{Z}, k)$. *Consider* $A = \{2m-1, 2m, 2m+1\} \subseteq \mathbb{Z}$. *Then,* $\gamma(\delta\widehat{\Omega})cl(A) = \{2m-2, 2m-1, 2m, 2m+1, 2m+2\}$ *and hence* $(\gamma(\delta\widehat{\Omega})cl(A))^c = \mathbb{Z}\backslash\{2m-2, 2m-1, 2m, 2m+1, 2m+2\}$. *For the point* $2m+3 \in \mathbb{Z}\backslash\{2m-2, 2m-1, 2m, 2m+1, 2m+2\}$, *every* δ*-open set* $U = \{2m+3\}$ *containing* $2m+3$ *is such that* $\gamma(U) = \{2m+2, 2m+3, 2m+4\} \nsubseteq \mathbb{Z}\backslash\{2m-2, 2m-1, 2m, 2m+1, 2m+2\}$. *Therefore,* $\mathbb{Z}\backslash\{2m-2, 2m-1, 2m, 2m+1, 2m+2\}$ *is not a* $\gamma_{\delta\widehat{\Omega}}$*-open set of* $\mathbb{Z}$.

Proposition 4.18. *Let* $\gamma : \widehat{\Omega}O(X) \to P(X)$ *be a quasi-operation on* $\widehat{\Omega}O(X)$ *in a space* X. *If* γ *is a* γ_{δ}*-open quasi-operation, then the following two statements hold for any subset* A *of* X.

i) $(\gamma(\delta\widehat{\Omega})cl(A))^c$ *is a* $\gamma_{\delta\widehat{\Omega}}$*–open set.*

ii) $\gamma(\delta\widehat{\Omega})int(A)$ *is a* $\gamma_{\delta\widehat{\Omega}}$*–open set.*

Proof. i) Let $x \in (\gamma(\delta\widehat{\Omega})cl(A))^c$ *be arbitrary. Then,* $x \notin \gamma(\delta\widehat{\Omega})cl(A)$. *By the definition of* $\gamma(\delta\widehat{\Omega})$*-closure, there exists* $U_x \in \delta O(X, x)$ *such that* $\gamma(U_x) \cap A = \emptyset$. *Since* γ *is* γ_{δ}*-open quasi-operation, there exists a* $\gamma_{\delta\widehat{\Omega}}$*-open set* V_x *containing x such that* $V_x \subseteq \gamma(U_x)$. *Then,* $V_x \cap A = \emptyset$. *It is proved that, for each* $x \in (\gamma(\delta\widehat{\Omega})cl(A))^c$, *there exists a* $\gamma_{\delta\widehat{\Omega}}$*-open set* V_x *containing* x *such that* $V_x \cap A = \emptyset$. *Since* V_x *is a* $\gamma_{\delta\widehat{\Omega}}$*-open set and arbitrary union of* $\gamma_{\delta\widehat{\Omega}}$*-open set is a* $\gamma_{\delta\widehat{\Omega}}$*-open set,* $\cup\{V_x / x \in (\gamma(\delta\widehat{\Omega})cl(A))^c\} = W$ *(say) is a* $\gamma_{\delta\widehat{\Omega}}$*-open subset of* X. *It is claiming that* $W = (\gamma(\delta\widehat{\Omega})cl(A))^c$. *Let* $x \in (\gamma(\delta\widehat{\Omega})cl(A))^c$ *be arbitrary. By the above argument, there exists a* $\gamma_{\delta\widehat{\Omega}}$*-open set* V_x *containing* x *such that* $V_x \cap A = \emptyset$. *Thus* $(\gamma(\delta\widehat{\Omega})cl(A))^c \subseteq W$.

On the other hand, let $y \in W = \cup\{V_x / x \in (\gamma(\delta\widehat{\Omega})cl(A))^c\}$ *be arbitrary. Then,* $y \in V_x$ *for some* $x \in (\gamma(\delta\widehat{\Omega})cl(A))^c$. *It is enough to prove* $V_x \cap \gamma(\delta\widehat{\Omega})cl(A) = \emptyset$. *If* $z \in V_x \cap \gamma(\delta\widehat{\Omega})cl(A)$, *then* $z \in V_x$ *and* $z \in \gamma(\delta\widehat{\Omega})cl(A)$. *Since* V_x *is* $\gamma_{\delta\widehat{\Omega}}$*-open subset of* X, *there exists a* δ*-open set* U_1 *containing* z *such that* $\gamma(U_1) \subseteq V_x$. *Since* $z \in \gamma(\delta\widehat{\Omega})cl(A)$, $\gamma(U_1) \cap A \neq \emptyset$. *Then,* $V_x \cap A \neq \emptyset$, *a contradiction to* $V_x \cap A = \emptyset$ *for each* $x \in (\gamma(\delta\widehat{\Omega})cl(A))^c$. *So assumption is wrong and hence* $V_x \cap \gamma(\delta\widehat{\Omega})cl(A) = \emptyset$. *Now,* $V_x \subseteq (\gamma(\delta\widehat{\Omega})cl(A))^c$. *As* $y \in V_x$, $y \in (\gamma(\delta\widehat{\Omega})cl(A))^c$. *That is,* $W \subseteq (\gamma(\delta\widehat{\Omega})cl(A))^c$. *Therefore,* $W = (\gamma(\delta\widehat{\Omega})cl(A))^c$ *is a* $\gamma_{\delta\widehat{\Omega}}$*-open subset of* X.

ii) Apply (i) for the set $X\backslash A = A^c$. $X\backslash\gamma(\delta\widehat{\Omega})cl(A^c) = \gamma(\delta\widehat{\Omega})int(A)$. *By i), L.H.S of the above equation is* $\gamma_{\delta\widehat{\Omega}}$*-open set. Therefore,* $\gamma(\delta\widehat{\Omega})int(A)$ *is a* $\gamma_{\delta\widehat{\Omega}}$*-open set.*

Proposition 4.19. *Let* $\gamma : \widehat{\Omega}O(X) \to P(X)$ *be a quasi-operation on* $\widehat{\Omega}O(X)$. *Then,* $\gamma(\delta\widehat{\Omega})cl(A) \subseteq \gamma_{\delta\widehat{\Omega}}cl(A)$ *for any subset* A *of* X.

Proof. Let $x \notin \gamma_{\delta\widehat{\Omega}}cl(A)$ *be arbitrary. Then,* $x \notin F$ *for some subset* F *of* X *such that* $A \subseteq F$; $X\backslash F \in \gamma_{\delta\widehat{\Omega}}O(X)$. *Put* $U = X\backslash F$. *Now,* U *is a* $\gamma_{\delta\widehat{\Omega}}$*-open set containing* x *such that* $U \subseteq A^c$. *By the definition of* $\gamma_{\delta\widehat{\Omega}}$*-open set, there exists* $V \in \delta O(X, x)$ *such that* $\gamma(V) \subseteq U = X\backslash F \subseteq X\backslash A$. *It is proved that there exists* $U \in \delta O(X, x)$ *such that* $\gamma(U) \cap A = \emptyset$. *By the definition of* $\gamma(\delta\widehat{\Omega})$*-closure,* $x \notin \gamma(\delta\widehat{\Omega})cl(A)$. *Therefore,* $\gamma(\delta\widehat{\Omega})cl(A) \subseteq \gamma_{\delta\widehat{\Omega}}cl(A)$.

Example 4.20. *For a quasi-operation* $\gamma : \widehat{\Omega}O(X) \to P(X)$ *and for a subset* A *of* X, $\gamma_{\delta\widehat{\Omega}}cl(A) \subseteq \gamma(\delta\widehat{\Omega})cl(A)$ *does not always hold. For an example, define* $\gamma : \widehat{\Omega}O(\mathbb{Z}, k) \to P(\mathbb{Z})$ *by* $\gamma(U) = \delta int(\delta cl(U)) \cap \{2n / n \in \mathbb{Z}\}$ *for all* $U \in \widehat{\Omega}O(\mathbb{Z}, k)$. *Consider* $A = \{2m+1, 2m+3\} \subseteq \mathbb{Z}$. *then,*

$\gamma_{\delta\widehat{\Omega}}cl(A) = \{2m, 2m+1, 2m+2, 2m+3, 2m+4\}$, but $\gamma(\delta\widehat{\Omega})cl(A) = \emptyset$. Therefore, $\gamma_{\delta\widehat{\Omega}}cl(A) \nsubseteq \gamma(\delta\widehat{\Omega})cl(A)$.

Proposition 4.21. *Let $:\widehat{\Omega}O(X) \to P(X)$ be a quasi-operation on $\widehat{\Omega}O(X)$. If γ is a γ_{δ}-open quasi-operation, then $\gamma_{\delta\widehat{\Omega}}cl(A) = \gamma(\delta\widehat{\Omega})cl(A)$ for any subset A of X.*

Proof. If $x \notin \gamma(\delta\widehat{\Omega})cl(A)$, then there exists $U \in \delta O(X,x)$ such that $\gamma(U) \cap A = \emptyset$ or $\gamma(U) \subseteq A^c$. By the definition of γ_{δ}-open quasi-operation, there exists a $\gamma_{\delta\widehat{\Omega}}$-open set V containing x such that $V \subseteq \gamma(U) \subseteq A^c$ and hence $V \cap A = \emptyset$. By the Proposition 3.11. (i), $x \notin \gamma_{\delta\widehat{\Omega}}cl(A)$. Therefore, $\gamma_{\delta\widehat{\Omega}}cl(A) \subseteq \gamma(\delta\widehat{\Omega})cl(A)$. Hence $\gamma_{\delta\widehat{\Omega}}cl(A) = \gamma(\delta\widehat{\Omega})cl(A)$.

Example 4.22. i) Here is an example for $A \nsubseteq \gamma(\delta\widehat{\Omega})cl(A)$ for a quasi-operation $:\widehat{\Omega}O(X) \to P(X)$ and for a subset A of X. Define $\gamma : \widehat{\Omega}O(\mathbb{Z},k) \to P(\mathbb{Z})$ by
$\gamma(U) = \delta int(\delta cl(U)) \cap \{2n+1/n \in \mathbb{Z}\}$ for all $U \in \widehat{\Omega}O(\mathbb{Z},k)$. Consider $A = \{2m\} \subseteq \mathbb{Z}$ then, $\gamma(\delta\widehat{\Omega})cl(A) = \emptyset$. Therefore, $A \nsubseteq \gamma(\delta\widehat{\Omega})cl(A)$.
iii) Here is an example for $\gamma(\delta\widehat{\Omega})cl(A)$ is not always a $\gamma(\delta\widehat{\Omega})$-closed set for a quasi-operation $:\widehat{\Omega}O(X) \to P(X)$ and for a subset A of X. Define $\gamma : \widehat{\Omega}O(\mathbb{Z},k) \to P(\mathbb{Z})$ by
$\gamma(U) = \delta int(\delta cl(U))$ for every $U \in \widehat{\Omega}O(\mathbb{Z},k)$. Consider $A = \{2m+1, 2m+2, 2m+3\} \subseteq \mathbb{Z}$ then, $\gamma(\delta\widehat{\Omega})cl(A) = \{2m, 2m+1, 2m+2, 2m+3, 2m+4\}$. Therefore, $\gamma(\delta\widehat{\Omega})cl(A) \neq A$. Hence $\gamma(\delta\widehat{\Omega})cl(A)$ is not a $\gamma(\delta\widehat{\Omega})$-closed set.
iv) Here is an example for $\gamma(U) = \emptyset$ for a non-empty δ-open subset U containing x where $\gamma : \widehat{\Omega}O(X) \to P(X)$ is a quasi-operation. Define $\gamma : \widehat{\Omega}O(\mathbb{Z},k) \to P(\mathbb{Z})$ by
$\gamma(U) = \delta int(\delta cl(U)) \cap \{2n/n \in \mathbb{Z}\}$ for all $U \in \widehat{\Omega}O(\mathbb{Z},k)$. For a δ-open set $U = \{2m-1\} \subseteq \mathbb{Z}$. $\gamma(U) = \emptyset$.
v) The statement $\gamma(\delta\widehat{\Omega})cl(X) = X$ is not always hold for a quasi-operation $:\widehat{\Omega}O(X) \to P(X)$. Define $\gamma : \widehat{\Omega}O(\mathbb{Z},k) \to P(\mathbb{Z})$ by $\gamma(U) = \delta cl(\delta int(U)) \cap \{\cap\, cl(\{x\})/x \in U\}$ for every $U \in \widehat{\Omega}O(\mathbb{Z},k)$. For a point $2m+3 \in \mathbb{Z}$, there exists a δ-open set $U = \{2m+1, 2m+3, 2m+5\}$ containing the point $2m+3$ such that $\gamma(U) = \emptyset$. Then, the point $2m+3 \notin \gamma(\delta\widehat{\Omega})cl(\mathbb{Z})$. Therefore, $\gamma(\delta\widehat{\Omega})cl(\mathbb{Z}) \neq \mathbb{Z}$.

Proposition 4.23. *Let $\gamma : \widehat{\Omega}O(X) \to P(X)$ be a quasi-operation on $\widehat{\Omega}O(X)$. If γ is a γ_{δ}-open quasi-operation, then the following statements hold for any subset A of X.*
i) $A \subseteq \gamma(\delta\widehat{\Omega})cl(A)$.
ii) $\gamma(\delta\widehat{\Omega})cl(\gamma(\delta\widehat{\Omega})cl(A)) = \gamma(\delta\widehat{\Omega})cl(A)$.
iii) $\gamma(\delta\widehat{\Omega})cl(A)$ is a $\gamma(\delta\widehat{\Omega})$-closed set.
iv) $\gamma(U) \neq \emptyset$ for every non-empty subset $U \in \delta O(X)$.
v) $\gamma(\delta\widehat{\Omega})cl(X) = X$.

Proof. i) By the Proposition 3.10.(i), $A \subseteq \gamma_{\delta\widehat{\Omega}}cl(A)$ and by the Proposition 4.21, $\gamma_{\delta\widehat{\Omega}}cl(A) = \gamma(\delta\widehat{\Omega})cl(A)$. Therefore, $A \subseteq \gamma(\delta\widehat{\Omega})cl(A)$.
ii) By the Proposition 4.18.(i), $(\gamma(\delta\widehat{\Omega})cl(A))^c$ is a $\gamma_{\delta\widehat{\Omega}}$-open subset of X. Then, $\gamma(\delta\widehat{\Omega})cl(A)$ is a $\gamma(\delta\widehat{\Omega})$-closed subset of X. By the definition of $\gamma(\delta\widehat{\Omega})$-closed set, $\gamma(\delta\widehat{\Omega})cl(\gamma(\delta\widehat{\Omega})cl(A)) = \gamma(\delta\widehat{\Omega})cl(A)$.
iii) Let $B = \gamma(\delta\widehat{\Omega})cl(A)$. By (ii), $\gamma(\delta\widehat{\Omega})cl(B) = B$. By the definition of $\gamma(\delta\widehat{\Omega})$-closed set, B is a $\gamma(\delta\widehat{\Omega})$-closed set in X.
iv) Let U be any non-empty δ-open subset of X. Choose $x \in U$. Since γ is a γ_{δ}-open quasi-operation, there exists a $\gamma_{\delta\widehat{\Omega}}$-open set W containing x such that $W \subseteq \gamma(U)$. Since $x \in W$, $x \in \gamma(U)$. Therefore, $\gamma(U) \neq \emptyset$.
v) Always $\gamma(\delta\widehat{\Omega})cl(X) \subseteq X$. Let $x \in X$ and $U \in \delta O(X,x)$ be arbitrary. By (iv), $\gamma(U) \neq \emptyset$. Then, $X \cap \gamma(U) = \gamma(U) \neq \emptyset$. Therefore, $x \in \gamma(\delta\widehat{\Omega})cl(X)$. Hence $\gamma(\delta\widehat{\Omega})cl(X) = X$.

Example 4.24. i) Here is an example for $\delta cl(A)$ is not always subset of $\gamma(\delta\widehat{\Omega})cl(A)$ for a quasi-operation

$\gamma :\widehat{\Omega}O(X) \to P(X)$ and for a subset A of X. Define $\gamma : \widehat{\Omega}O(\mathbb{Z},k) \to P(\mathbb{Z})$ by
$\gamma(U) = \delta int(\delta cl(U)) \cap \{2n/n \in \mathbb{Z}\}$ for every $U \in \widehat{\Omega}O(\mathbb{Z},k)$. Consider $A = \{2m-1, 2m, 2m+1\} \subseteq \mathbb{Z}$ then $\delta cl(A) = \{2m-2, 2m-1, 2m, 2m+1, 2m+2\}$ and $\gamma(\delta\widehat{\Omega})cl(A) = \{2m\}$. Therefore, $\delta cl(A) \nsubseteq \gamma(\delta\widehat{\Omega})cl(A)$.
ii) Here is an example for $\gamma(\delta\widehat{\Omega})int(A)$ is not always subset of $\delta int(A)$ for a quasi-operation $:\widehat{\Omega}O(X) \to P(X)$ and for a subset A of X. Define $\gamma : \widehat{\Omega}O(\mathbb{Z},k) \to P(\mathbb{Z})$ by

$\gamma(U) = \delta cl(\delta int(U))$ for every $U \in \widetilde{\Omega}O(\mathbb{Z}, k)$. *Consider* $A = \{2m-2\} \subseteq \mathbb{Z}$ *then,*
$\gamma(\delta\widetilde{\Omega})int(A) = X\backslash\gamma(\delta\widetilde{\Omega})cl(A) = X\backslash\{2m-4, 2m-3, 2m-2, 2m-1, 2m\}$, *but* $\delta int(A) = \{2m-2\}$.
Therefore, $\gamma(\delta\widetilde{\Omega})int(A) \nsubseteq \delta int(A)$.
iii) Here is an example for $\gamma(\delta\widetilde{\Omega})int(\emptyset)$ *is not always an empty set for a quasi-operation* $:\widetilde{\Omega}O(X) \to P(X)$ *and for a subset* A *of* X. *Define* $\gamma : \widetilde{\Omega}O(\mathbb{Z}, k) \to P(\mathbb{Z})$ *by*
$\gamma(U) = \delta int(\delta cl(U)) \cap \{2n/n \in \mathbb{Z}\}$ for every $U \in \widetilde{\Omega}O(\mathbb{Z}, k)$. *Consider* $A = \{2m-3, 2m-1\} \subseteq \mathbb{Z}$ *then,*
$\gamma(\delta\widetilde{\Omega})int(A) = X\backslash\gamma(\delta\widetilde{\Omega})cl(A) = X$. *Therefore,* $\gamma(\delta\widetilde{\Omega})int(\emptyset) \neq \emptyset$.

Proposition 4.25. *Let* $\gamma : \widetilde{\Omega}O(X) \to P(X)$ *be an operation on* $\widetilde{\Omega}O(X)$. *Then, the following statements hold for any subset* A *of* X.
i) $A \subseteq \delta cl(A) \subseteq \gamma(\delta\widetilde{\Omega})cl(A)$.
ii) $\gamma(\delta\widetilde{\Omega})int(A) \subseteq \delta int(A) \subseteq A$.
iii) $\gamma(\delta\widetilde{\Omega})int(\emptyset) = \emptyset$.
iv) Every $\gamma_{\delta\widetilde{\Omega}} - open\ set\ is\ the\ union\ of\ some\ \delta - open\ subset$.
Proof. i) Always $A \subseteq \delta cl(A)$. *Let* $x \in \delta cl(A)$ *and* $U \in \delta O(X, x)$ *be arbitrary. Then,* $U \cap A \neq \emptyset$. *Since* γ *is a operation on* $\widetilde{\Omega}O(X)$, $\gamma(U) \cap A \neq \emptyset$. *Then,* $x \in \gamma(\delta\widetilde{\Omega})cl(A)$. *Therefore,* $\delta cl(A) \subseteq \gamma(\delta\widetilde{\Omega})cl(A)$. *Hence* $A \subseteq \delta cl(A) \subseteq \gamma(\delta\widetilde{\Omega})cl(A)$.
ii) Always $\delta int(A) \subseteq A$. *By (i),* $\delta cl(X\backslash A) \subseteq \gamma(\delta\widetilde{\Omega})cl(X\backslash A)$. *Then,* $X\backslash\delta int(A) \subseteq X\backslash\gamma(\delta\widetilde{\Omega})int(A)$.
Therefore, $\gamma(\delta\widetilde{\Omega})int(A) \subseteq \delta int(A)$. *Hence* $\gamma(\delta\widetilde{\Omega})int(A) \subseteq \delta int(A) \subseteq A$.
iii) Apply ii) for an empty set $\emptyset$. *Then* $\gamma(\delta\widetilde{\Omega})int(\emptyset) \subseteq \emptyset$. *Therefore,* $\gamma(\delta\widetilde{\Omega})int(\emptyset) = \emptyset$..
iv) Let A *be any* $\gamma_{\delta\widetilde{\Omega}}$*-open set in* X *and*
$\mathcal{B} = \{U_y / for\ each\ y \in A, there\ exists\ U_y \in \delta O(X, y) such\ that\ \gamma(U_y) \subseteq A\}$. *Since* $A \in \mathcal{B}, \mathcal{B} \neq \emptyset$.
Clearly $\mathcal{B} \subseteq \delta O(X)$. *Let* $U = \bigcup_{y \in A}\{U_y / U_y \in \mathcal{B}\}$. *It is enough to prove* $A \subseteq U$. *Let* $x \in A$ *be arbitrary. Since* A *is a* $\gamma_{\delta\widetilde{\Omega}}$*-open set, there exists* $U_x \in \delta O(X, x)$ *such that* $\gamma(U_x) \subseteq A$ *which gives* $U_x \in \mathcal{B}$. *Then,* $U_x \subseteq U$.
That is, $x \in U$.
On the other hand, Let $x \in U$ *be arbitrary. Then,* $x \in U_y$ *for some* $U_y \in \mathcal{B}$. *Clearly,* $\{x, y\} \subseteq U_y$ *and* $U_y \in \delta O(X)$ *such that* $\gamma(U_y) \subseteq A$. *Now,* $x \in U_y \subseteq \gamma(U_y) \subseteq A$. *Then,* $U \subseteq A$. *Therefore,* $A = U$.

VI. $\gamma(\delta\widetilde{\Omega})$-REGULAR SPACE

Definition 5.1. *A space* (X, τ) *with a quasi-operation* γ *on* $\widetilde{\Omega}O(X)$ *is said to be* $\boldsymbol{\gamma(\delta\widetilde{\Omega})}$***-regular space****, if for each* $x \in X$ *and for each subset* $U \in \widetilde{\Omega}O(X, x)$ *there exists a subset* $W \in \delta O(X, x)$ *such that* $\gamma(W) \subseteq U$.

Example 5.2. *For a quasi-operation* $\gamma : \widetilde{\Omega}O(X) \to P(X)$ *and for a subset A of X,* $\widetilde{\Omega}cl(A)$ *is not always a subset of* $\gamma(\delta\widetilde{\Omega})cl(A)$. *For an example, define* $\gamma : \widetilde{\Omega}O(\mathbb{Z}, k) \to P(\mathbb{Z})$ *by*
$\gamma(U) = \delta int(\delta cl(U)) \cap \{2n+1/n \in \mathbb{Z}\}$ for every $U \in \widetilde{\Omega}O(\mathbb{Z}, k)$. *Consider* $A = \{2m\} \subseteq \mathbb{Z}$ *then,*
$\widetilde{\Omega}cl(A) = \{2m\}$, *but* $\gamma(\delta\widetilde{\Omega})cl(A) = \emptyset$. *Therefore,* $\widetilde{\Omega}cl(A) \nsubseteq \gamma(\delta\widetilde{\Omega})cl(A)$. *Hence*
$\delta cl(A) \nsubseteq \widetilde{\Omega}cl(A) \nsubseteq \gamma(\delta\widetilde{\Omega})cl(A)$.

Proposition 5.3. *Let* $\gamma : \widetilde{\Omega}O(X) \to P(X)$ *be a quasi-operation on* $\widetilde{\Omega}O(X)$. *Then, the following statements hold for any subset* A *of* X.
i) If X *is a* $\gamma(\delta\widetilde{\Omega})$*-regular space then,* $\gamma(\delta\widetilde{\Omega})cl(A) \subseteq \widetilde{\Omega}cl(A) \subseteq \delta cl(A)$ *for any set* A *of* X.
ii) The set X *is a* $\gamma(\delta\widetilde{\Omega})$*-regular space iff* $\widetilde{\Omega}O(X) \subseteq \gamma_{\delta\widetilde{\Omega}}O(X)$.
Proof. i) If $x \notin \widetilde{\Omega}cl(A)$, *then there exists* $U \in \widetilde{\Omega}O(X, x)$ *such that* $U \cap A = \emptyset$. *By the definition of* $\gamma(\delta\widetilde{\Omega})$*-regular space, there exists* $W \in \delta O(X, x)$ *such that* $\gamma(W) \subseteq U$ *and hence* $\gamma(W) \cap A \subseteq U \cap A = \emptyset$. *That is,* $\gamma(W) \cap A = \emptyset$. *Then,* $x \notin \gamma(\delta\widetilde{\Omega})cl(A)$. *Therefore,* $\gamma(\delta\widetilde{\Omega})cl(A) \subseteq \widetilde{\Omega}cl(A)$. *By the Proposition 2.7,* $\widetilde{\Omega}cl(A) \subseteq \delta cl(A)$. *Hence* $\gamma(\delta\widetilde{\Omega})cl(A) \subseteq \widetilde{\Omega}cl(A) \subseteq \delta cl(A)$.
ii) Assume that X *is a* $\gamma(\delta\widetilde{\Omega})$*-regular space. Let* $U \in \widetilde{\Omega}O(X)$ *and* $x \in U$ *be arbitrary. By hypothesis, there exists* $W \in \delta O(X, x)$ *such that* $\gamma(W) \subseteq U$ *and hence* $U \in \gamma_{\delta\widetilde{\Omega}}O(X)$. *Therefore,* $\widetilde{\Omega}O(X) \subseteq \gamma_{\delta\widetilde{\Omega}}O(X)$.
Conversely, assume that every $\widetilde{\Omega}$*-open set is a* $\gamma_{\delta\widetilde{\Omega}}$*-open set. Let* $x \in X$ *and* U *be an* $\widetilde{\Omega}$*-open set containing* x.
By the definition of $\gamma_{\delta\widetilde{\Omega}}$*-open set, there exists a* δ*-open set* V *containing* x *such that* $\gamma(V) \subseteq U$.

Proposition 5.4. *Let X be a $\gamma(\delta\tilde{\Omega})$-regular space with operation γ on $\tilde{\Omega}O(X)$. Then, $\gamma(\delta\tilde{\Omega})cl(A) = \tilde{\Omega}cl(A) = \delta cl(A)$ for any subset A of X.*

Proof. By the Proposition 5.3.(i), $\gamma(\delta\tilde{\Omega})cl(A) \subseteq \tilde{\Omega}cl(A) \subseteq \delta cl(A)$. By the Proposition 4.25.(i), $\delta cl(A) \subseteq \gamma(\delta\tilde{\Omega})cl(A)$. Now, $\gamma(\delta\tilde{\Omega})cl(A) \subseteq \tilde{\Omega}cl(A) \subseteq \delta cl(A) \subseteq \gamma(\delta\tilde{\Omega})cl(A)$. Therefore, $\gamma(\delta\tilde{\Omega})cl(A) = \tilde{\Omega}cl(A) = \delta cl(A)$.

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