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Fractional Reduced Differential Transform Method To Analytical Solution Of Fractional Order Biological Population Model

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Abstract - *In this paper, we obtain analytical solution to the non-linear fractional order biological population model by using fractional reduced differential transform method. We presented some examples are provided to check the effectiveness, accuracy and performance of proposed work. The results and figure show that the proposed method is very convenient.*

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Keywords:, *Biological population model, Caputo derivative, Initial value problem , Mittag-Leffler function, Reduced differential transform method .*

1. INTRODUCTION

Nowadays there is increasing attention paid to fractional order differential equations and their broad applications in mathematics, physics and engineering [2],[3],[11],[12],[13],[17] such as anomalous transport disordered system, earthquake modeling the diffusion problem, the phenomena in electromagnetic acoustic viscoelasticity etc. have been widely spread in the recent years. In the present paper, fractional reduced differential transform method is considered. This method for the fractional order differential equations provides the analytical solutions for both the linear and non-linear fractional order differential equations in the form of power series. The method was presented by Keskin and Oturanc [6], Srivastava V.K.et al. [16] and they applied the fractional reduced differential transform method to non-linear fractional order differential equations.

We extend the fractional reduced differential transform method to time fractional-order biological population model. The representive generalized time fractional-ordered nonlinear biological population diffusion equation is given as

$$D_t^{\alpha}u(x,y,t)=\left(u^2(x,y,t)\right)_{xx}+\left(u^2(x,y,t)\right)_{yy}+f\left(u(x,y,t)\right) \qquad (1.1)$$

with the initial condition

$$u(x,y,0)=f(x,y), \qquad (1.2)$$

where $t>0, x,y\in R$ (set of real numbers) and $u(x,y,t)$ denotes the population density and $f(u)$ represents the population supply due to births and deaths. Also, $f(u)=hu^{a}\left(1-ru^{b}\right)$ with h,a,r,b are real numbers. If we take special value of numbers then

(i) For $f(u) = cu,$ c (constant), we get *Malthusian Law.*

(ii) For $f(u) = c_1 u - c_2 u^2$ where c_1, c_2 are positive constants, we get *Verhulst Law.*

(iii) For $f(u) = -cu^p, (c \geq 0, 0 < p < 1)$ we get *Parous Media.*

2. Fractional Calculus

In this section, we present some formulae and properties of fractional integrals and fractional derivatives which are useful in our further investigations (see Kilbas, Srivastava and Trujillo [5])

Riemann- Liouville Fractional Integral and Fractional Derivative

Let $\Omega = [a,b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$. The *Riemann-Liouville fractional integral* $I_a^\alpha f$ of order $\alpha \in \mathbb{C}$, $(\Re e(\alpha) > 0)$ is defined by (2.1)

$$\left(I_a^\alpha f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}} \quad (x > a;\ \Re e(\alpha) > 0).$$

The *Riemann-Liouville fractional derivative* $D_a^\alpha f$ of order $\alpha \in \mathbb{C}$, $(\Re e(\alpha) > 0)$ is defined by

$$\left(D_a^\alpha f\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_a^{n-\alpha} f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{f(t)\,dt}{(x-t)^{\alpha-n+1}}, x > a, \tag{2.2}$$

$$\left(n = \{\Re e(\alpha)\} + 1;\ x > a\right),$$

where $\{\Re e(\alpha)\}$ means the integral part of $\Re e(\alpha)$ and $\Gamma(.)$ is the well-know gamma functions.

The fractional derivative in the Caputo's sense is defined

$${}_a^C D_x^\alpha f(x) = {}_a I_x^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} \left(\frac{d}{dt}\right)^n f(t)\,dt,$$

(2,3)

where $n-1 < \alpha < n,\ \alpha > 0, t > 0,\ n \in \mathbb{N}$.

Property. For $f(x) = A$ (constant) (2.3) reduces to

$${}_a^C D_x^\alpha A = 0. \tag{2.4}$$

while as for $f(x) = A$ the *Riemann-Liouville fractional derivative* $D_a^\alpha f$ from (2.2) is reduces to

$${}_0 D_x^\alpha A = \frac{At^{-\alpha}}{\Gamma(1-\alpha)}, \alpha \neq 1, 2, 3.... . \tag{2.5}$$

3. Fractional Reduced Differential transforms method

The fractional reduced differential transform method [4], [6], [16] is applicable to solve a large class of nonlinear and linear problems with approximation that convert rapidly to the exact solutions.

Definition 3.1 Let the *k*th derivative of function $f(x,t)$ in two variables be an analytic function that continuously differentiable with respect t and x in domain of integers and exist in some neighborhood of $t=a$,then

$$F_k(x)=\frac{1}{\Gamma(k\alpha+1)}\left[\left(\frac{\partial^{\alpha k}}{\partial t^{\alpha k}}\right)f(x,t)\right]_{t=a}, \tag{3.1}$$

where α is a parameter describing the order of the time-fractional derivative and t

dimensional spectrum function $F_k(x)$ (the fractional reduced transformed function of original function $f(x,t)$).

The differential inverse transform of $F_k(x)$ is defined as follows

$$f(x,t)=\sum_{k=0}^{\infty}F_k(x)(x-a)^{\alpha k}. \tag{3.2}$$

Eqn. (3.2) is known as the Taylor series expansion of $f(x,t)$ at $t=a$, using (3.1) and (3.2), we get

$$f(x,t)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\left(\frac{\partial^{\alpha k}}{\partial t^{\alpha k}}\right)f(x,t)\right]_{t=a}(x-a)^{\alpha k}. \tag{3,3}$$

When $a=0,$ (3.3) reduces to

$$f(x,t)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\alpha+1)}\left[\left(\frac{\partial^{\alpha k}}{\partial t^{\alpha k}}\right)f(x,t)\right]_{t=0}x^{\alpha k},$$

$$=\sum_{k=0}^{\infty}F_k(x)x^{\alpha k}. \tag{3.4}$$

From the above definition it is easily possible to verify that the concept of the fractional reduced differential transform method is obtained from the power series expansion of a function. Some basic properties of the fractional reduced differential transform method

are introduced blow.

Theorem 3.2 If $f(x,t)=\lambda_1 f_1(x,t)\pm\lambda_2 f_2(x,t)$, then $F_k(x)=\lambda_1 F_{1k}(x)\pm\lambda_2 F_{2k}(x)$, where λ_1,λ_2 are constant.

Theorem 3.3 If $f(x,t)=g(x,t)h(x,t)$, then $F_k(x)=\sum_{n=0}^{k}G_{k-n}(x)H_n(x)$.

Theorem 3.4 If $f(x,t)=D_t^{N\alpha}g(x,t)$ then $F_k(x)=\frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}G_{k+n}(x)$.

4. Analysis of Method by Some Numerical Examples

In this section, we solve our problem stated in (1.1) and (1.2) through some examples with analytical solution to show the accuracy and efficiency of the fractional reduced differential transform method described in the previous section-3.

Example-1 If we take $a=1,b=1$, in (1.1), we have the following fractional biological population equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2}+hu(1-ru), \qquad (\textit{Verhulst law}) \tag{4.1}$$

with the initial condition

$$u_0=e^{\sqrt{\frac{hr}{8}}(x+y)}. \tag{4.2}$$

With the help of the basic properties of the fractional reduced differential transform method given in the section-3, linear time fractional-order biological population equation (4.1) with the initial condition (4.2) may be written as

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x,y)=\frac{\partial^2}{\partial x^2}\left[\sum_{m=0}^{k}U_m(x,y)U_{k-m}(x,y)\right]+\frac{\partial^2}{\partial y^2}\left[\sum_{m=0}^{k}U_m(x,y)U_{k-m}(x,y)\right] \quad (4.3)$$

$$+\; hU_k(x,y)-hr\sum_{m=0}^{k}U_m(x,y)U_{k-m}(x,y),$$

where $U_k(x,y)$ fractional reduced transformed function of original function $u(x,y,t)$ defined in (3.1).

An appeal to (4.2), show that

$$U_0(x,y)=e^{\sqrt{\frac{hr}{8}}(x+y)}, \quad (4,4)$$

$$U_1(x,y)=\frac{h}{\Gamma(1+\alpha)}e^{\sqrt{\frac{hr}{8}}(x+y)}, \quad (4,5)$$

$$U_2(x,y)=\frac{h^2}{\Gamma(1+2\alpha)}e^{\sqrt{\frac{hr}{8}}(x+y)}, \quad (4.6)$$

$$U_k(x,y)=\frac{h^k}{\Gamma(1+k\alpha)}e^{\sqrt{\frac{hr}{8}}(x+y)}, \quad (4.7)$$

Using the differential inverse transform of $U_k(x)$ defined in 3.4), then the approximate solution in a series form is

$$u(x,y,t)=\sum_{n=0}^{\infty}U_k(x,y)t^{n\alpha} \quad = e^{\sqrt{\frac{hr}{8}}(x+y)}\sum_{n=0}^{\infty}\frac{(ht^{\alpha})^n}{\Gamma(1+n\alpha)} \quad = e^{\sqrt{\frac{hr}{8}}(x+y)}E_\alpha(ht^\alpha), \quad (4.8)$$

where $E_\alpha(.)$ is Mittag-Leffler function. Eqn (4.8) can be written in term of the *H*-function as

$$u(x,y,t)=e^{\sqrt{\frac{hr}{8}}(x+y)}H_{1\,2}^{1\,1}\left[-ht^{\alpha}:\begin{matrix}(0,1)\\(0,1)\quad(0,\alpha)\end{matrix}\right] \quad (4.9)$$

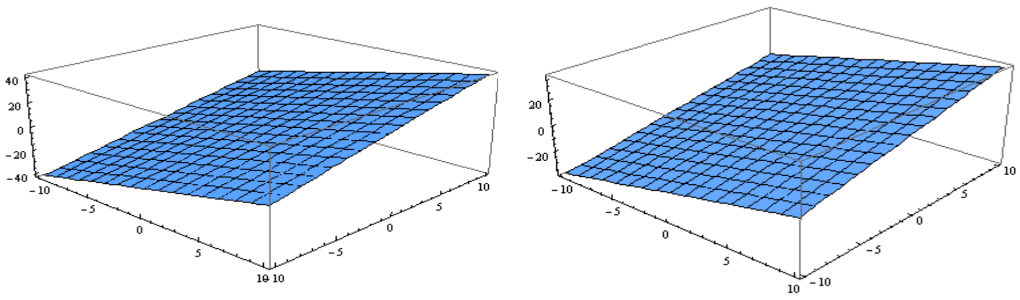
where $h>0,\alpha>0.$

For $\alpha\to1$, we have

$$u(x,y,t)=e^{\sqrt{\frac{hr}{8}}(x+y)+ht}, \quad (4.10)$$

which is an exact solution of the integer order biological population.

(a) **(b)**



The surfaces (a) and (b) show the solution of $u(x,y,t)$ for problem (4.1) and (4.2): **(a)**, $\alpha=1$; **(b)** $\alpha=0.95$, when $h=0.1,\ \mathbf{t}=10$.

Now we consider the mittag-leffler function of rational order $\alpha=\frac{p}{q}$, with $p=1, q=2,3....$ relative prime then we have (see Mathai and Haubold [11, p. 84, Eq. (2.2.11)]

$$e^{ht+\sqrt{\frac{hr}{8}}(x+y)}\left[1+\sum_{n=1}^{q-1}\frac{\gamma\left(1-\frac{n}{q},ht\right)}{\Gamma\left(1-\frac{n}{q}\right)}\right]. \tag{4.11}$$

$\gamma(\alpha,z)$ is the incomplete gamma function, defined by $\gamma(\alpha,z)=\int_0^z e^{-u}u^{\alpha-1}du.$

Example-2 If we take $a=1, r=0$, in (1.1), then we have the following fractional biological population equation,

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2}+hu\ , \tag{4.12}$$

with the initial condition

$$u_0=\sqrt{xy}\ . \tag{4.13}$$

Applying the reduced differential transform method to Eqn. (4.12), we get

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x,y)=\frac{\partial^2}{\partial x^2}\left[\sum_{m=0}^{k}U_m(x,y)U_{k-m}(x,y)\right]+\frac{\partial^2}{\partial y^2}\left[\sum_{m=0}^{k}U_m(x,y)U_{k-m}(x,y)\right]+hU_k(x,y). \quad (4,14)$$

An appeal to (4.13), show that

$$U_0(x,y)=\sqrt{xy} \tag{4.15}$$

$$U_1(x,y)=\frac{h}{\Gamma(1+\alpha)}\sqrt{xy}, \tag{4.16}$$

$$U_2(x,y)=\frac{h^2}{\Gamma(1+2\alpha)}\sqrt{xy}, .., \tag{4.17}$$

$$U_k(x,y)=\frac{h^k}{\Gamma(1+k\alpha)}\sqrt{xy}. \tag{4.18}$$

Using the differential inverse transform of $U_k(x)$ defined in (3.4), then the approximate solution in a series form is

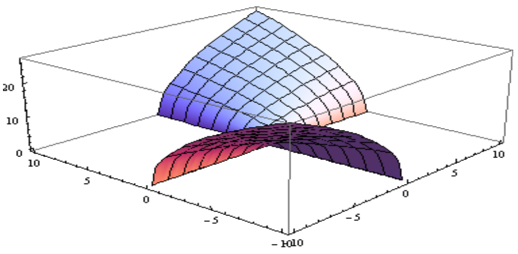
$$u(x,y,t)=\sum_{n=0}^{\infty}U_k(x,y)t^{n\alpha}\ =\sqrt{xy}\sum_{n=0}^{\infty}\frac{\left(ht^\alpha\right)^n}{\Gamma(1+n\alpha)}\ =\sqrt{xy}E_\alpha\left(ht^\alpha\right). \tag{4.19}$$

For $\alpha\to 1$, we have

$$u(x,y,t)=\sqrt{xy}e^{ht}, \tag{4.20}$$

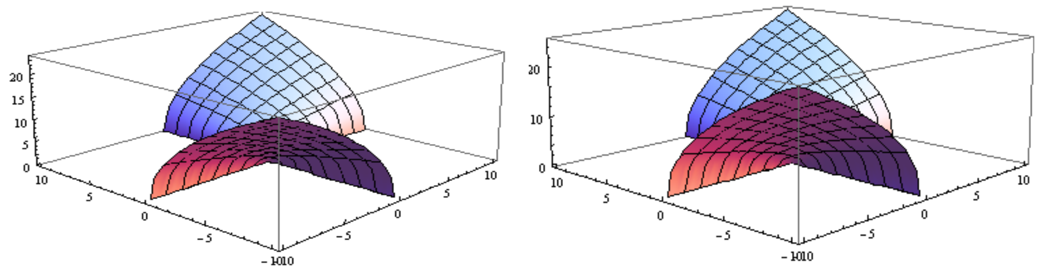
which is an exact solution of the integer order biological population.

(a)



The surface (a) show the exact solution of $u(x,y,t)$ for (4.20), **(a)** $\alpha=1$.

(b) **(c)**



The surfaces (b) and (c) show the approximate solution of $u(x,y,t)$ for (4.19): **(b)** $\alpha=0.90$, **(c)** $\alpha=0.95$, when $h=0.1,\ \mathrm{t}=10$.

Example-3 If we take $a=1, r=0, h=1$, (1.1), then we have the following fractional biological population equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2}+u\,, \tag{4.21}$$

with the initial condition

$$u_0=\sqrt{\sin x \sinh y}\,. \tag{4.22}$$

Applying the reduced differential transform method to Eqn. (4.21), we get

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x,y)=\frac{\partial^2}{\partial x^2}\left[\sum_{m=0}^{k}U_m(x,y)U_{k-m}(x,y)\right]+\frac{\partial^2}{\partial y^2}\left[\sum_{m=0}^{k}U_m(x,y)U_{k-m}(x,y)\right]+U_k(x,y) \quad (4.23)$$

An appeal to (4.22), show that

$$U_0(x,y)=\sqrt{\sin x \sinh y} \tag{4.24}$$

,

$$U_1(x,y)=\frac{1}{\Gamma(1+\alpha)}\sqrt{\sin x \sinh y}, \quad (4,25)$$

$$U_1(x,y)=\frac{1}{\Gamma(1+\alpha)}\sqrt{\sin x \sinh y}, \tag{4.26}$$

$$U_2(x,y)=\frac{1}{\Gamma(1+2\alpha)}\sqrt{\sin x \sinh y}, \tag{4.27}$$

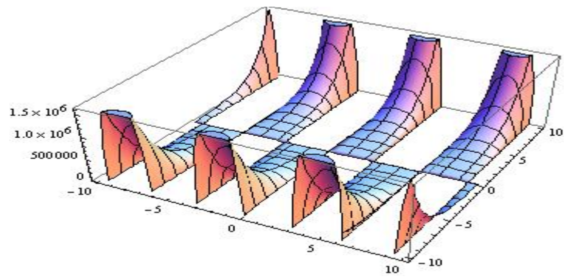
Using the differential inverse transform of $U_k(x)$ defined in Eqn. (3.4), then the approximate solution in a series form is

$$u(x,y,t)=\sum_{n=0}^{\infty}U_k(x,y)t^{n\alpha}=\sqrt{\sin x\sinh y}\sum_{n=0}^{\infty}\frac{\left(t^{\alpha}\right)^{n}}{\Gamma(1+n\alpha)}=\sqrt{\sin x\sinh y}E_{\alpha}\left(t^{\alpha}\right). \qquad (4,28)$$

For $\alpha\to 1$, we have

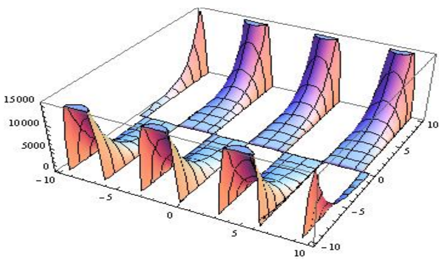
$$u(x,y,t)=\left(\sqrt{\sin x\sinh y}\right)e^{t}. \qquad (4,29)$$

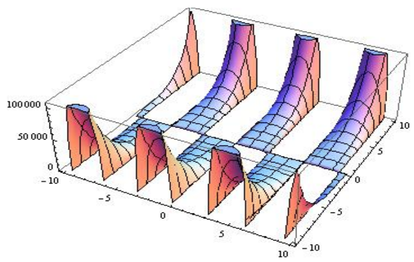
(a)



The surface (a) show the exact solution of $u(x,y,t)$ for (4.29), (a) $\alpha=1$.

(b) **(c)**





The surfaces (b) and (c) show the approximate solution of $u(x,y,t)$ for (4.28): **(b)** $\alpha=0.90$, **(c)** $\alpha=0.95$, when $h=1,\ \mathbf{t}=10$.

CONCLUSION

We employ the fractional reduced differential transform method to the non-linear fractional order biological population subject to some initial conditions. The results of some examples show that the results are identical to the results obtained through HPM, VIM and ADM. The corresponding solutions and figures are obtained according to results using *Mathematica*.

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