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Side Length of the Extension of Napoleon's Outer Theorem on Triangle

Silvia Riani, Mashadi*, Leli Deswita

Department of Mathematics,University of Riau, Pekanbaru, 28293, Indonesia

Abstract — *This paper discusses the side lengths of the development of Napoleon's outer theorem on triangles by applying the special case Miquel's theorem. If in Napoleon's theorem the outer triangle uses an equilateral triangle, in this paper it has been modified by using isosceles right triangles as the outer triangle. The new triangle formed has a shape similar to the original triangle. Furthermore, determination of the side length of the new triangle formed utilizes the Pythagorean formula and cosine rule.*

Keywords — *Miquel point, Napoleon's theorem, outer Napoleon.*

I. INTRODUCTION

Napoleon's theorem on a triangle was discovered by Napoleon Bonaparte (1789-1821). After four years of Napoleon's death, this theorem was first published by W. Rutherford in the New Mathematical Question [2] [8] [10] [16]. Napoleon's theorem states that if an equilateral triangle is constructed on any side of any triangle, then the three centers of the equilateral triangle will form a new equilateral triangle [7] [13] [14]. Napoleon's theorem in triangles can be proved by geometry and trigonometric algebra [2] [6] [15].

Several developments related to Napoleon's Theorem have been made by several authors [6] [7] [8]. Furthermore, in [12] [13] [14] it is discussed the development of Napoleon's theorem on rectangles that point outward and inward, provided that it only applies to rectangles having two pairs of parallel sides such as a square, a rhombus, a rectangle, and a parallelogram. The proof is done by using the concept of congruence and trigonometry [2] [15]. Furthermore, there is also paper that discusses the development of Napoleon's theorem on the hexagon. The results of his research also show that if a hexagon with three pairs of opposite sides which is the same length as a regular hexagon on each side, then the diagonal points of the six hexagons, if connected, will form a new flat hexagon [16].

Seeing the development of Napoleon's theorem, authors are interested in looking for other developments of Napoleon's theorem on the triangle. In this paper, the authors develop Napoleon's outer theorem on triangles. If in Napoleon's theorem the outer triangle uses an equilateral triangle, the author modifies it by using an isosceles right triangle. Using the special case of Miquel's theorem yields the Miquel points on each of the outward-facing right triangles. When these points are connected, it will form a new triangle similar to the original triangle.

II. NAPOLEON'S THEOREM AND MIQUEL'S POINTS

In the field of geometry, among the theorems that discuss triangles is the Napoleon's theorem. Napoleon's theorem states that if an equilateral triangle is constructed on each side of an arbitrary triangle, whether built in or out, then from the three central points of an equilateral triangle a new equilateral triangle is formed [3] [5] [7] [10]. The following is stated in Theorem 2.1 and Figure 1.

Theorem 2.1 Given ΔABC is any triangle. On each side of ΔABC, the equilateral triangles ΔABD, ΔACE and ΔBCF are constructed pointing outward. Let P, Q and R be the respective centers of the equilateral triangle constructed. If the three central points are connected, an equilateral triangle will be formed.

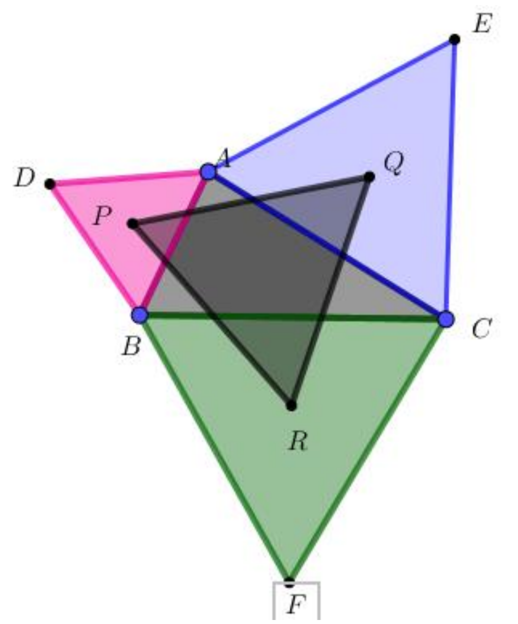

Fig 1. Napoleon's Theorem on a Triangle

Miquel's theorem is one of the theorems studied in the field of geometry. This theorem was originally applied to triangles discussed by Walles [1] [11] in 1799 and later proved by Miquel in 1838. A Miquel point is constructed on each triangle then any point is selected on each side, then the circle passes through each point. The points that are on adjacent sides intersect at a point, namely the Miquel point [4] [9] [11]. The following is stated in Theorem 2.2 and Figure 2.

Theorem 2.2. Given ΔABC is any triangle, if any point is chosen on each side, namely point D on side AB, point E on side AC and point F on side BC, then the circle passing through the adjacent points will intersect at one point, namely the Miquel point (M).

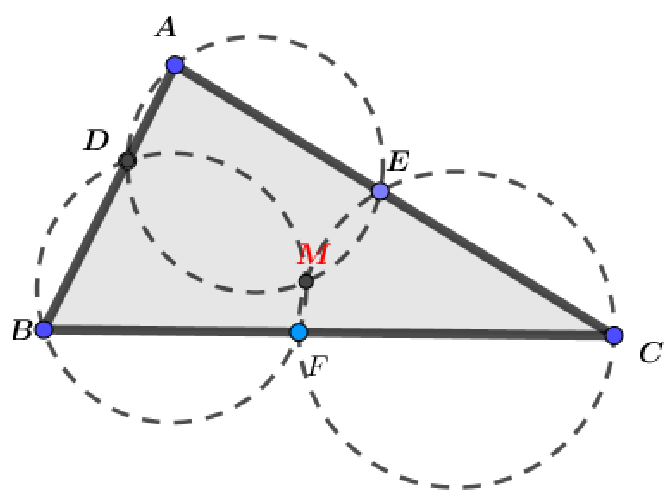

Fig 2. Miquel's Points on Triangle

III. SIDE LENGTH OF DEVELOPMENT OUTER NAPOLEON THEOREM ON TRIANGLE

In this paper the authors discuss the determination of the side lengths in the development of Napoleon's outer theorem on any triangle, isosceles triangle and equilateral triangle. By using the special case in Miquel's theorem, if the Miquel points on each of the isosceles right triangles are connected, a triangle $M_1M_2M_3$ will be formed as described in Theorem 3.1 and illustrated in Figure 3.

Theorem 3.1. If on any side of triangle ΔABC, isosceles right triangles are constructed, namely ΔABD on the AB side, ΔAEC on the AC side and ΔBCF on the BC side pointing outward, then each side of the triangle is divided into two, then when the Miquel points on each of the isosceles right triangles are connected to form any triangle $M_1M_2M_3$. Then the length of the sides is

$$M_1M_2 = \sqrt{\tfrac{1}{2}(b^2+c^2) - bc.\cos\,(90^0+\alpha)}$$
$$M_2M_3 = \sqrt{\tfrac{1}{2}(c^2+a^2) - ca.\cos\,(90^0+\beta)}$$
$$M_1M_3 = \sqrt{\tfrac{1}{2}(a^2+b^2) - ab.\cos\,(90^0+\gamma)}.$$

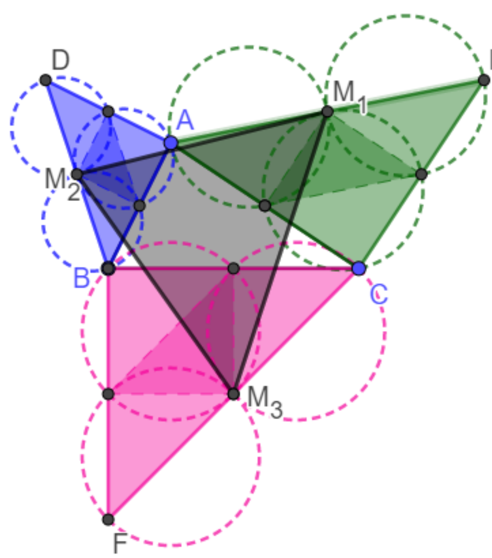

Fig 3. Development of the Outer Napoelon Theorem on Any Triangle Divided by Two Each Side

PROOF. Proving the lengths of M_1M_2, M_2M_3 dan M_1M_3 make uses the concepts of trigonometry and similarity. ΔABC is any triangle so that AB=c BC=a AC=b. ΔABD is an isosceles right triangle at point A so that $AB = AD$, then it is obtained $BD = c\sqrt{2}$. The point M_2 is the Miquel point and the midpoint of the side BD so that the length of $DM_2 = \frac{c\sqrt{2}}{2}$. ΔAM_2B is an isosceles right triangle at point M_2. Because $\Delta AM_2B \cong \Delta AM_2D$ so that $DM_2 = BM_2 = AM_2 = \frac{c\sqrt{2}}{2}$. ΔAEC is an isosceles right triangle at point C so that $AC = CE$, then it yields $AE = b\sqrt{2}$. The point M_1 is the Miquel point and the midpoint of the side AE so that the length of $AM_1 = \frac{b\sqrt{2}}{2}$. ΔAM_1C is the isosceles right triangle at point M_1. Because $\Delta AM_1C \cong \Delta EM_1C$ so that $AM_1 = EM_1 = CM_1 = \frac{b\sqrt{2}}{2}$. ΔBCF is is an isosceles right triangle at point B so that $BC = BF$, then it is obtained $CF = a\sqrt{2}$. The point M_3 is the Miquel point and the midpoint of the CF side so that the length $CM_3 = \frac{a\sqrt{2}}{2}$. ΔBM_3C is an isosceles right triangle at point M_3. Because $\Delta BM_3C \cong \Delta BM_3F$ so that $CM_3 = FM_3 = BM_3 = \frac{a\sqrt{2}}{2}$.

Next to determine $\angle M_2AM_1$, suppose $\angle BAC = \alpha, \angle ABC = \beta$ and $\angle BCA = \gamma$. $\Delta M_2AB \sim \Delta M_1AC \sim \Delta M_3BC$ so that $\angle M_2AB = \angle M_1AC = \angle M_3BC = 45^0$,

$$\begin{aligned}\cos\angle M_2AM_1 &= \cos(\angle M_2AB + \angle ABC + \angle CBM_3)\\ &= \cos(45^0 + \alpha + 45^0)\\ &= \cos(90^0+\alpha)\end{aligned}$$

Using the same method gives

$$\cos\angle M_2BM_3 \quad = \cos(90^0+\beta)$$
$$\cos\angle M_1CM_3 \quad = \cos(90^0+\gamma)$$

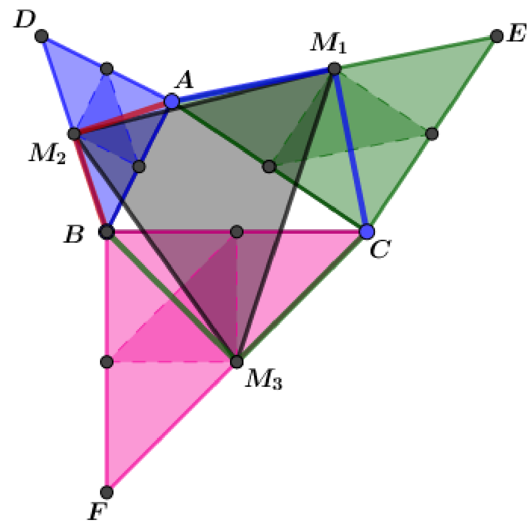

Fig 4. Triangle $M_1M_2M_3$ on Development of the Napoelon's Outer Theorem on Any Triangle Divided by Two Each Side

Furthermore, using the cosine rule, the evaluation of the length of M_1M_2, M_2M_3 dan M_1M_3 as in Figure 4 yields

$$(M_1M_2)^2 = (AM_1)^2+(AM_2)^2-2.AM_1.AM_2\cos\ (2x+\alpha)$$
$$= \left(\frac{b\sqrt{2}}{2}\right)^2+\left(\frac{c\sqrt{2}}{2}\right)^2-2\frac{b\sqrt{2}}{2}.\frac{c\sqrt{2}}{2}.\cos\ (90^0+\alpha)$$
$$(M_1M_2)^2 = \frac{1}{2}(b^2+c^2)-bc.\cos\ (90^0+\alpha)$$
$$M_1M_2 = \sqrt{\frac{1}{2}(b^2+c^2)-bc.\cos\ (90^0+\alpha)}\ .$$

Using the same method produces

$$M_2M_3 = \sqrt{\frac{1}{2}(c^2+a^2)-ca.\cos\ (90^0+\beta)}$$
$$M_1M_3 = \sqrt{\frac{1}{2}(a^2+b^2)-ab.\cos\ (90^0+\gamma)}$$

From the evidence obtained, it can be seen that $M_1M_3 \neq M_2M_3 \neq M_1M_2$ so that it is proven that $\Delta M_1M_2M_3$ is any riangle. ∎

Corrolarry 3.2. If on each side of triangle ΔABC is isosceles, isosceles right triangles are constructed, namely ΔABD on the *AB* side, ΔAEC on the *AC* side and ΔBCF on the *BC* side pointing outward, then each side of the triangle is divided into two, then when the Miquel points on each of the isosceles right triangles are connected any triangle $M_1M_2M_3$ is formed. Then the length of the sides is

$$M_1M_2 = \sqrt{\frac{1}{2}(b^2+c^2)-bc.\cos\ (90^0+\alpha)}$$
$$M_2M_3 = M_1M_3 = \sqrt{\frac{1}{2}(c^2+a^2)-ca.\cos\ (90^0+\beta)}$$

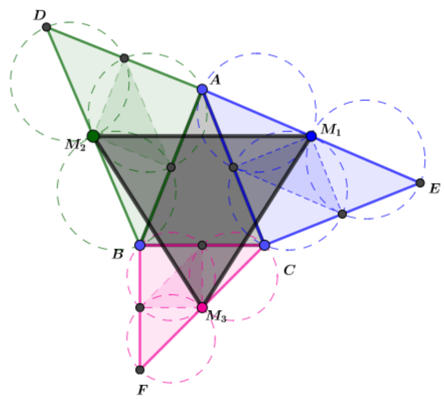

Fig 5. Development of Napoleon's Outer Theorem on a Isosceles Triangle, Each Side Divided by Two

PROOF. Proving the lengths $M_1\ M_2,\ M_2M_3$ dan M_1M_3 utilizes the concepts of trigonometry and similarity. ΔABC is isosceles triangle so that $AB = c\ ,BC = a,\ AC = b.\ AB = AC = b$ and $BC = a$ dan $\angle ABC = \angle BCA = \beta$.

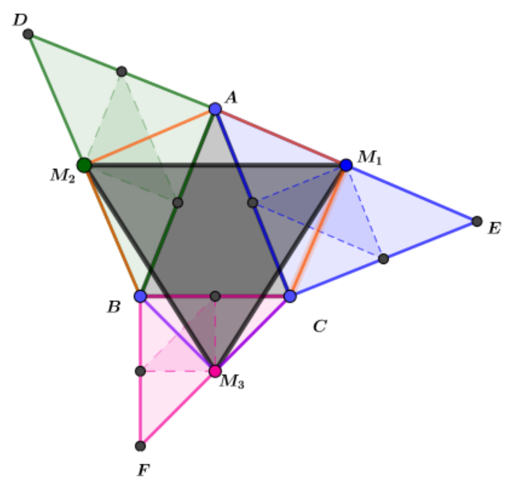

Fig 6. Triangle $M_1M_2M_3$ on the development of Napoleon's Outer Theorem on a Isosceles Triangle Whose Sides are Divided by Two

Further using the same method of Theorem 3.1 gives

$$M_1M_2 = \sqrt{\frac{1}{2}(b^2 + c^2) - bc.\cos\ (90^0 + \alpha)}$$

$$M_2M_3 = \sqrt{\frac{1}{2}(c^2 + a^2) - ca.\cos\ (90^0 + \beta)}$$

$$M_1M_3 = \sqrt{\frac{1}{2}(c^2 + a^2) - ca.\cos\ (90^0 + \beta)}.$$

From the evidence obtained, it can be seen that $M_2M_3 = M_1M_2$ so it is proven that $\Delta M_1M_2M_3$ is an isosceles triangle. ■

Corrolarry 3.3. If on each side of triangle ΔABC is equilateral, isosceles right triangles are constructed, namely ΔABD on the AB side, ΔAEC on the AC side and ΔBCF on the BC side pointing outward, then each side of the triangle is divided into two, then when the Miquel points on each of the isosceles right triangles are connected an equilateral triangle $M_1M_2M_3$ is formed. Then the length of the sides is

$$M_1M_2 = M_2M_3 = M_1M_3 = b\sqrt{1 - \cos\ (90^0 + \alpha)}.$$

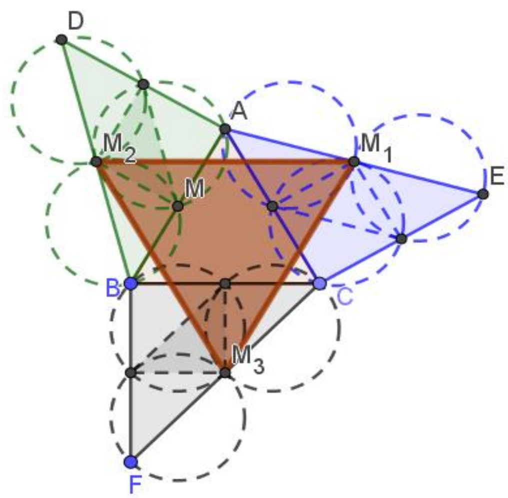

Fig 7. Development of Napoleon's Outer Theorem on an Equilateral Triangle, Each Side Divided by Two

PROOF. Proof of the lengths of $M_1 M_2$, M_2M_3 and M_1M_3 is carried out using the concepts of trigonometry and similarity. Triangle ABC is an equilateral triangle so that $AB = AC = BC = b$ and $\angle BAC = \angle ABC = \angle BCA = \alpha$.

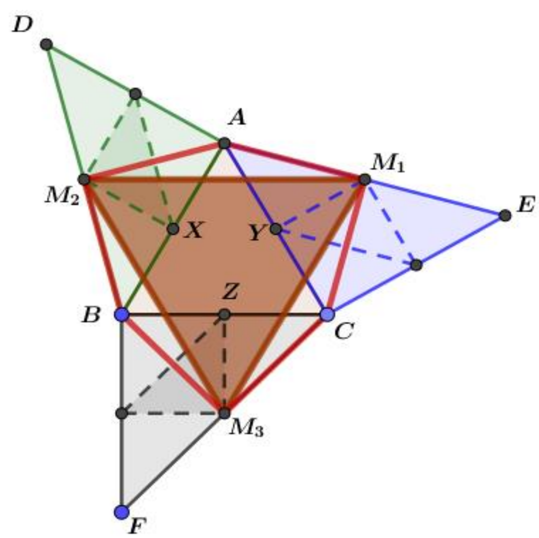

Fig 8. Triangle $M_1M_2M_3$ on the development of Napoleon's Outer Theorem on a Equilateral Triangle Whose Sides are Divided by Two

Further using the same method of Theorem 3.1 yields

$$M_1M_2 = \sqrt{\frac{1}{2}(b^2 + b^2) - bb\cos(90^0 + \alpha)}$$
$$M_1M_2 = b\sqrt{1 - \cos\ (90^0 +\ \alpha)}$$
$$M_2M_3 = b\sqrt{1 - \cos\ (90^0 +\ \alpha)}$$
$$M_1M_3 = b\sqrt{1 - \cos\ (90^0 +\ \alpha)}\,.$$

From the results obtained, it can be seen that $M_1M_3 = M_2M_3 = \ M_1M_2$ so it is proven that $\Delta M_1M_2M_3$ is an isosceles triangle. ■

VI. CONCLUSIONS

From the development of Napoleon's outer theorem using a right triangle isosceles as the outer triangle using a special case of Miquel's theorem, there is a relationship between the original triangle and the newly formed triangle. If the original triangle is any triangle then the new triangle formed is any triangle. If the original triangle is isosceles, the new triangle is isosceles. If the original triangle is an equilateral triangle, the new triangle that is formed is also an equilateral triangle.

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