Parameter Estimation of Dagum Distribution

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Abstract - *Dagum distribution is considered. Bayesian method of estimation is employed in order to estimate the shape parameter of Dagum distribution by using quasi and gamma priors. In this paper, the Bayes estimators of the shape parameter have been obtained under squared error, precautionary and weighted loss functions.*

Keywords - Dagum distribution, Bayesian method, quasi and gamma priors, squared error, precautionary and weighted loss functions.

I. INTRODUCTION

The Dagum distribution was proposed by Camilo Dagum [1]. This distribution is very useful to represent the distribution of income, actuarial data as well for survival analysis. Naqash et al. [2] consider the problem of Bayesian analysis of Dagum distribution for the complete case. An interesting aspect of Dagum distribution is that it admits a mixture representation in terms of generalized gamma and inverse Weibull distributions. The cumulative distribution function of Dagum distribution is given by

$$F(x;\theta) = \left[1 + cx^{-a}\right]^{-\theta} \quad ; x \ge 0,\ \theta > 0. \tag{1}$$

Therefore, the probability density function of Dagum distribution is given by

$$f(x;\theta) = ac\theta x^{-(a+1)}\left[1 + cx^{-a}\right]^{-(\theta+1)} \quad ; x \ge 0,\ \theta > 0. \tag{2}$$

The joint density function or likelihood function of (2) is given by

$$f(\underline{x};\theta) = (ac)^n \theta^n \left(\prod_{i=1}^{n} x_i^{-(a+1)}\right) e^{-(\theta+1)\sum_{i=1}^{n} log\left(1+cx_i^{-a}\right)}. \tag{3}$$

The log likelihood function is given by

$$log\ f(\underline{x};\theta) = n\, log\, ac + n\, log\,\theta + log\left(\prod_{i=1}^{n} x_i^{-(a+1)}\right) - (\theta+1)\sum_{i=1}^{n} log\left(1+cx_i^{-a}\right) \tag{4}$$

Differentiating (4) with respect to θ and equating to zero, we get the maximum likelihood estimator of θ which is given as

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} log\left(1+cx_i^{-a}\right)}. \tag{5}$$

II. BAYESIAN METHOD OF ESTIMATION

In Bayesian analysis the fundamental problem are that of the choice of prior distribution g (θ) and a loss function $L\left(\hat{\theta},\theta\right)$. The squared error loss function for the parameter θ are defined as

$$L\left(\hat{\theta},\theta\right)=\left(\hat{\theta}-\theta\right)^2. \tag{6}$$

The Bayes estimator under the above loss function, say, $\hat{\theta}_s$ is the posterior mean, i.e,

$$\hat{\theta}_S = E(\theta). \tag{7}$$

This loss function is often used because it does not lead to extensive numerical computations but several authors (Zellner [3], Basu and Ebrahimi [4]) have recognized that the inappropriateness of using symmetric loss function. J.G. Norstrom [5] introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss functions with quadratic loss function as a special case. A very useful and simple asymmetric precautionary loss function is given as

$$L\left(\hat{\theta},\theta\right)=\frac{\left(\hat{\theta}-\theta\right)^2}{\hat{\theta}}. \tag{8}$$

The Bayes estimator under precautionary loss function is denoted by $\hat{\theta}_P$ and is obtained by solving the following equation.

$$\hat{\theta}_P = \left[E\left(\theta^2\right)\right]^{1/2}. \tag{9}$$

Weighted loss function (Ahamad et al. [6]) is given as

$$L\left(\hat{\theta},\theta\right)=\frac{\left(\hat{\theta}-\theta\right)^2}{\theta}. \tag{10}$$

The Bayes estimator under weighted loss function is denoted by $\hat{\theta}_W$ and is obtained as

$$\hat{\theta}_W = \left[E\left(\frac{1}{\theta}\right)\right]^{-1}. \tag{11}$$

Let us consider two prior distributions of θ to obtain the Bayes estimators.

(i) Quasi-prior: For the situation where the experimenter has no prior information about the parameter θ, one may use the quasi density as given by

$$g_1(\theta)=\frac{1}{\theta^d}\ ;\ \theta>0,\ d\geq 0, \tag{12}$$

where $d = 0$ leads to a diffuse prior and $d = 1$, a non-informative prior.

(ii) Gamma prior: The most widely used prior distribution of θ is the gamma distribution with parameters α and $\beta\,(>0)$ with probability density function given by

$$g_2(\theta)=\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}\ ;\ \theta>0. \tag{13}$$

III. BAYES ESTIMATORS UNDER $g_1(\theta)$

The posterior density of θ under $g_1(\theta)$, on using (3), is given by

$$f(\theta/\underline{x}) = \frac{(ac)^n \theta^n \left(\prod_{i=1}^{n} x_i^{-(a+1)}\right) e^{-(\theta+1)\sum_{i=1}^{n}\log\left(1+cx_i^{-a}\right)} \theta^{-d}}{\int_0^\infty (ac)^n \theta^n \left(\prod_{i=1}^{n} x_i^{-(a+1)}\right) e^{-(\theta+1)\sum_{i=1}^{n}\log\left(1+cx_i^{-a}\right)} \theta^{-d}\, d\theta}$$

$$= \frac{\theta^{n-d}\, e^{-\theta\sum_{i=1}^{n}\log\left(1+cx_i^{-a}\right)}}{\int_0^\infty \theta^{n-d}\, e^{-\theta\sum_{i=1}^{n}\log\left(1+cx_i^{-a}\right)}\, d\theta}$$

$$= \frac{\left(\sum_{i=1}^{n}\log\left(1+cx_i^{-a}\right)\right)^{n-d+1}}{\Gamma(n-d+1)} \theta^{n-d}\, e^{-\theta\sum_{i=1}^{n}\log\left(1+cx_i^{-a}\right)} \qquad (14)$$

Theorem 1. Assuming the squared error loss function, the Bayes estimate of the shape parameter θ, is of the form

$$\hat{\theta}_S = (n-d+1)\left[\sum_{i=1}^{n}\log\left(1+cx_i^{-a}\right)\right]^{-1}. \qquad (15)$$

Proof. From equation (7), on using (14),

$$\hat{\theta}_S = E(\theta) = \int \theta\, f(\theta/\underline{x})\, d\theta$$

$$= \frac{\left(\sum_{i=1}^{n}\log\left(1+cx_i^{-a}\right)\right)^{n-d+1}}{\Gamma(n-d+1)} \int_0^\infty \theta^{n-d+1}\, e^{-\theta\sum_{i=1}^{n}\log\left(1+cx_i^{-a}\right)}\, d\theta$$

$$= \frac{\left(\sum_{i=1}^{n}\log\left(1+cx_i^{-a}\right)\right)^{n-d+1}}{\Gamma(n-d+1)} \frac{\Gamma(n-d+2)}{\left(\sum_{i=1}^{n}\log\left(1+cx_i^{-a}\right)\right)^{n-d+2}}$$

or, $$\hat{\theta}_S = (n-d+1)\left[\sum_{i=1}^{n}\log\left(1+cx_i^{-a}\right)\right]^{-1}.$$

Theorem 2. Assuming the precautionary loss function, the Bayes estimate of the shape parameter θ, is of the form

$$\hat{\theta}_P = \left[(n-d+2)(n-d+1)\right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} \log\left(1+cx_i^{-a}\right)\right]^{-1} \tag{16}$$

Proof. From equation (9), on using (14),

$$\left(\hat{\theta}_P\right)^2 = E\left(\theta^2\right) = \int \theta^2 f\left(\theta/\underline{x}\right) d\theta$$

$$= \frac{\left(\sum_{i=1}^{n} \log\left(1+cx_i^{-a}\right)\right)^{n-d+1}}{\Gamma(n-d+1)} \int_0^{\infty} \theta^{n-d+2} e^{-\theta \sum_{i=1}^{n} \log\left(1+cx_i^{-a}\right)} d\theta$$

$$= \frac{\left(\sum_{i=1}^{n} \log\left(1+cx_i^{-a}\right)\right)^{n-d+1}}{\Gamma(n-d+1)} \frac{\Gamma(n-d+3)}{\left(\sum_{i=1}^{n} \log\left(1+cx_i^{-a}\right)\right)^{n-d+3}}$$

$$= \frac{(n-d+2)(n-d+1)}{\left(\sum_{i=1}^{n} \log\left(1+cx_i^{-a}\right)\right)^{2}}$$

$$\Rightarrow \qquad \hat{\theta}_P = \left[(n-d+2)(n-d+1)\right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} \log\left(1+cx_i^{-a}\right)\right]^{-1}.$$

Theorem 3. Assuming the weighted loss function, the Bayes estimate of the shape parameter θ, is of the form

$$\hat{\theta}_W = (n-d)\left[\sum_{i=1}^{n} \log\left(1+cx_i^{-a}\right)\right]^{-1}. \tag{17}$$

Proof. From equation (11), on using (14),

$$\hat{\theta}_W = \left[E\left(\frac{1}{\theta}\right)\right]^{-1} = \left[\int \frac{1}{\theta} f\left(\theta/\underline{x}\right) d\theta\right]^{-1}$$

$$= \left[\frac{\left(\sum_{i=1}^{n} \log\left(1+cx_i^{-a}\right)\right)^{n-d+1}}{\Gamma(n-d+1)} \int_0^{\infty} \theta^{n-d-1} e^{-\theta \sum_{i=1}^{n} \log\left(1+cx_i^{-a}\right)} d\theta\right]^{-1}$$

$$= \left[\frac{\left(\sum_{i=1}^{n} \log\left(1+cx_i^{-a}\right)\right)^{n-d+1}}{\Gamma(n-d+1)} \frac{\Gamma(n-d)}{\left(\sum_{i=1}^{n} \log\left(1+cx_i^{-a}\right)\right)^{n-d}}\right]^{-1}$$

$$= \left[\frac{\sum_{i=1}^{n} \log\left(1 + c x_i^{-a}\right)}{n - d} \right]^{-1}$$

or, $$\hat{\theta}_W = (n - d)\left[\sum_{i=1}^{n} \log\left(1 + c x_i^{-a}\right) \right]^{-1}.$$

IV. BAYES ESTIMATORS UNDER $g_2(\theta)$

Under $g_2(\theta)$, the posterior density of θ, using equation (3), is obtained as

$$f(\theta/\underline{x}) = \frac{(ac)^n \theta^n \left(\prod_{i=1}^{n} x_i^{-(a+1)}\right) e^{-(\theta+1)\sum_{i=1}^{n} \log\left(1+c x_i^{-a}\right)} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}}{\int_0^\infty (ac)^n \theta^n \left(\prod_{i=1}^{n} x_i^{-(a+1)}\right) e^{-(\theta+1)\sum_{i=1}^{n} \log\left(1+c x_i^{-a}\right)} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} d\theta}$$

$$= \frac{\theta^{n+\alpha-1} e^{-\left(\beta + \sum_{i=1}^{n} \log\left(1+c x_i^{-a}\right)\right)\theta}}{\int_0^\infty \theta^{n+\alpha-1} e^{-\left(\beta + \sum_{i=1}^{n} \log\left(1+c x_i^{-a}\right)\right)\theta} d\theta}$$

$$= \frac{\theta^{n+\alpha-1} e^{-\left(\beta + \sum_{i=1}^{n} \log\left(1+c x_i^{-a}\right)\right)\theta}}{\Gamma(n+\alpha) \Big/ \left(\beta + \sum_{i=1}^{n} \log\left(1 + c x_i^{-a}\right)\right)^{n+\alpha}}$$

$$= \frac{\left(\beta + \sum_{i=1}^{n} \log\left(1 + c x_i^{-a}\right)\right)^{n+\alpha}}{\Gamma(n+\alpha)} \theta^{n+\alpha-1} e^{-\left(\beta + \sum_{i=1}^{n} \log\left(1+c x_i^{-a}\right)\right)\theta} \tag{18}$$

Theorem 4. Assuming the squared error loss function, the Bayes estimate of the shape parameter θ, is of the form

$$\hat{\theta}_S = (n + \alpha)\left[\beta + \sum_{i=1}^{n} \log\left(1 + c x_i^{-a}\right) \right]^{-1}. \tag{19}$$

Proof. From equation (7), on using (18),

$$\hat{\theta}_S = E(\theta) = \int \theta f(\theta/\underline{x}) d\theta$$

$$= \frac{\left(\beta + \sum_{i=1}^{n} log\left(1 + cx_i^{-a}\right)\right)^{n+\alpha}}{\Gamma(n+\alpha)} \int_0^\infty \theta^{n+\alpha} e^{-\left(\beta + \sum_{i=1}^{n} log\left(1+cx_i^{-a}\right)\right)\theta} d\theta$$

$$= \frac{\left(\beta + \sum_{i=1}^{n} log\left(1 + cx_i^{-a}\right)\right)^{n+\alpha}}{\Gamma(n+\alpha)} \frac{\Gamma(n+\alpha+1)}{\left(\beta + \sum_{i=1}^{n} log\left(1 + cx_i^{-a}\right)\right)^{n+\alpha+1}}$$

or, $$\hat{\theta}_S = (n+\alpha)\left[\beta + \sum_{i=1}^{n} log\left(1 + cx_i^{-a}\right)\right]^{-1}.$$

Theorem 5. Assuming the precautionary loss function, the Bayes estimate of the shape parameter θ, is of the form

$$\hat{\theta}_P = \left[(n+\alpha+1)(n+\alpha)\right]^{\frac{1}{2}} \left[\beta + \sum_{i=1}^{n} log\left(1 + cx_i^{-a}\right)\right]^{-1}. \tag{20}$$

Proof. From equation (9), on using (18),

$$\left(\hat{\theta}_P\right)^2 = E\left(\theta^2\right) = \int \theta^2 f\left(\theta / \underline{x}\right) d\theta$$

$$= \frac{\left(\beta + \sum_{i=1}^{n} log\left(1 + cx_i^{-a}\right)\right)^{n+\alpha}}{\Gamma(n+\alpha)} \int_0^\infty \theta^{n+\alpha+1} e^{-\left(\beta + \sum_{i=1}^{n} log\left(1+cx_i^{-a}\right)\right)\theta} d\theta$$

$$= \frac{\left(\beta + \sum_{i=1}^{n} log\left(1 + cx_i^{-a}\right)\right)^{n+\alpha}}{\Gamma(n+\alpha)} \frac{\Gamma(n+\alpha+2)}{\left(\beta + \sum_{i=1}^{n} log\left(1 + cx_i^{-a}\right)\right)^{n+\alpha+2}}$$

$$= \frac{(n+\alpha+1)(n+\alpha)}{\left(\beta + \sum_{i=1}^{n} log\left(1 + cx_i^{-a}\right)\right)^{2}}$$

or, $$\hat{\theta}_P = \left[(n+\alpha+1)(n+\alpha)\right]^{\frac{1}{2}} \left[\beta + \sum_{i=1}^{n} log\left(1 + cx_i^{-a}\right)\right]^{-1}.$$

Theorem 6. Assuming the weighted loss function, the Bayes estimate of the shape parameter θ, is of the form

$$\hat{\theta}_W = (n+\alpha-1)\left[\beta + \sum_{i=1}^{n} log\left(1 + cx_i^{-a}\right)\right]^{-1} \tag{21}$$

Proof. From equation (11), on using (18),

$$\hat{\theta}_W = \left[E\left(\frac{1}{\theta}\right)\right]^{-1} = \left[\int \frac{1}{\theta} f\left(\theta / \underline{x}\right) d\theta\right]^{-1}$$

$$= \left[\frac{\left(\beta + \sum_{i=1}^{n} log\left(1 + cx_i^{-a}\right)\right)^{n+\alpha}}{\Gamma(n+\alpha)} \int_0^{\infty} \theta^{n+\alpha-2} e^{-\left(\beta + \sum_{i=1}^{n} log\left(1+cx_i^{-a}\right)\right)\theta} d\theta\right]^{-1}$$

$$= \left[\frac{\left(\beta + \sum_{i=1}^{n} log\left(1 + cx_i^{-a}\right)\right)^{n+\alpha}}{\Gamma(n+\alpha)} \frac{\Gamma(n+\alpha-1)}{\left(\beta + \sum_{i=1}^{n} log\left(1 + cx_i^{-a}\right)\right)^{n+\alpha-1}}\right]^{-1}$$

$$= \left[\frac{\beta + \sum_{i=1}^{n} log\left(1 + cx_i^{-a}\right)}{n+\alpha-1}\right]^{-1}$$

or, $$\hat{\theta}_W = (n+\alpha-1)\left[\beta + \sum_{i=1}^{n} log\left(1 + cx_i^{-a}\right)\right]^{-1}.$$

V. CONCLUSION

In this paper, we have obtained a number of estimators of parameter of Dagum distribution. In equation (15), (16) and (17) we have obtained the Bayes estimators under squared error, precautionary and weighted loss functions using quasi prior. In equation (19), (20) and (21) we have obtained the Bayes estimators under squared error, precautionary and weighted loss functions using gamma prior. In the above equation, it is clear that the Bayes estimators depend upon the parameters of the prior distribution.

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