International Journal of Mathematics Trends and Technology
ISSN: 2231 – 5373 /doi:10.14445/22315373/IJMTT-V67I1P506 Volume 67 Issue 1, 36-43, January, 2021

Stability of Splitting Methods For Systems of Nonlinear Ordinary Differential Equations

Lin Lin Hteik

Associate Professor, Department of Mathematics, Mandalay University of Distance Education

Abstract: *In this paper, we present some numerical methods for the approximate solution of system of nonlinear ordinary differential equations. We discuss alternating direction implicit methods and approximate matrix factorization methods, both of which are splitting methods. Stability of methods is also investigated.*

Keywords: *The Peaceman-Rachford method, the Douglas method and Approximate Matrix Factorization (AMF) methods.*

INTRODUCTION

Linear systems are almost the only large class of ordinary differential equations for which there exists a definite theory. The theory of linear differential equations is also useful as a first approximation to the study of nonlinear problems. We introduced Euler's method to solve the initial value problems for systems of ordinary differential equations numerically. We give some definitions and basic properties of norms of matrices and discuss perturbations on ordinary differential equation systems. We devote to the study of Runge-Kutta method, Rosenbrock method; alternating direction implicit method and approximate matrix factorization method are presented.

I. ALTERNATING DIRECTION IMPLICIT (ADI) METHODS

ADI methods were developed by Douglas, Gunn, Peaceman and Rachford. An important application field at that time was formed by two- and three-dimensional parabolic problems.

A. The Peaceman-Rachford method

The general formulation of an initial value problem for a system of ordinary differential equations is

$$\mathbf{w}'(t) = \mathbf{F}(\mathbf{w}(t), t), \quad t > 0,$$
$$\mathbf{w}(0) = \mathbf{w}_0,$$

with given $\mathbf{F} : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ and $\mathbf{w}_0 \in \mathbb{R}^m$. We consider numerical approximations $\mathbf{w}_n$ to the exact solution values $\mathbf{w}(t_n)$ at the points $t_n = n\Delta t$, $n = 0,1,2,...,$ with $\Delta t > 0$ being the step size. For simplicity this step size Δt is taken to be constant. Convergence properties of the numerical methods will only be considered on bounded time intervals [0, T].
The ADI method of Peaceman and Rachford is one of the very first splitting methods to be proposed in the literature. For the nonlinear ordinary differential equations system

$$\mathbf{w}'(t) = \mathbf{F}(\mathbf{w}(t),t) \tag{1}$$

with the two-term splitting

$$\mathbf{F}(\mathbf{w}(t),t) = \mathbf{F}_1(\mathbf{w}(t),t) + \mathbf{F}_2(\mathbf{w}(t),t) \tag{2}$$

thePeaceman-Rachford method reads

$$\left.\begin{aligned}
\mathbf{w}_{n+\frac{1}{2}} &= \mathbf{w}_n + \frac{1}{2}\Delta t\,\mathbf{F}_1(\mathbf{w}_n, t_n) + \frac{1}{2}\Delta t\,\mathbf{F}_2(\mathbf{w}_{n+\frac{1}{2}}, t_{n+\frac{1}{2}}),\\
\mathbf{w}_{n+1} &= \mathbf{w}_{n+\frac{1}{2}} + \frac{1}{2}\Delta t\,\mathbf{F}_1(\mathbf{w}_{n+1}, t_{n+1}) + \frac{1}{2}\Delta t\,\mathbf{F}_2(\mathbf{w}_{n+\frac{1}{2}}, t_{n+\frac{1}{2}}).
\end{aligned}\right\} \tag{3}$$

This method could be viewed as being obtained by Strang-type operator splitting with alternate use of explicit and implicit Euler in a symmetrical fashion to get second-order. However, it is more natural to consider it as a method of its own. Because of this alternate implicit use of $\mathbf{F}_1$ and $\mathbf{F}_2$, in the framework of dimension splitting, the method is called alternating direction implicit.
Although we will not restrict ourselves to dimension splitting, the name ADI will still be employed for methods like (3) in which each stage is consistent. We can note that the method returns stationary solutions exactly:
for autonomous problems we have

$$\mathbf{w}_n = \tilde{\mathbf{w}}, \ \mathbf{F}(\tilde{\mathbf{w}}) = \mathbf{0} \text{ implies } \mathbf{w}_{n+\frac{1}{2}} = \mathbf{w}_{n+1} = \tilde{\mathbf{w}}. \tag{4}$$

B. Stability of the Peaceman-Rachford method

If $\mathbf{F}_1$ and $\mathbf{F}_2$ are linear, $\mathbf{F}_j(\mathbf{w}(t),t) = \mathbf{A}_j\,\mathbf{w}(t),\ j = 1,2,$ then the Peaceman-Rachford method gives $\mathbf{w}_{n+1} = \mathbf{R}\mathbf{w}_n$ with

$$\mathbf{R} = (\mathbf{I} - \frac{1}{2}\Delta t\,\mathbf{A}_1)^{-1}(\mathbf{I} + \frac{1}{2}\Delta t\,\mathbf{A}_2)(\mathbf{I} - \frac{1}{2}\Delta t\,\mathbf{A}_2)^{-1}(\mathbf{I} + \frac{1}{2}\Delta t\,\mathbf{A}_1). \qquad (5)$$

If $\mathbf{A}_1$ and $\mathbf{A}_2$ commute, it is identical to the locally one-dimensional (LOD) method:

$$\left.\begin{aligned}
&\mathbf{v}_0 = \mathbf{w}_n,\\
&\mathbf{v}_1 = \mathbf{v}_0 + \frac{1}{2}\Delta t\,\mathbf{F}_1(\mathbf{v}_0, t_n) + \frac{1}{2}\Delta t\,\mathbf{F}_1(\mathbf{v}_1, t_{n+\frac{1}{2}}),\\
&\mathbf{v}_2 = \mathbf{v}_1 + \frac{1}{2}\Delta t\,\mathbf{F}_2(\mathbf{v}_1, t_{n+\frac{1}{2}}) + \frac{1}{2}\Delta t\,\mathbf{F}_2(\mathbf{v}_2, t_{n+1}),\\
&\mathbf{w}_{n+1} = \mathbf{v}_2.
\end{aligned}\right\} \qquad (6)$$

In more general linear case we can write

$$\mathbf{w}_n = (\mathbf{I} - \frac{1}{2}\Delta t\,\mathbf{A}_1)^{-1}\tilde{\mathbf{R}}^n(\mathbf{I} - \frac{1}{2}\Delta t\,\mathbf{A}_1)\mathbf{w}_0, \qquad (7)$$

where

$$\tilde{\mathbf{R}} = (\mathbf{I} + \frac{1}{2}\Delta t\,\mathbf{A}_2)(\mathbf{I} - \frac{1}{2}\Delta t\,\mathbf{A}_2)^{-1}(\mathbf{I} + \frac{1}{2}\Delta t\,\mathbf{A}_1)(\mathbf{I} - \frac{1}{2}\Delta t\,\mathbf{A}_1)^{-1}.$$

Hence it is possible to rewrite (3) in the form (6), and a similar transformation is also possible for nonlinear problems. The condition number of $\mathbf{A} \in C^{m\times m}$ is defined as

$$\mathrm{cond}(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{A}^{-1}\|.$$

If we have

$$\mathrm{cond}(\mathbf{I} - \frac{1}{2}\Delta t\,\mathbf{A}_1) \le K_1, \qquad \|\tilde{\mathbf{R}}^n\| \le K_2, \qquad n \ge 0,$$

then $\|\mathbf{R}^n\| \le K = K_1K_2$ for all $n \ge 0$. In general, boundedness of $\left\|\mathbf{I} - \frac{1}{2}\Delta t\mathbf{A}_1\right\|$ will only be valid under very strict conditions on the time step, such as $\Delta t = O(h^2)$. Practically, such restrictions are not obeyed, but the Peaceman-Rachford method appears to be unconditionally stable for problems with smooth coefficients.

C. The Douglas method

Suppose we have a splitting

$$\mathbf{F}(\mathbf{w}(t),t) = \mathbf{F}_0(\mathbf{w}(t),t) + \mathbf{F}_1(\mathbf{w}(t),t) + \cdots + \mathbf{F}_s(\mathbf{w}(t),t), \qquad (8)$$

where it is assumed that $\mathbf{F}_0$ is non stiff so that this term can be treated explicitly. All the others may be stiff and they are treated implicitly in a sequential fashion below:

$$\left.\begin{aligned}
&\mathbf{v}_0 = \mathbf{w}_n + \Delta t\,\mathbf{F}(\mathbf{w}_n, t_n)\\
&\mathbf{v}_j = \mathbf{v}_{j-1} + \theta\,\Delta t\,\{\mathbf{F}_j(\mathbf{v}_j, t_{n+1}) - \mathbf{F}_j(\mathbf{w}_n, t_n)\}, \quad j = 1,2,\ldots,s,\\
&\mathbf{w}_{n+1} = \mathbf{v}_s.
\end{aligned}\right\} \qquad (9)$$

The method is of order one if $\theta = 1$ and of order two if $\theta = \frac{1}{2},\ \mathbf{F}_0 = \mathbf{0}$.

As the first explicit Euler stage is followed by implicit stages which serve to stabilize this first explicit stage, methods of the type in (9) are also known as stabilizing correction methods. A nice property of (9) is that all internal vectors $\mathbf{v}_j$ are consistent approximations to $\mathbf{w}(t_{n+1})$. Furthermore; this method returns stationary solutions exactly, similar to (4) which can be seen by considering the consecutive $\mathbf{v}_j,\ j = 1,2,\ldots,s.$

D. Stability of the Douglas method

The Douglas method lends itself less easily to stability analysis than the LOD methods and the same observations for applying the Peaceman-Rachford method. Here we can consider the simplest situation from the analysis point of view: linear problem with commuting, normal matrices $\mathbf{A}_j$.

We now consider

$$\mathbf{w}'(t) = \lambda_0\mathbf{w}(t) + \lambda_1\mathbf{w}(t) + \ldots + \lambda_s\mathbf{w}(t), \qquad (10)$$

where the term $\lambda_0\, \mathbf{w}(t)$ is included to take the explicit term $\mathbf{F}_0$ into account. Applying the Douglas method to (10) shows a recursion $\mathbf{w}_{n+1} = R\ \mathbf{w}_n$ with

$$=R = R\,(z_0, z_1, ..., z_s) = 1 + \left(\prod_{j=1}^{s}(1-\theta z_j)\right)^{-1}\sum_{j=0}^{s} z_j \qquad (11)$$

as stability function.

For the stability analysis we will consider the wedge, for $\alpha > 0$,

$$W_\alpha = \left\{\zeta \in C \mid \zeta = 0 \text{ or } |\arg(-\zeta)| \le \alpha\right\}$$

Inthe complex plane and examine stability for $z_j \in W_\alpha,\ j \ge 1$.

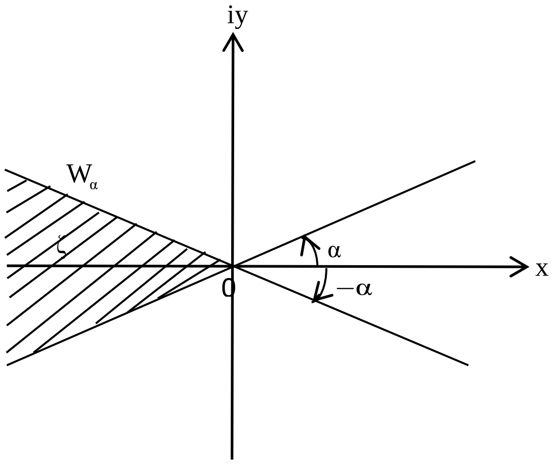

Figure 1.1: The graph of a wedge W_α

E. Definition

The stability region of the method is the set in the complex plane. A method that has the property that S contains the left half-plane is called **A-stable**.

A method with stability function R is said to be strongly **A-stable** if it is A-stable with $\lim_{z\to\infty}|R\,(z)| < 1$, and it is said to be L**-stable** if we have $\lim_{z\to\infty}|R\,(z)| = 0$ in addition.

F. Theorem

Let R be defined by (11). Suppose $z_0 = 0$ and $s \ge 2,\ 1 \le r \le s-1$. For any $\theta \ge \frac{1}{2}$ we have

$$\alpha \le \frac{1}{s-1}\frac{\pi}{2} \Leftrightarrow \left[|R| \le 1 \quad \text{for all } z_j \in W_\alpha,\ j = 1,2,...,s\right],$$

$$\alpha \le \frac{1}{s-r}\frac{\pi}{2} \Leftrightarrow \begin{bmatrix} |R| \le 1 \quad \text{for all } z_1,...,z_{s-r} \in W_\alpha \text{ and} \\ \mathrm{Re}(z_{s-r+1}) \le 0, ..., \mathrm{Re}(z_s) \le 0 \end{bmatrix}.$$

Proof

See [2].

II. APPROXIMATE MATRIX FACTORIZATION (AMF) METHODS

In this section we will discuss a number of splitting methods derived from the Rosenbrock methods with inexact Jacobian matrices. Splitting is realized here by choosing special approximations to the Jacobian matrix. The basic idea is to simplify and economize the linear system with involving matrix $\mathbf{I} - \gamma \Delta t\, \mathbf{A}$. This is achieved by approximating entries in the Jacobian matrix and in particular by factorizing the matrix $\mathbf{I} - \gamma \Delta t\, \mathbf{A}$. The common name for this technique is AMF method.

We primarily focus on autonomous systems

$$\mathbf{w}'(t) = \mathbf{F}(\mathbf{w}(t)) = \mathbf{F}_0(\mathbf{w}(t)) + \mathbf{F}_1(\mathbf{w}(t)) + \ldots + \mathbf{F}_s(\mathbf{w}(t)),$$

where, $\mathbf{F}_0$ is the non stiff term. Time-dependent terms will be treated by a transformation to an augmented autonomous form.

A. One-stage methods of order one and two

We first consider the one-stage Rosenbrock method:

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \Delta t\, \mathbf{B}^{-1}\mathbf{F}(\mathbf{w}_n), \qquad (12)$$

Where $\mathbf{B} = \mathbf{I} - \gamma \Delta t\, \mathbf{A}$ and $\mathbf{A}$ is an approximation to $\mathbf{F}'(\mathbf{w}_n) = \frac{\partial \mathbf{F}}{\partial \mathbf{w}}(\mathbf{w}_n)$. Now, let us assume

$$\mathbf{F}_j'(\mathbf{w}_n) = \frac{\partial \mathbf{F}_j}{\partial \mathbf{w}}(\mathbf{w}_n) = \mathbf{A}_j + O(\Delta t), \quad j = 1, 2, \ldots, s, \qquad (13)$$

And replace $\mathbf{B}$ in (12) by the approximate factorization

$$\mathbf{B} = (\mathbf{I} - \gamma \Delta t\, \mathbf{A}_1)(\mathbf{I} - \gamma \Delta t\, \mathbf{A}_2)\ldots(\mathbf{I} - \gamma \Delta t\, \mathbf{A}_s). \qquad (14)$$

So, the resulting method reads

$$\mathbf{w}_{n+1} = \mathbf{w}_n + \Delta t\, (\mathbf{I} - \gamma \Delta t\, \mathbf{A}_s)^{-1} \ldots (\mathbf{I} - \gamma \Delta t\, \mathbf{A}_1)^{-1}\mathbf{F}(\mathbf{w}_n). \qquad (15)$$

We will note that the non stiff Jacobian $\frac{\partial \mathbf{F}_0}{\partial \mathbf{w}}(\mathbf{w}_n)$ is not present here. So, the $\mathbf{F}_0$ term is treated explicitly and the other terms linearly implicit one after another. Because $\mathbf{F}$ is not split, stationary solutions $\tilde{\mathbf{w}}$ satisfying $\mathbf{F}(\tilde{\mathbf{w}}) = \mathbf{0}$ are maintained. Also we note that in a concrete case a change in sequence of the factors $\mathbf{I} - \gamma \Delta t\, \mathbf{A}_j$ gives a different algorithm, unless the matrices $\mathbf{A}_j$ commute.

With this approximate matrix factorization, the order is one for any γ in general. We will get order two if $\gamma = \frac{1}{2}$ and $\mathbf{F}_0 = \mathbf{0}$. When the problem is linear; this one-stage Rosenbrock AMF method is identical to the Douglas method.
To apply the method to a non autonomous problem $\mathbf{w}'(t) = \mathbf{F}(\mathbf{w}(t), t)$, first we rewrite this in the augmented autonomous form

$$\mathbf{u}'(t) = \mathbf{G}(\mathbf{u}(t)) \qquad (16)$$

with

$$\mathbf{u}(t) = \begin{bmatrix} \mathbf{w}(t) \\ t \end{bmatrix}, \quad \mathbf{G}(\mathbf{u}(t)) = \begin{bmatrix} \mathbf{F}(\mathbf{w}(t), t) \\ 1 \end{bmatrix},$$

to which the method can be applied. Then t is formally considered as an unknown, but it is easily seen that the approximations t_n found with this method still equal $n\Delta t$. Reformulating in terms of $\mathbf{w}_n$, the methods will involve approximations to the time derivatives $\frac{\partial \mathbf{F}}{\partial t}(\mathbf{w}, t)$. Take that

$$\mathbf{A}_j \approx \frac{\partial \mathbf{F}_j}{\partial \mathbf{w}}(\mathbf{w}_n, t_{n+\gamma}) \in \mathbf{R}^{m \times m},$$

$$\mathbf{b}_j \approx \frac{\partial \mathbf{F}_j}{\partial t}(\mathbf{w}_n, t_{n+\gamma}) \in \mathbf{R}^{m}$$

and

$$\mathbf{B}_j^{-1} = \begin{bmatrix} (\mathbf{I} - \gamma \Delta t \mathbf{A}_j)^{-1} \gamma\, \Delta t\, \mathbf{b}_j & (\mathbf{I} - \gamma \Delta t \mathbf{A}_j)^{-1} \\ 1 & \mathbf{0}^T \end{bmatrix} \in \mathbf{R}^{(m+1)\times(m+1)}, \quad j = 1, 2, \ldots, s,$$

the factorized Rosenbrock method (12) reads

$$\begin{bmatrix} \mathbf{w}_{n+1} \\ t_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{w}_n \\ t_n \end{bmatrix} + \mathbf{B}_s^{-1} \ldots \mathbf{B}_2^{-1}\mathbf{B}_1^{-1} \begin{bmatrix} \Delta t\, \mathbf{F}(\mathbf{w}_n, t_n) \\ \Delta t \end{bmatrix}. \qquad (17)$$

So, we will have $t_{n+1} = t_n + \Delta t$, as it should be, and the computation of $\mathbf{w}_{n+1}$ can be written in the more transparent recursive form with increments $d\mathbf{v}_j$,

$$\left.\begin{aligned} d\mathbf{v}_0 &= \Delta t\,\mathbf{F}(\mathbf{w}_n, t_n), \\ d\mathbf{v}_j &= (\mathbf{I} - \gamma\,\Delta t\,\mathbf{A}_j)^{-1}(d\mathbf{v}_{j-1} + \gamma\,(\Delta t)^2\mathbf{b}_j),\ \ j=1,2,...,s, \\ \mathbf{w}_{n+1} &= \mathbf{w}_n + d\mathbf{v}_s. \end{aligned}\right\} \tag{18}$$

Setting $\mathbf{v}_j = \mathbf{w}_n + d\mathbf{v}_j$, We can once more see the close relation with the Douglas method for linear problems.

B. Two-stage methods of order two and three

Now, we can consider the two-stage method

$$\left.\begin{aligned} \tilde{\mathbf{c}}_1 &= \Delta t\,\mathbf{F}(\mathbf{w}_n) + \gamma\,\Delta t\,\mathbf{A}\tilde{\mathbf{c}}_1, \\ \tilde{\mathbf{c}}_2 &= \Delta t\,\mathbf{F}(\mathbf{w}_n + \alpha_1\,\tilde{\mathbf{c}}_1) + \alpha_2\,\Delta t\,\mathbf{A}\tilde{\mathbf{c}}_1 + \gamma\,\Delta t\,\mathbf{A}\tilde{\mathbf{c}}_2, \\ \mathbf{w}_{n+1} &= \mathbf{w}_n + b_1\,\tilde{\mathbf{c}}_1 + b_2\,\tilde{\mathbf{c}}_2, \end{aligned}\right\} \tag{19}$$

where

$$b_2 \neq 0,\ b_1 = 1 - b_2,\ \alpha_1 = \frac{1}{2b_2},\ \alpha_2 = -\frac{\gamma}{b_2}.$$

Before we introduce the approximate factorization we remove the matrix vector multiplication $\mathbf{A}\tilde{\mathbf{c}}_1$ in the second stage. We substitute $\tilde{\mathbf{c}}_1 = \mathbf{c}_1,\ \tilde{\mathbf{c}}_2 = \mathbf{c}_2 - \frac{\alpha_2}{\gamma}\mathbf{c}_1$ to that part. When we impose the relations for order two, it gives the method

$$\left.\begin{aligned} \mathbf{B}\mathbf{c}_1 &= \Delta t\,\mathbf{F}(\mathbf{w}_n), \\ \mathbf{B}\mathbf{c}_2 &= \Delta t\,\mathbf{F}(\mathbf{w}_n + \frac{1}{2b_2}\mathbf{c}_1) - \frac{1}{b_2}\mathbf{c}_1, \\ \mathbf{w}_{n+1} &= \mathbf{w}_n + (2 - b_2)\mathbf{c}_1 + b_2\,\mathbf{c}_2, \end{aligned}\right\} \tag{20}$$

where $\mathbf{B} = \mathbf{I} - \gamma\Delta t\,\mathbf{A}$ and γ, b_2 are free parameters. With (14) this method remains of order two, even with $\mathbf{F}_0 \neq \mathbf{0}$. Moreover, we will take $\gamma \geq \frac{1}{4}$ and use (14). The choice of γ is important for stability.

Method in (20) provided with approximate matrix factorization returns stationary solutions because splitting of $\mathbf{F}$ itself is not used. The AMF methods can also be applied to non-autonomous problems $\mathbf{w}'(t) = \mathbf{F}(\mathbf{w}(t), t)$ in the same way as for the one-stage method. For both stages this leads to a recursion of the type (18).

An AMF counterpart of the third order method

$$\left.\begin{aligned} \mathbf{c}_1 &= \Delta t\,\mathbf{F}(\mathbf{w}_n) + \gamma\,\Delta t\,\mathbf{A}\mathbf{c}_1, \\ \mathbf{c}_2 &= \Delta t\,\mathbf{F}(\mathbf{w}_n + \frac{2}{3}\mathbf{c}_1) - \frac{4}{3}\gamma\,\Delta t\,\mathbf{A}\mathbf{c}_1 + \gamma\,\Delta t\,\mathbf{A}\mathbf{c}_2, \\ \mathbf{w}_{n+1} &= \mathbf{w}_n + \frac{1}{4}\mathbf{c}_1 + \frac{3}{4}\mathbf{c}_2, \end{aligned}\right\} \tag{21}$$

could be obtained with $b_2 = \frac{3}{4},\ \gamma = \frac{1}{2} + \frac{1}{6}\sqrt{3}$. But, for order three it is necessary to require $\mathbf{F}_0 = \mathbf{0}$, since it was supposed that $\mathbf{A} \approx \frac{\partial \mathbf{F}}{\partial \mathbf{w}}(\mathbf{w}_n)$ in the order condition for (21). With the factorization in (14) this will be satisfied only if $\mathbf{F}_0 = \mathbf{0}$.

C. Stability of two-stage methods

The method in (20), which is non-factorized, is A-stable for any $\gamma \geq \frac{1}{4}$. With approximate matrix factorization the stability properties change. As for the Douglas method, we consider the scalar test model $w'(t) = \lambda_0 w(t) + \lambda_1 w(t) + ... + \lambda_s w(t)$. Applying (20) to this test model gives the stability function

$$R\,(z_0, z_1, ..., z_s) = 1 + \frac{2z}{\omega} - \frac{z}{\omega^2} + \frac{z^2 - 2z}{2\omega^2}, \tag{22}$$

where

$$z = \sum_{j=0}^{s} z_j,\quad \omega = \prod_{j=1}^{s}(1 - \gamma\,z_j),$$

and γ is still free. The parameter b_2 has cancelled in this expression. If $s=0$ we get

$$R(z_0) = 1 + z_0 + \frac{1}{2}z_0^2.$$

If we put $\omega = 1-\gamma\xi$ in (22) we have the stability function

$$R(\xi) = \frac{1+(1-2\gamma)\xi + (\frac{1}{2}-2\gamma+\gamma^2)\xi^2}{(1-\gamma\xi)^2}$$

of the unfactorized methods.

As we consider the wedge W_α and examine stability under the condition for the Douglas method,

$$z_j \in W_\alpha,\ j=1,2,...,s \text{ with either } z_0 = 0 \text{ or} \left|1+z_0+\frac{1}{2}z_0^2\right| \le 1.$$

If s =1, which means if we have one explicitly and one implicitly treated term, then A-stability for the implicit term is preserved with $\gamma = \frac{1}{2}$. This is surprising in the view of the fact that without factorization the stability function is A-stable and it does not vanish at infinity. This result can be shown as follows

For s =1 we can write $R = R(z_0, z_1)$ as

$$R = \frac{(1+z_0+\frac{1}{2}z_0^2)+(1-2\gamma)(1+z_0)z_1+(\frac{1}{2}-2\gamma+\gamma^2)z_1^2}{(1-\gamma z_1)^2}. \tag{23}$$

For $\gamma = \frac{1}{2}$, it becomes

$$|R| = \left|\frac{(1+z_0+\frac{1}{2}z_0^2)-\frac{1}{4}z_1^2}{(1-\frac{1}{2}z_1)^2}\right|.$$

For $\left|1+z_0+\frac{1}{2}z_o^2\right| \le 1$, we can write

$$\begin{aligned} |R| &\le \frac{1+\frac{1}{4}|z_1|^2}{\left|1-\frac{1}{2}z_1\right|^2} \\ &= \frac{1+\frac{1}{4}(x_1^2+y_1^2)}{1-x_1+\frac{1}{4}(x_1^2+y_1^2)} \\ &\le 1 \end{aligned}$$

Whenever $x_1 = \mathrm{Re}(z_1) \le 0.$

D. Remark

Let $z_0 = 0$ and $s \ge 2$. Angle barriers for stability were already encountered in the inequality

$$\alpha \le \frac{1}{s-1}\frac{\pi}{2} \Leftrightarrow \left[|R| \le 1 \text{ for all } z_j \in W_\alpha,\ j=1,2,...,s\right], \tag{24}$$

for the Douglas method and consequently also for the one-stage method in (15) with approximate matrix factorization. This barrier is for AMF methods always true.

To show that, let $z = z_1 + z_2 + ... + z_s$ and consider the function $R = R(z_1, z_2, ..., z_s)$ given by

$$R = 1 + \frac{\varphi(z_1, z_2, ..., z_s)}{\psi_1(z_1)\psi_2(z_2)...\psi_s(z_s)} z, \tag{25}$$

where φ is a polynomial and then ψ_j are polynomials without zero in the left half-plane. This is the general form of a stability function for a one-step method with approximate matrix factorization of implicit terms.

Now we consider $z_j = -\tau e^{i\beta},\ j=1,2,...,s,$ with $0 \le \beta \le \alpha$ and assume that

$$\frac{\varphi(-\tau e^{i\beta}, -\tau e^{i\beta}, \ldots, -\tau e^{i\beta})}{\psi_1(-\tau e^{i\beta})\,\psi_2(-\tau e^{i\beta})\ldots\psi_s(-\tau e^{i\beta})} = C(\tau e^{i\beta})^{-r} + O(\tau^{-r-1}), \qquad \text{as } \tau \to \infty,$$

where r is an integer and C is a non zero constant. Then, as $\tau \to \infty$,

$$R = 1 - s\,C\,\tau^{1-r} e^{i(1-r)\beta} + O(\tau^{-r}).$$

Hence stability for all $\beta \le \alpha$ requires $C > 0$ and $|r-1|\alpha \le \frac{\pi}{2}$.

Stability for fixed $\mathrm{Re}(z_k) < 0$ and $\mathrm{Re}(z_j) \to -\infty,\ j \ne k,$ implies that the degree of φ in z_j is less than the degree of ψ_j. Consequently $r \ge s,\ j \ne k,$ and so we get the condition

$$\alpha \le \frac{1}{s-1}\frac{\pi}{2},$$

which is the same upper bound as in the first statement of Theorem 1.6.

E. A three-stage method of order two

Second order explicit Runge-Kutta methods are found as:

$$\left.\begin{aligned} \mathbf{v}_0 &= \mathbf{w}_n, \\ \mathbf{v}_j &= \mathbf{v}_{j-1} + \frac{1}{s}\Delta t\,\mathbf{F}(\mathbf{v}_{j-1}, t_n + \frac{j-1}{s}\Delta t),\ \text{j=1,2,...,s}, \\ \mathbf{w}_{n+1} &= \frac{1}{s+1}\mathbf{w}_n + \left\{\frac{s}{s+1}\mathbf{v}_s + \frac{1}{s}\Delta t\,\mathbf{F}(\mathbf{v}_s, t_{n+1})\right\}. \end{aligned}\right\} \tag{26}$$

Taking the three-stage member of this class Gerisch and Verwer have constructed a three-stage Rosenbrock method which has order two for arbitrary Jacobian approximations **A**, similar to the method in (20).

In standard form, when we use the notation from (25)-(26), where **A** is the Jacobian matrix $\frac{\partial \mathbf{F}}{\partial \mathbf{w}}(\mathbf{w}_n)$, this Rosenbrock method shows

$$\begin{aligned} (\mathbf{I} - \gamma\,\Delta t\mathbf{A})\,\tilde{\mathbf{c}}_1 &= \Delta t\,\mathbf{F}(\mathbf{w}_n), \\ (\mathbf{I} - \gamma\,\Delta t\,\mathbf{A})\tilde{\mathbf{c}}_j &= \Delta t\,\mathbf{F}(\mathbf{w}_n + \frac{1}{2}\sum_{\ell=1}^{j-1}\tilde{\mathbf{c}}_\ell) + \Delta t\,\mathbf{A}\sum_{\ell=1}^{j-1}\gamma_{j,\ell}\tilde{\mathbf{c}}_\ell, \quad j = 2,3, \\ \mathbf{w}_{n+1} &= \mathbf{w}_n + \frac{1}{3}(\tilde{\mathbf{c}}_1 + \tilde{\mathbf{c}}_2 + \tilde{\mathbf{c}}_3), \end{aligned}$$

where

$$\begin{aligned} \gamma_{3,2} &= \frac{1}{2} - 3\gamma, \\ \gamma_{3,1} &= \frac{-1}{1 + 2\gamma_{3,2}}\left(6\gamma^3 - 12\gamma^2 + 6(1-\gamma_{3,2})\gamma + 2\gamma_{3,2}^2 - \frac{1}{2}\right), \\ \gamma_{2,1} &= -(3\gamma + \gamma_{3,1} + \gamma_{3,2}), \end{aligned}$$

with

$$\gamma = 1 - \frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{6}}{2}\sin\theta, \qquad \theta = \frac{1}{3}\arctan(\frac{\sqrt{2}}{4}).$$

The value of γ is approximately 0.43586652. The parameters are chosen in such a way that the method is L-stable and of order three for homogeneous linear problems with constant coefficients. Now we change $\tilde{\mathbf{c}}_j$ to $\mathbf{c}_j$ by

$\tilde{\mathbf{c}}_1 = \mathbf{c}_1,\ \ \tilde{\mathbf{c}}_2 = \mathbf{c}_2 - \frac{1}{\gamma}\gamma_{2,1}\tilde{\mathbf{c}}_1,\ \ \tilde{\mathbf{c}}_3 = \mathbf{c}_3 - \frac{1}{\gamma}\gamma_{3,1}\tilde{\mathbf{c}}_1 - \frac{1}{\gamma}\gamma_{3,2}\tilde{\mathbf{c}}_2.$ After that, one can apply the method with the approximate matrix factorization in the same manner, replacing the matrix $\mathbf{B} = \mathbf{I} - \gamma\Delta t\,\mathbf{A}$ by its factorized counterpart defined by $(\mathbf{I} - \gamma\,\Delta t\,\mathbf{A}_1)(\mathbf{I} - \gamma\,\Delta t\,\mathbf{A}_2)\ldots(\mathbf{I} - \gamma\,\Delta t\,\mathbf{A}_s)$.

III. CONCLUSIONS

We have found that the Peaceman-Rachford method is unconditionally stable for some particular problems. The approximate matrix factorization method is of order one in general. Three-stage Rosenbrock method has order two for arbitrary Jacobian approximation.

ACKNOWLEDGEMENT

First, I would like to express my gratitude to my supervisor, Dr Saw Win Maung, part-time lecturer, Department of Mathematics, University of Mandalay, who has given me invaluable advice and patient guidance that helped my research work to accomplish. In addition, I would like to express my thanks to all my teachers throughout student life.

I acknowledge with gratitude of assistance and help of many friends, colleagues. I am indebted to all my friends for helping me in various ways while writing this paper. Finally, I would like to thank my parents, U Aung Nyunt and Daw Thein Nyunt, my husband, Ko Zaw Naing Htun for I have received invaluable help from them.

REFERENCES

[1] J. Frank, W.H. Hundsdorfer and J.G. Verwer, Stability of Implicit-Explicit Linear Multistep Methods, Research Report NM-R9623, ISSN 0169-0388, (1996), Amsterdam, Netherlands.

[2] W. Hundsdorfer, and J.G Verwer, Numerical Solution of Time-Dependent Advection-Diffusion-Reaction Equations, (2003), Springer-Verlag, Berlin.

[3] J.W. Thomas, Numerical Partial Differential Equations: Conservation Laws and Elliptic Equations, (1999), Springer-Verlag Inc., New York.