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Ruscheweyh Derivative and a New Generalized Operator Involving Convolution

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Abstract *- In this present investigation, we introduce a generalized differential operator $E^m_{\alpha,\beta}(\mu,\varphi,t)$via convolution approach. Using this operator, we further introduced a new generalized differential operator $RE^m_{\alpha,\beta,\delta}(\sigma,\varphi,t)f(z)$ obtained as a linear combination of Ruscheweyh derivative and the operator $E^m_{\alpha,\beta}(\mu,\varphi,t)$. With the aid of the new generalized differential operator, a new subclass $\mathcal{M}^{m,\lambda,\mu,\sigma,\varphi,t}_{\alpha,\beta,\delta}(\rho)$ of analytic functions in the open unit disk is introduced and investigated. Characterization and other properties of this class are studied. In particular, Coefficient estimates, distortion theorems of functions with negative coefficients belonging to this class are also determined. Some relevant remarks and useful connections of the main results are also pointed out.*

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I. INTRODUCTION

As usual, we let $\mathcal{A}$ to denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disk $U = \{z \in C: |z| < 1\}$ and normalized under the condition of $f(0) = f'(0) - 1 = 0$. We also denote by T, the subclass of functions of $\mathcal{A}$ of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0 \tag{2}$$

Ruscheweyh [1] introduced the operator R^m for $m \in \mathbb{N}_0, f \in \mathcal{A}$ and is defined by

$$R^m: \mathcal{A} \longrightarrow \mathcal{A},$$
$$R^0 f(z) = f(z), R^1 f(z) = zf'(z), \dots, (m+1)R^{m+1} f(z) = z\left(R^m f(z)\right)' + mR^n f(z), z \in U.$$

Remark 1.1 Let $f \in \mathcal{A}$, then

$$R^m f(z) = z + \sum_{k=2}^{\infty} B_k(m) a_k z^k, \quad z \in U, \tag{3}$$

where

$$B_k(m) = C^m_{m+k-1} = B(m,k) = \binom{m+k-1}{n} = \frac{(m+1)(m+2)\dots(m+k-1)}{(k-1)!}$$

$$= \frac{(m+1)_{k-1}}{(1)_{k-1}}. \quad (4)$$

In the year 2012, Swamy [2] introduced a new generalized multiplier differential operator as follows.

Let $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then for $f \in \mathcal{A}$, a new generalized multiplier operator $I_{\alpha,\beta}^m f(z)$ was defined by

$$I_{\alpha,\beta}^0 f(z) = f(z), I_{\alpha,\beta}^1 f(z) = \frac{\alpha f(z) + \beta z f'(z)}{\alpha + \beta}, \ldots, I_{\alpha,\beta}^m \left(I_{\alpha,\beta}^{m-1} f(z) \right).$$

Remark 1.2 $I_{\alpha,\beta}^m f(z)$ is a linear operator such that for $f \in \mathcal{A}$,

$$I_{\alpha,\beta}^m f(z) = z + \sum_{k=2}^{\infty} A_k(\alpha, \beta, m) a_k z^k, \quad (5)$$

where

$$A_k(\alpha, \beta, m) = \left(\frac{\alpha + k\beta}{\alpha + \beta} \right)^m. \quad (6)$$

We note that for various suitable choices of the parameters involve, operator $I_{\alpha,\beta}^m f(z)$ reduces to operators defined in [3, 4, 5, 6, 7]. Of note is the Salagean differential operator $D^m f(z)$ introduced in [7] and also considered for $m \geq 0$ in [8]:

$$D^m f(z) \equiv I_{0,1}^m f(z) = z + \sum_{k=2}^{\infty} k^m a_k z^k, \ z \in U. \quad (7)$$

For detail information on $D^m f(z)$, see [7].

Consequently, by making use of (3) and (6), Swamy [9], further introduced the linear operator $RI_{\alpha,\beta,\delta}^m f(z)$ as follow.

Definition 1.3 [9] Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \delta \geq 0, \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$. The operator $RI_{\alpha,\beta,\delta}^m f(z)$ is given by

$$RI_{\alpha,\beta,\delta}^m : \mathcal{A} \longrightarrow \mathcal{A},$$
$$RI_{\alpha,\beta,\delta}^m f(z) = (1-\delta) R^m f(z) + \delta I_{\alpha,\beta}^m f(z), z \in U.$$

Remark 1.4 Let $f(z)$ of the form (1) be in $\mathcal{A}$, then we have

$$RI_{\alpha,\beta,\delta}^m f(z) = z + \sum_{k=2}^{\infty} \{(1-\delta) B_k(m) + \delta A_k(\alpha, \beta, m) a_k z^k\}, \qquad z \in U, \quad (8)$$

where $B_k(m)$ and $A_k(\alpha, \beta, m)$ are as defined in (4) and (6) respectively. Other related works on the operator (8) can be found in [10], [11] and [12].

By making use of the differential operator (8), Swamy [13] introduced and investigated the following class.

Definition 1.5 [13] Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \geq 0, 0 \leq \lambda \leq 1, \lambda \leq \mu, \delta \geq 0, 0 \leq \rho < 1, \ \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then $f(z) \in \aleph_{\alpha,\beta,\delta}^{m,\lambda,\mu}(\rho)$ if and only if

$$Re\left(\frac{z\left(RI_{\alpha,\beta,\delta}^m f(z)\right)' + \ \mu z^2 \left(RI_{\alpha,\beta,\delta}^m f(z)\right)''}{(1-\lambda) RI_{\alpha,\beta,\delta}^m f(z) + \lambda z \left(RI_{\alpha,\beta,\delta}^m f(z)\right)'} \right) > \rho, z \in U. \quad (9)$$

When $\lambda = 0$ in Definition 1.5, we have

Definition 1.6 [13] Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \geq 0,\ \delta \geq 0, 0 \leq \rho < 1,\ \beta \geq 0, \alpha$ a real number such that $\alpha + \beta > 0$. Then $f(z) \in \Omega_{\alpha,\beta,\delta}^{m,\lambda,\mu}(\rho)$ if and only if

$$Re\left(\frac{z\left(RI_{\alpha,\beta,\delta}^{m} f(z)\right)'}{RI_{\alpha,\beta,\delta}^{m} f(z)}\right)\left(1+\mu\frac{z\left(RI_{\alpha,\beta,\delta}^{m} f(z)\right)''}{\left(RI_{\alpha,\beta,\delta}^{m} f(z)\right)'}\right) > \rho, z \in U. \qquad (10)$$

The relevance of the class $\aleph_{\alpha,\beta,\delta}^{m,\lambda,\mu}(\rho)$with some known classes are pointed out in [13], interested readers may as well confirm these in [14], [10] and [11].

Recently in the year 2017, Opoola [15] introduced the generalized differential operator $D^m(\mu,\beta,t)f(z)$ for a function $f \in \mathcal{A}$, with

$$D_t f(z) = 1 + \sum_{k=2}^{\infty} \quad [1 + (k + \beta - \mu - 1)t]a_k z^{k-1},\ 0 \leq \mu \leq \beta, t \geq 0,$$

such that

$$D^0(\mu,\beta,t)f(z) = f(z)$$
$$D^1(\mu,\beta,t)f(z) = zD_t f(z) =$$
$$ztf'(z) - zt(\mu - \beta) + (1 + (\mu - \beta - 1)t)f(z)$$
$$\cdots$$

$$D^n(\mu,\beta,t)f(z) = zD_t[D^{n-1}(\mu,\beta,t)f(z)],\quad n \in \mathbb{N} \cup \{0\},\qquad 0 \leq \beta \leq \mu, t \geq 0$$

and $m \in \mathbb{N}\cup\{0\}$.

Remark 1. 7 $D^m(\mu,\beta,t)f(z)$ is a linear operator such that for $f \in \mathcal{A}$

$$D^m(\mu,\beta,t)f(z) = z + \sum_{k=2}^{\infty}[1 + (k + \beta - \mu - 1)t]^m a_k z^k, \qquad (11)$$

$0 \leq \beta \leq \mu, t \geq 0$, and $m \in \mathbb{N}\cup\{0\}$.

For convenience, we shall in the sequel,
let

$$D^m\left(\sigma, \varphi, t\right)f(z) = z + \sum_{k=2}^{\infty}\left[1 + \left(k + \varphi - \sigma - 1\right)t\right]^m a_k z^k, \qquad (12)$$

$0 \leq \sigma \leq \varphi, t \geq 0$, and $m \in \mathbb{N}\cup\{0\}$.

It turns out that the differential operator $D^m\left(\sigma, \varphi, t\right)f(z)$ reduces to the Salagean and Al-Oboudi differential operators for suitably varied parameters [7, 3].

In section 2, we shall recall a fundamental (necessary) definition and also give other definitions to acquaint the readers with the main content

II. PRELIMINARIES

Definition 2.1 (Hadamard product or convolution) [16, 17, 18]:
The Hadamard product or convolution of two analytic functions $f(z)$ given by (1) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is given by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \qquad z \in U. \qquad (13)$$

By employing Definition 2.1 on (5) and (12), we have the following:

Definition 2.2 (A new generalized operator involving convolution**).** Let $f \in \mathcal{A}$ then from (5) and (12) we obtain a generalized operator involving convolution as follows:

$$E^m_{\alpha,\beta}(\sigma,\varphi,t)f(z) = \left(D^m(\sigma,\varphi,t)f(z) * I^m_{\alpha,\beta}f(z)\right)$$
$$= z + \sum_{k=2}^{\infty} A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m a_k z^k, \qquad (14)$$

for $\beta \geq 0, \alpha \in \mathbb{R}, \alpha+\beta > 0, 0 \leq \sigma \leq \varphi, t \geq 0$, and $m \in \mathbb{N}\cup\{0\}$.

Remark 2.3 Here, we observe the following with respect to $E^m_{\alpha,\beta}(\sigma,\varphi,t)f(z)$:

(i) When $t = 0$, we have $E^m_{\alpha,\beta}(\sigma,\varphi,0)f(z) \equiv I^m_{\alpha,\beta}f(z)$ defined by (4).

(ii) When $\beta = 0$, we have $E^m_{\alpha,0}(\sigma,\varphi,t)f(z) \equiv D^m\left(\sigma,\varphi,t\right)f(z)$ defined by (11) or (12).

(iii) When $\varphi = \sigma, t = 1$, we have $E^m_{\alpha,\beta}(\sigma,\sigma,1)f(z) \equiv D^m f(z)$, which is the Salagean differential operator studied in [7].

(iv) When $\varphi = \sigma, \beta = 0,\ t = \delta$, we have $E^m_{\alpha,\beta}(\sigma,\sigma,\delta)f(z) \equiv D^m_\delta f(z)$,which is the Al-Oboudi differential operator studied in [3].

(v) When $\varphi = \sigma$, we have

$$E^m_{\alpha,\beta}(\sigma,\sigma,t)f(z) \equiv E^m_{\alpha,\beta}(t)f(z)$$
$$= z + \sum_{k=2}^{\infty} A_k(\alpha,\beta,m)[1+(k-1)t]^m a_k z^k,$$

for $\beta \geq 0, \alpha \in \mathbb{R}, \alpha+\beta > 0, t \geq 0$, and $m \in \mathbb{N}\cup\{0\}$,which is a new class.

(vi) When $t = 1$ in (v), we have $E^m_{\alpha,\beta}(1)f(z) \equiv E^m_{\alpha,\beta}f(z)$

$$= z + \sum_{k=2}^{\infty} A_k(\alpha,\beta,m)k^m a_k z^k,$$

for $\beta \geq 0, \alpha \in \mathbb{R}, \alpha+\beta > 0$ and $m \in \mathbb{N}\cup\{0\}$, which is another new class.

We note here that recently in 2020, Oyekan and Awolere [19] defined a new operator involving convolution on the operator $D^m\left(\sigma,\varphi,t\right)f(z)$.

Next, we shall give an analogue of the Definition 1.3 as follows:

Definition 2.4(Combination of Ruscheweyh Differential Operator and a new generalized Operator involving convolution). We combine the new generalized operator involving convolution defined by (14) and the Ruscheweyh operator defined by (2) to obtain a certain linear operator defined as:

$$RE^m_{\alpha,\beta,\delta}(\sigma,\varphi,t)f(z) = (1-\delta)R^m f(z) + \delta E^m_{\alpha,\beta}(\sigma,\varphi,t)f(z), z \in U.$$

Remark 2. 5 If $f \in \mathcal{A}$, then we have

$$RE^m_{\alpha,\beta,\delta}(\sigma,\varphi,t)f(z) = z + \sum_{k=2}^{\infty}\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m\}a_k z^k,\ z \in U, \qquad (15)$$

where $B_k(m)$ and $A_k(\alpha,\beta,m)$ are as defined in (4) and (6) respectively.

Remark 2. 6 We note the following in respect of $RE^m_{\alpha,\beta,\delta}(\sigma,\varphi,t)f(z)$:

(i) When $t = 0$, we have $RE^m_{\alpha,\beta,\delta}(\sigma,\varphi,0)f(z) \equiv RI^m_{\alpha,\beta,\delta}f(z)$ defined by (8)

(ii) When $\beta = 0$, we have $RE^m_{\alpha,0,\delta}(\sigma,\varphi,t)f(z)$

$$= z + \sum_{k=2}^{\infty}\{(1-\delta)B_k(m) + \delta[1+(k+\varphi-\sigma-1)t]^m\}a_k z^k,\ z \in U,$$

which is a new class.

(iii) When $\varphi = \sigma, t = 1$, we have

$$RE^{m}_{\alpha,\beta,\delta}(\sigma,\sigma,1)f(z) = z + \sum_{k=2}^{\infty}\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)k^m\}a_k z^k,\ \ z \in U,$$

which is a new class

(iv) When $\varphi = \sigma$, we have

$$RE^{m}_{\alpha,\beta,\delta}(\sigma,\sigma,t)f(z) =$$

$$= z + \sum_{k=2}^{\infty}\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)[1+(k-1)t]^m\}a_k z^k,$$

for $\beta \geq 0, \alpha \in \mathbb{R}, \alpha + \beta > 0,\ t \geq 0$, and $m \in \mathbb{N}\cup\{0\}$. Which is new.

Motivated by each of the above definitions and in particular Definitions 1.5 and Definition 1.6, we now define the following classes of analytic functions by making use of the generalized operator $RE^{m}_{\alpha,\beta,\delta}(\sigma,\varphi,t)f(z)$.

Definition 2.7 Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}, \mu \geq 0, 0 \leq \lambda \leq 1, \lambda \leq \mu, \delta \geq 0, 0 \leq \rho < 1,\ \beta \geq 0, t \geq 0,\ \sigma \in [0,\varphi]$, and α a real number such that $\alpha + \beta > 0$. Then $f(z) \in \mathcal{M}^{m,\lambda,\mu,\sigma,\varphi,t}_{\alpha,\beta,\delta}(\rho)$ if and only if

$$Re\left(\frac{z\left(RE^{m}_{\alpha,\beta,\delta}(\sigma,\varphi,t)f(z)\right)' + \mu z^2\left(RE^{m}_{\alpha,\beta,\delta}(\sigma,\varphi,t)f(z)\right)''}{(1-\lambda)RE^{m}_{\alpha,\beta,\delta}(\sigma,\varphi,t)f(z) + \lambda z\left(RE^{m}_{\alpha,\beta,\delta}(\sigma,\varphi,t)f(z)\right)'}\right) > \rho,\quad z \in U. \tag{16}$$

When $\lambda = 0$ in Definition 2.7, we have

Definition 2.8 Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}, \mu \geq 0, \delta \geq 0, 0 \leq \rho < 1,\ \beta \geq 0, t \geq 0,\ \sigma \in [0,\varphi]$, and α a real number such that $\alpha + \beta > 0$. Then $f(z) \in \mathrm{N}^{m,\mu,\sigma,\varphi,t}_{\alpha,\beta,\delta}(\rho)$ if and only if

$$Re\left(\frac{z\left(RE^{m}_{\alpha,\beta,\delta}(\sigma,\varphi,t)f(z)\right)'}{RE^{m}_{\alpha,\beta,\delta}(\sigma,\varphi,t)f(z)}\right)\left(1 + \mu\frac{z\left(RE^{m}_{\alpha,\beta,\delta}(\sigma,\varphi,t)f(z)\right)''}{\left(RE^{m}_{\alpha,\beta,\delta}(\sigma,\varphi,t)f(z)\right)'}\right) > \rho, z \in U. \tag{17}$$

Remark 2. 9 i) $t = 0$, we have $\mathcal{M}^{m,\lambda,\mu,\sigma,\varphi}_{\alpha,\beta,\delta}(\rho) = \aleph^{m,\lambda,\mu}_{\alpha,\beta,\delta}(\rho)$ and $\mathrm{N}^{m,\mu,\sigma,\varphi}_{\alpha,\beta,\delta}(\rho) = \Omega^{m,\mu}_{\alpha,\beta,\delta}(\rho)$, i.e [13, Definition 1.8] and [13, Definition 1.9] respectively, which are Definition 1.5 and Definition 1.6 above.

Remark 2. 10 We note that: i) $\mathcal{M}^{m,0,0,\sigma,\varphi,0}_{\alpha,\beta,\delta}(\rho) = S^{m}_{\alpha,\beta,\delta}(\rho)$ ([11]), ii) $\mathcal{M}^{m,\frac{1}{2},\frac{1}{2},\sigma,\varphi,0}_{\alpha,\beta,\delta}(\rho) = K^{m}_{\alpha,\beta,\delta}(\rho)$ ([11]), iii) $\mathcal{M}^{m,1,1,\sigma,\varphi,0}_{\alpha,\beta,\delta}(\rho) = C^{m}_{\alpha,\beta,\delta}(\rho)$ ([11]), iv) $\mathcal{M}^{m,0,\frac{1}{2},\sigma,\varphi,0}_{\alpha,\beta,\delta}(\rho) = \mathfrak{I}^{m}_{\alpha,\beta,\delta}(\rho)$ ([10]), v) $\mathcal{M}^{m,0,1,\sigma,\varphi,0}_{\alpha,\beta,\delta}(\rho) = \mathcal{R}^{m}_{\alpha,\beta,\delta}(\rho)$ ([10]), vi) $\mathcal{M}^{m,\frac{1}{2},1,\sigma,\varphi,0}_{\alpha,\beta,\delta}(\rho) = \ell^{m}_{\alpha,\beta,\delta}(\rho)$ ([10]), vii) $\mathcal{M}^{m,0,0,\sigma,\varphi,0}_{\alpha,\beta,\delta}(\rho) = S^{m}_{\beta,\delta}(\rho)$ ([14]) and viii) $\mathcal{M}^{m,1,1,\sigma,\varphi,0}_{\alpha,\beta,\delta}(\rho) = C^{m}_{\beta,\delta}(\rho)$ ([14]).

III. MAIN RESULTS

In this section, we make use of the techniques in [20] to give characterization properties for the function $f(z)$ of the form (1) to belong to the class $\mathcal{M}^{m,\lambda,\mu,\sigma,\varphi,t}_{\alpha,\beta,\delta}(\rho)$ by obtaining the coefficient bounds. Also, we state and prove the distortion theorem of functions belonging to this class and lastly, analytic functions with negative coefficients belonging to the above class are considered.

3.1 Characterization Properties for the function $f \in \mathcal{M}^{m,\lambda,\mu,\sigma,\varphi,t}_{\alpha,\beta,\delta}(\rho)$.

We begin by finding a sufficient condition for functions $f(z) \in \mathcal{A}$ of the form (1) to belong to the class $\mathcal{M}^{m,\lambda,\mu,\sigma,\varphi,t}_{\alpha,\beta,\delta}(\rho)$.

Theorem 3.1.1 Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}, \mu \geq 0, 0 \leq \lambda \leq 1, \lambda \leq \mu, \delta \geq 0, 0 \leq \rho < 1,\ \beta \geq 0, t \geq 0,\ \sigma \in [0,\varphi]$, and α a real number such that $\alpha + \beta > 0$. If

$$\sum_{k=2}^{\infty}\{[(k-\rho) + (k-1)(k\mu - \rho\lambda)][(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m]\}|a_k| \leq 1-\rho, \tag{18}$$

then $f(z) \in \mathcal{M}_{\alpha,\beta,\delta}^{m,\lambda,\mu,\sigma,\varphi,t}(\rho)$. The result (18) is sharp.

Proof. For the proof, we need to show that

$$\left|\frac{z\left(RE_{\alpha,\beta,\delta}^{m}(\sigma,\varphi,t)f(z)\right)' + \mu z^2\left(RE_{\alpha,\beta,\delta}^{m}(\sigma,\varphi,t)f(z)\right)''}{(1-\lambda)RE_{\alpha,\beta,\delta}^{m}(\sigma,\varphi,t)f(z) + \lambda z\left(RE_{\alpha,\beta,\delta}^{m}(\sigma,\varphi,t)f(z)\right)'} - 1\right| \le 1-\rho. \quad (19)$$

We have that

$$\left|\frac{z\left(RE_{\alpha,\beta,\delta}^{m}(\sigma,\varphi,t)f(z)\right)' + \mu z^2\left(RE_{\alpha,\beta,\delta}^{m}(\sigma,\varphi,t)f(z)\right)''}{(1-\lambda)RE_{\alpha,\beta,\delta}^{m}(\sigma,\varphi,t)f(z) + \lambda z\left(RE_{\alpha,\beta,\delta}^{m}(\sigma,\varphi,t)f(z)\right)'} - 1\right|$$

$$= \left|\frac{\sum_{k=2}^{\infty}(k-1)(1+\mu k-\lambda)[(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m]a_k z^k}{z+\sum_{k=2}^{\infty}(1+\mu k-\lambda)[(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m]a_k z^k}\right|$$

$$\le \frac{\sum_{k=2}^{\infty}(k-1)(1+\mu k-\lambda)[(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m]|a_k||z|^{k-1}}{1-\sum_{k=2}^{\infty}(1+\mu k-\lambda)[(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m]\,|a_k||z|^{k-1}}$$

$$\le \frac{\sum_{k=2}^{\infty}(k-1)(1+\mu k-\lambda)[(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m]|a_k|}{1-\sum_{k=2}^{\infty}(1+\mu k-\lambda)[(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m]|a_k|}.$$

The last expression is bounded above by $1-\rho$ if

$$\sum_{k=2}^{\infty}(k-1)(1+\mu k-\lambda)(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m|a_k|$$
$$\le (1-\rho)\left(1-\sum_{k=2}^{\infty}(1+\mu k-\lambda)[(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m]|a_k|\right)$$

which is equivalent to (17). Hence,

$$\left|\frac{\boldsymbol{z}\left(RE_{\alpha,\beta,\delta}^{m}(\sigma,\varphi,t)f(z)\right)' + \mu z^2\left(RE_{\alpha,\beta,\delta}^{m}(\sigma,\varphi,t)f(z)\right)''}{(1-\lambda)RE_{\alpha,\beta,\delta}^{m}(\sigma,\varphi,t)f(z) + \lambda z\left(RE_{\alpha,\beta,\delta}^{m}(\sigma,\varphi,t)f(z)\right)'} - 1\right| \le 1-\rho.$$

The proof is complete.

The result is sharp and the extremal function is given by

$$f(z) = z + \sum_{k=2}^{\infty}\frac{1-\rho}{[(k-\rho)+(k-1)(k\mu-\rho\lambda)]\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m\}}z^k.$$

Theorem 3.1.2 Let the hypothesis of Theorem 3.1.1be satisfied. Then

$$|a_k| \le \frac{1-\rho}{[(k-\rho)+(k-1)(k\mu-\rho\lambda)]\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^{m\}}}, k \ge 2.$$

Theorem 3.1.3 Let $0 \le \rho_1 \le \rho_2 < 1$. Then

$$\mathcal{M}_{\alpha,\beta,\delta}^{m,\lambda,\mu,\sigma,\varphi,t}(\rho_2) \subseteq \mathcal{M}_{\alpha,\beta,\delta}^{m,\lambda,\mu,\sigma,\varphi,t}(\rho_1).$$

Taking $\lambda = 0$ in Theorem 3.1 we obtain

Corollary 3.1.4 Let$f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \ge 0,\ \delta \ge 0, 0 \le \rho < 1,\ \beta \ge 0, t \ge 0,\ \sigma \in [0,\varphi],$ and α a real number such that $\alpha + \beta > 0$. If

$$\sum_{\boldsymbol{k=2}}^{\infty}\{[(k-\rho)+(k-1)k\mu][(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m]\}|a_k| \le 1-\rho, \quad (20)$$

then $f(z) \in \mathrm{N}_{\alpha,\beta,\delta}^{m,\mu,\sigma,\varphi,t}(\rho)$. The result (17) is sharp, the extremal function being

$$f_1(z) = z$$

$$+\sum_{k=2}^{\infty}\frac{1-\rho}{[(k-\rho)+(k-1)k\mu]\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m\}}z^k, z\in U.$$

Setting $t=0$ in Theorem 3.1.1, we have the following:

Corollary 3.1.5 Let $f\in\mathcal{A}, m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}, \mu\geq 0, 0\leq\lambda\leq 1, \lambda\leq\mu, \delta\geq 0, 0\leq\rho<1, \beta\geq 0,$ and α a real number such that $\alpha+\beta>0$. If

$$\sum_{k=2}^{\infty}\{[(k-\rho)+(k-1)(k\mu-\rho\lambda)][(1-\delta)B_k+\delta A_k(\alpha,\beta,m)]\}|\boldsymbol{a_k}|\leq 1-\rho,$$

then $f(z)\in\aleph_{\alpha,\beta,\delta}^{m,\lambda,\mu}(\rho)$.This result is due to Swamy in [13, Theorem 2.1].

Letting $t=0,\lambda=0$ in Theorem 3.1.1, we have the following:

Corollary 3.1.6 Let $f\in\mathcal{A}, m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}, \mu\geq 0, \delta\geq 0, 0\leq\rho<1, \beta\geq 0,$ and α a real number such that $\alpha+\beta>0$. If

$$\sum_{k=2}^{\infty}\{[(k-\rho)+(k-1)k\mu][(1-\delta)B_k+\delta A_k(\alpha,\beta,m)]\}|a_k|\leq 1-\rho,$$

then $f(z)\in\Omega_{\alpha,\beta,\delta}^{m,\lambda,\mu}(\rho)$. This result is due to Swamy in [13, Corollary2.4]

3.2 Distortion Theorems

Theorem 3.2.1 Let$f\in\mathcal{A}, m\in\mathbb{N}_{\mathbf{0}}=\mathbb{N}\cup\{0\}, \mu\geq 0, 0\leq\lambda\leq 1, \lambda\leq\mu, \delta\geq 0, 0\leq\rho<1, \beta\geq 0, t\geq 0, \sigma\in[0,\varphi],$ and α a real number such that $\alpha+\beta>0$. If

$$\sum_{k=2}^{\infty}\left\{\begin{matrix}[(k-\rho)+(k-1)(k\mu-\rho\lambda)][(1-\delta)B_k(m)\\ +\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m]\end{matrix}\right\}|a_k| \leq 1-\rho, \quad (21)$$

then

$$|z|-\frac{1-\rho}{(2-\rho)+(2\mu-\rho\lambda)}|z|^2\leq\left|RE_{\alpha,\beta,\delta}^m(\sigma,\varphi,t)f(z)\right|\leq|z|+\frac{1-\rho}{(2-\rho)(2\mu-\rho\lambda)}|z|^2, z\in U.$$

Proof. From Theorem 3.1.1 we have that

$$[(2-\rho)+(2\mu-\rho\lambda)]\sum_{k=2}^{\infty}\{[(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k-1)t]^m]\}|a_k|\leq$$

$$\sum_{k=2}^{\infty}\{[(k-\rho)+(k-1)(k\mu-\rho\lambda)][(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m]\}|a_k|\leq 1-\rho,$$

Thus

$$\sum_{k=2}^{\infty}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m\}|a_k|\leq\frac{1-\rho}{(2-\rho)(2\mu-\rho\lambda)}.$$

Therefore,

$$\left|RE_{\alpha,\beta,\delta}^m(\sigma,\varphi,t)f(z)\right|$$

$$\leq|z|+\sum_{k=2}^{\infty}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m\}|a_k|\,|z|^k$$

$$\leq|z|+|z|^2\sum_{k=2}^{\infty}\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m\}|a_k|$$

$$\leq|z|+\frac{1-\rho}{(2-\rho)(2\mu-\rho\lambda)}|z|^2.$$

Similarly,

$$\left|RE_{\alpha,\beta,\delta}^m(\sigma,\varphi,t)f(z)\right|$$

$$\geq |z| - \sum_{k=2}^{\infty}\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m\}|a_k|\,|z|^k$$
$$\geq |z| - |z|^2\sum_{k=2}^{\infty}\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m\}|a_k|$$
$$\geq |z| - \frac{1-\rho}{(2-\rho)(2\mu-\rho\lambda)}|z|^2.$$

The proof is complete.

Taking $t = 0$ in Theorem 3.2.1, we get

Corollary 3.2.2 Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \geq 0, 0 \leq \lambda \leq 1, \lambda \leq \mu, \delta \geq 0, 0 \leq \rho < 1,\ \beta \geq 0,$ and α a real number such that $\alpha + \beta > 0$. If

$$\sum_{k=2}^{\infty}\{[(k-\rho)+(k-1)(k\mu-\rho\lambda)][(1-\delta)B_k + \delta A_k(\alpha,\beta,m)]\}|a_k| \leq 1-\rho,$$

then,

$$|z| - \frac{1-\rho}{(2-\rho)+(2\mu-\rho\lambda)}|z|^2 \leq \left|RE^m_{\alpha,\beta,\delta}(\sigma,\varphi,0)f(z)\right| \leq |z| + \frac{1-\rho}{(2-\rho)(2\mu-\rho\lambda)}|z|^2, z \in U,$$

where $RE^m_{\alpha,\beta,\delta}(\sigma,\varphi,0)f(z) \equiv RI^m_{\alpha,\beta,\delta}f(z)$. Hence, we have Theorem 3.1 in Swamy [13].

Putting $t = \lambda = 0$ in Theorem 3.2.1, we get

Corollary 3.2.3 Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \geq 0, 0 \leq \lambda \leq 1, \lambda \leq \mu, \delta \geq 0, 0 \leq \rho < 1,\ \beta \geq 0,$ and α a real number such that $\alpha + \beta > 0$. If

$$\sum_{k=2}^{\infty}\{[(k-\rho)+(k-1)k\mu][(1-\delta)B_k + \delta A_k(\alpha,\beta,m)]\}|a_k| \leq 1-\rho,$$

then

$$|z| - \frac{1-\rho}{2(1+\mu)-\rho}|z|^2 \leq \left|RE^m_{\alpha,\beta,\delta}(\sigma,\varphi,0)f(z)\right| \leq |z| + \frac{1-\rho}{2(1+\mu)-\rho}|z|^2, z \in U.$$

This is equivalent to
$|z| - \frac{1-\rho}{2(1+\mu)-\rho}|z|^2 \leq \left|RI^m_{\alpha,\beta,\delta}f(z)\right| \leq |z| + \frac{1-\rho}{2(1+\mu)-\rho}|z|^2, z \in U,$ which is Corollary 3.2 in Swamy [13].

Letting $\lambda = 0$ in Theorem 3.2.1, we get

Corollary 3.2.4 Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \geq 0, \delta \geq 0, 0 \leq \rho < 1,\ \beta \geq 0, t \geq 0,$ and α a real number such that $\alpha + \beta > 0$. If

$$\sum_{k=2}^{\infty}\{[(k-\rho)+(k-1)k\mu][(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m]\}|a_k| \leq 1-\rho,$$

,

then

$$|z| - \frac{1-\rho}{2(1+\mu)-\rho}|z|^2 \leq \left|RE^m_{\alpha,\beta,\delta}(\sigma,\varphi,t)f(z)\right| \leq |z| + \frac{1-\rho}{2(1+\mu)-\rho}|z|^2, z \in U.$$

Theorem 3.2.5 Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \geq 0, 0 \leq \lambda \leq 1, \lambda \leq \mu, \delta \geq 0, 0 \leq \rho < 1,\ \beta \geq 0, t \geq 0,\ \sigma \in [0,\varphi],$ and α a real number such that $\alpha + \beta > 0$. If

$$\sum_{k=2}^{\infty}\{[(k-\rho)+(k-1)(k\mu-\rho\lambda)][(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m]\}|a_k| \leq 1-\rho,$$

then

$$|f(z)| \geq |z| - \frac{(1-\rho)(\alpha+\beta)^m}{[(2-\rho)+(2\mu-\rho\lambda)]\{(m+1)(1-\delta)(\alpha+\beta)^m + \delta(\alpha+2\beta)^m[1+(1+\varphi-\sigma)t]^m\}}|z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{(1-\rho)(\alpha+\beta)^m}{[(2-\rho)+(2\mu-\rho\lambda)]\{(m+1)(1-\delta)(\alpha+\beta)^m+\delta(\alpha+2\beta)^m[1+(1+\varphi-\sigma)t]^m\}}|z|^2. z \in U$$

Proof. From Theorem 3.1.1, we have

$$\frac{[(2-\rho)+(2\mu-\rho\lambda)]\{(m+1)(1-\delta)(\alpha+\beta)^m+\delta(\alpha+2\beta)^m[1+(1+\varphi-\sigma)t]^m\}}{(\alpha+\beta)^m}\sum_{k=2}^{\infty}|a_k| \leq$$

$$\sum_{k=2}^{\infty}\{[(k-\rho)+(k-1)(k\mu-\rho\lambda)][(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m]\}|a_k| \leq 1-\rho.$$

Therefore,

$$\sum_{k=2}^{\infty}|a_k| \leq \frac{(1-\rho)(\alpha+\beta)^m}{[(2-\rho)+(2\mu-\rho\lambda)]\{(m+1)(1-\delta)(\alpha+\beta)^m+\delta(\alpha+2\beta)^m[1+(1+\varphi-\sigma)t]^m\}}.$$

Hence,

$$|f(z)| \leq |z| + |z|^2\sum_{k=2}^{\infty}|a_k|$$

$$\leq |z| + \frac{(1-\rho)(\alpha+\beta)^m}{[(2-\rho)+(2\mu-\rho\lambda)]\{(m+1)(1-\delta)(\alpha+\beta)^m+\delta(\alpha+2\beta)^m[1+(1+\varphi-\sigma)t]^m\}}|z|^2.$$

Similarly,

$$|f(z)| \geq |z| - |z|^2\sum_{k=2}^{\infty}|a_k| \leq |z|$$

$$-\frac{(1-\rho)(\alpha+\beta)^m}{[(2-\rho)+(2\mu-\rho\lambda)]\{(m+1)(1-\delta)(\alpha+\beta)^m+\delta(\alpha+2\beta)^m[1+(1+\varphi-\sigma)t]^m\}}|z|^2.$$

Taking $t = 0$ in Theorem 3.2.5, we get

Corollary 3.2.6 Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \geq 0, 0 \leq \lambda \leq 1, \lambda \leq \mu, \delta \geq 0, 0 \leq \rho < 1,\ \beta \geq 0,$ and α a real number such that $\alpha + \beta > 0$. If

$$\sum_{k=2}^{\infty}\{[(k-\rho)+(k-1)(k\mu-\rho\lambda)][(1-\delta)(\alpha+\beta)^m+\delta(\alpha+2\beta)^m]\}a_k \leq 1-\rho,$$

then

$$|f(z)| \geq |z| - \frac{(1-\rho)(\alpha+\beta)^m}{[(2-\rho)+(2\mu-\rho\lambda)]\{(m+1)(1-\delta)(\alpha+\beta)^m+\delta(\alpha+2\beta)^m\}}|z|^2, \qquad z \in U.$$

and

$$|f(z)| \leq |z| + \frac{(1-\rho)(\alpha+\beta)^m}{[(2-\rho)+(2\mu-\rho\lambda)]\{(m+1)(1-\delta)(\alpha+\beta)^m+\delta(\alpha+2\beta)^m\}}|z|^2, \qquad z \in U.$$

Allowing $t = \lambda = 0$ in Theorem 3.2.5, we get

Corollary 3.2.7 Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \geq 0,\ \delta \geq 0, 0 \leq \rho < 1,\ \beta \geq 0,$ and α a real number such that $\alpha + \beta > 0$. If

$$\sum_{k=2}^{\infty}\{[(k-\rho)+(k-1)k\mu][(1-\delta)(\alpha+\beta)^m+\delta(\alpha+2\beta)^m]\}a_k \leq 1-\rho,$$

then

$$|f(z)| \geq |z| - \frac{(1-\rho)(\alpha+\beta)^m}{[2(1+\mu)-\rho]\{(m+1)(1-\delta)(\alpha+\beta)^m+\delta(\alpha+2\beta)^m\}}|z|^2, \qquad z \in U.$$

and

$$|f(z)| \leq |z| + \frac{(1-\rho)(\alpha+\beta)^m}{[2(1+\mu)-\rho]\{(m+1)(1-\delta)(\alpha+\beta)^m+\delta(\alpha+2\beta)^m\}}|z|^2, \qquad z \in U.$$

If $\lambda = 0$ in Theorem 3.2.5, we get

Corollary 3.2.8 Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \geq 0,\ \delta \geq 0, 0 \leq \rho < 1,\ \beta \geq 0, t \geq 0,\ \sigma \in [0, \varphi]$, and α a real number such that $\alpha + \beta > 0$. If

$$\sum_{k=2}^{\infty}\{[(k-\rho) + (k-1)k\mu][(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)[1 + (k+\varphi-\sigma-1)t]^m]\}a_k \leq 1-\rho,$$

then

$$|f(z)| \geq |z| - \frac{(1-\rho)(\alpha+\beta)^m}{[2(1+\mu)-\rho]\{(m+1)(1-\delta)(\alpha+\beta)^m + \delta(\alpha+2\beta)^m[1+(1+\varphi-\sigma)t]^m\}}|z|^2, \qquad z \in U,$$

and

$$|f(z)| \leq |z| + \frac{(1-\rho)(\alpha+\beta)^m}{[2(1+\mu)-\rho]\{(m+1)(1-\delta)(\alpha+\beta)^m + \delta(\alpha+2\beta)^m[1+(1+\varphi-\sigma)t]^m\}}|z|^2, \qquad z \in U.$$

3.3 Functions with negative coefficients

We denote by $T\mathcal{M}_{\alpha,\beta,\delta}^{m,\lambda,\mu,\sigma,\varphi,t}(\rho)$ and $T\mathbb{N}_{\alpha,\beta,\delta}^{m,\mu,\sigma,\varphi,t}(\rho)$, the classes of functions $f \in T$ satisfying (16) and (17) respectively. Clearly (i) $T\mathcal{M}_{\alpha,\beta,\delta}^{m,\lambda,\mu,\sigma,\varphi,t}(\rho) = \mathcal{M}_{\alpha,\beta,\delta}^{m,\lambda,\mu,\sigma,\varphi,t}(\rho) \cap T$ (ii) $T\mathbb{N}_{\alpha,\beta,\delta}^{m,\mu,\sigma,\varphi,t}(\rho) = \mathbb{N}_{\alpha,\beta,\delta}^{m,\mu,\sigma,\varphi,t}(\rho) \cap T$.

Next, we shall show that the Theorem 3.1.1 is also a necessary condition for $f \in T$ to be in the class $T\mathcal{M}_{\alpha,\beta,\delta}^{m,\lambda,\mu,\sigma,\varphi,t}(\rho)$ by following the technique of Silverman [20].

Theorem 3.3.1 Let $f \in T, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \geq 0, 0 \leq \lambda \leq 1, \lambda \leq \mu, \delta \geq 0, 0 \leq \rho < 1,\ \beta \geq 0, t \geq 0,\ \sigma \in [0, \varphi]$, and α a real number such that $\alpha + \beta > 0$. Then$f(z) \in T\mathcal{M}_{\alpha,\beta,\delta}^{m,\lambda,\mu,\sigma,\varphi,t}(\rho)$ If and only if

$$\sum_{k=2}^{\infty}\{[(k-\rho) + (k-1)(k\mu - \rho\lambda)][(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)[1 + (k+\varphi-\sigma-1)t]^m]\}|a_k| \leq 1-\rho.$$

The result is sharp.

Proof. In view of Theorem 3.1.1, we only need to prove the only if part of Theorem 3.3.1.

Since $T\mathcal{M}_{\alpha,\beta,\delta}^{m,\lambda,\mu,\sigma,\varphi,t}(\rho) \subset \mathcal{M}_{\alpha,\beta,\delta}^{m,\lambda,\mu,\sigma,\varphi,t}(\rho)$, for functions $f \in T\mathcal{M}_{\alpha,\beta,\delta}^{m,\lambda,\mu,\sigma,\varphi,t}(\rho)$, we can write

$$Re\left(\frac{\boldsymbol{z}\left(RE_{\alpha,\beta,\delta}^m(\sigma,\varphi,t)f(z)\right)' + \mu z^2\left(RE_{\alpha,\beta,\delta}^m(\sigma,\varphi,t)f(z)\right)''}{(1-\lambda)RE_{\alpha,\beta,\delta}^m(\sigma,\varphi,t)f(z) + \lambda z\left(RE_{\alpha,\beta,\delta}^m(\sigma,\varphi,t)f(z)\right)'}\right)$$
$$= Re\left\{\frac{z - \sum_{k=2}^{\infty}\big(1+\mu(k-1)\big)[(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)](1+(k+\varphi-\sigma-1)t)^m k a_k z^k}{z - \sum_{k=2}^{\infty}\big(1+\lambda(k-1)\big)[(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)](1+(k+\varphi-\sigma-1)t)^m a_k z^k}\right\}$$
$$> \rho. \tag{22}$$

Clearing the denominator in (22) and letting $z \to 1^-$ through the real values, we obtain

$$z - \sum_{k=2}^{\infty}\{\big(1+\mu(k-1)\big)[(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)](1+(k+\varphi-\sigma-1)t)^m\}a_k \geq$$
$$z - \sum_{k=2}^{\infty}\{\big(1+\lambda(k-1)\big)[(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)](1+(k+\varphi-\sigma-1)t)^m\}a_k.$$

So that

$$\sum_{k=2}^{\infty}\{[(k-\rho) + (k-1)(\rho\lambda - k\mu)][(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)[1 + (k+\varphi-\sigma-1)t]^m]\}a_k \geq \rho - 1.$$

Hence,

$$\sum_{k=2}^{\infty}\{[(k-\rho) + (k-1)(k\mu - \rho\lambda)][(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)[1 + (k+\varphi-\sigma-1)t]^m]\}|a_k| \leq 1-\rho.$$

The result is sharp and the extremal function is given by

$$f_k(z) = z - \sum_{k=2}^{\infty} \frac{1-\rho}{[(2-\rho)+(2\mu-\rho\lambda)][(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma-1)t]^m]} z^k,$$
$$k \geq 2, \qquad z \in U.$$

Remark 3.3.2 i)Taking $t = 0$ in Theorem 3.3.2 we get Theorem 4.1 of Swamy [13].ii)Taking $\lambda = t = 0$ in Theorem 3.3.2 we obtain Corollary 4.1 of Swamy [13].

With the above coefficient bounds, one obtain the following distortion theorem for the class $T\mathcal{M}_{\alpha,\beta,\delta}^{m,\lambda,\mu,\sigma,\varphi,t}(\rho)$:

Theorem 3.3.3 Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \geq 0, 0 \leq \lambda \leq 1, \lambda \leq \mu, \delta \geq 0, 0 \leq \rho < 1,\ \beta \geq 0, t \geq 0,\ \sigma \in [0,\varphi]$, and α a
real number such that $\alpha + \beta > 0$. If $f(z) \in T\mathcal{M}_{\alpha,\beta,\delta}^{m,\lambda,\mu,\sigma,\varphi,t}(\rho)$, then

$$|f(z)| \geq |z| - \frac{(1-\rho)(\alpha+\beta)^m}{[(2-\rho)+(2\mu-\rho\lambda)]\{(m+1)(1-\delta)(\alpha+\beta)^m+\delta(\alpha+2\beta)^m[1+(1+\varphi-\sigma)t]^m\}}|z|^2, z \in U$$

and

$$|f(z)| \leq |z| + \frac{(1-\rho)(\alpha+\beta)^m}{[(2-\rho)+(2\mu-\rho\lambda)]\{(m+1)(1-\delta)(\alpha+\beta)^m+\delta(\alpha+2\beta)^m[1+(1+\varphi-\sigma)t]^m\}}|z|^2, z \in U$$

with

$$f(z) = z - \sum_{k=2}^{\infty} \frac{1-\rho}{[(2-\rho)+(2\mu-\rho\lambda)]\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma)t]^m\}} z^2, z \in U$$

$(|z| = r)$, for equality.

Allowing $t = 0, \lambda = 0$ in Theorem 3.3.3, we obtain

Corollary 3.3.4 Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \geq 0, \lambda \leq \mu, \delta \geq 0, 0 \leq \rho < 1,\ \beta \geq 0$, and α a real number such that $\alpha + \beta > 0$. If $f \in T\mathcal{M}_{\alpha,\beta,\delta}^{m,\mu,0,\sigma,\varphi,0}(\rho)$, then

$$|f(z)| \geq |z| - \frac{(1-\rho)(\alpha+\beta)^m}{[(2-\rho)+2\mu]\{(m+1)(1-\delta)(\alpha+\beta)^m+\delta(\alpha+2\beta)^m\}}|z|^2, \qquad z \in U.$$

and

$$|f(z)| \leq |z| + \frac{(1-\rho)(\alpha+\beta)^m}{[(2-\rho)+2\mu]\{(m+1)(1-\delta)(\alpha+\beta)^m+\delta(\alpha+2\beta)^m\}}|z|^2, \qquad z \in U,$$

with

$$f(z) = z - \sum_{k=2}^{\infty} \frac{1-\rho}{[(2-\rho)+2\mu]\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)\}} z^2, z \in U, (|z| = r),$$

for equality.

Putting $t = 0$ in Theorem 3.3.5, we get

Corollary 3.3.5 Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \geq 0, 0 \leq \lambda \leq 1, \lambda \leq \mu, \delta \geq 0, 0 \leq \rho < 1,\ \beta \geq 0,\ \sigma \in [0,\varphi]$, and α a real number such that $\alpha + \beta > 0$. If $f \in T\mathcal{M}_{\alpha,\beta,\delta}^{m,\mu,\lambda,\sigma,\varphi,t}(\rho)$, then

$$|f(z)| \geq |z| - \frac{(1-\rho)(\alpha+\beta)^m}{[(2-\rho)+(2\mu-\rho\lambda)]\{(m+1)(1-\delta)(\alpha+\beta)^m+\delta(\alpha+2\beta)^m\}}|z|^2, \qquad z \in U$$

and

$$|f(z)| \leq |z| + \frac{(1-\rho)(\alpha+\beta)^m}{[(2-\rho)+(2\mu-\rho\lambda)]\{(m+1)(1-\delta)(\alpha+\beta)^m+\delta(\alpha+2\beta)^m\}}|z|^2, \qquad z \in U,$$

with

$$f(z) = z - \sum_{k=2}^{\infty} \frac{1-\rho}{[(2-\rho)+(2\mu-\rho\lambda)]\{(1-\delta)B_k(m)+\delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma)t]^m\}} z^2,$$

$z \in U, (|z| = r)$, for equality.

Theorem 3.3.5 would yield the following corollary,when $\lambda = \mu = 0$.

Corollary 3.3.6 Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\delta \geq 0, 0 \leq \rho < 1$, $\beta \geq 0, t \geq 0$, $\sigma \in [0, \varphi]$, and α a real number such that $\alpha + \beta > 0$. If $f \in T\mathcal{M}_{\alpha,\beta,\delta}^{m,0,0,\sigma,\varphi,t}(\rho)$, then

$$|f(z)| \geq |z| - \frac{(1-\rho)(\alpha+\beta)^m}{(2-\rho)\{(m+1)(1-\delta)(\alpha+\beta)^m + \delta(\alpha+2\beta)^m[1+(1+\varphi-\sigma)t]^m\}}|z|^2, \qquad z \in U$$

and

$$|f(z)| \leq |z| + \frac{(1-\rho)(\alpha+\beta)^m}{(2-\rho)\{(m+1)(1-\delta)(\alpha+\beta)^m + \delta(\alpha+2\beta)^m[1+(1+\varphi-\sigma)t]^m\}}|z|^2, \qquad z \in U,$$

with

$$f(z) = z - \sum_{k=2}^{\infty} \frac{1-\rho}{(2-\rho)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)[1+(k+\varphi-\sigma)t]^m\}} z^2, \qquad z \in U, \qquad (|z| = r),$$

for equality.

We conclude the below result by taking $\mu = \frac{1}{2}, \lambda = t = 0$ in Theorem 3.3.5.

Corollary 3.3.7 Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \geq 0, \delta \geq 0, 0 \leq \rho < 1$, $\beta \geq 0$, and α a real number such that $\alpha + \beta > 0$. If $f(z) \in T\mathcal{M}_{\alpha,\beta,\delta}^{m,\frac{1}{2},0,\sigma,\varphi,0}(\rho)$,then

$$|f(z)| \geq |z| - \frac{(1-\rho)(\alpha+\beta)^m}{(3-\rho)\{(m+1)(1-\delta)(\alpha+\beta)^m + \delta(\alpha+2\beta)^m\}}|z|^2, \qquad z \in U.$$

and

$$|f(z)| \leq |z| + \frac{(1-\rho)(\alpha+\beta)^m}{(3-\rho)\{(m+1)(1-\delta)(\alpha+\beta)^m + \delta(\alpha+2\beta)^m\}}|z|^2, \qquad z \in U,$$

with

$$f(z) = z - \sum_{k=2}^{\infty} \frac{1-\rho}{(3-\rho)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)^m\}} z^2, \quad z \in U, (|z| = r)$$

for equality. This corrects Theorem 4.2 (i) in Swamy [10] and it is in agreement with the corrigendum of the paper: New classes containing Ruscheweyh derivative and a new generalized multiplier differential operator, *American International Journal of Research in Science, Technology, Engineering & Mathematics*, 11(1) June – August (2015) 65-71 by S R Swamy, which is available in ResearchGate.

Corollary 3.3.8 asserts aconsequence of Theorem 3.3.3, when $\lambda = \frac{1}{2}\mu = 1, t = 0$.

Corollary 3.3.8 Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \geq 0, \delta \geq 0, 0 \leq \rho < 1$, $\beta \geq 0$, and α a real number such that $\alpha + \beta > 0$. If $f(z) \in T\mathcal{M}_{\alpha,\beta,\delta}^{m,1,0,\sigma,\varphi,0}(\rho)$,then

$$|f(z)| \geq |z| - \frac{2(1-\rho)(\alpha+\beta)^m}{(8-3\rho)\{(m+1)(1-\delta)(\alpha+\beta)^m + \delta(\alpha+2\beta)^m\}}|z|^2, \qquad z \in U$$

and

$$|f(z)| \leq |z| + \frac{2(1-\rho)(\alpha+\beta)^m}{(8-3\rho)\{(m+1)(1-\delta)(\alpha+\beta)^m + \delta(\alpha+2\beta)^m\}}|z|^2, \qquad z \in U,$$

with

$$f(z) = z - \sum_{k=2}^{\infty} \frac{2(1-\rho)}{(8-3\rho)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)^m\}} z^2, \qquad z \in U, (|z| = r),$$

for equality. This corresponds Theorem 4.2 (ii**)** in Swamy [10].

Setting $\mu = 1, \lambda = t = 0$ in Theorem 3.3.3, we arrive at the following:

Corollary 3.3.9 Let $f \in \mathcal{A}, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mu \geq 0, \delta \geq 0, 0 \leq \rho < 1$, $\beta \geq 0$, and α a real number such that $\alpha + \beta > 0$. If $f(z) \in T\mathcal{M}_{\alpha,\beta,\delta}^{m,1,0,\sigma,\varphi,0}(\rho)$, then

$$|f(z)| \geq |z| - \frac{(1-\rho)(\alpha+\beta)^m}{(4-\rho)\{(m+1)(1-\delta)(\alpha+\beta)^m + \delta(\alpha+2\beta)^m\}}|z|^2, \qquad z \in U$$

and

$$|f(z)| \leq |z| + \frac{(1-\rho)(\alpha+\beta)^m}{(4-\rho)\{(m+1)(1-\delta)(\alpha+\beta)^m + \delta(\alpha+2\beta)^m\}}|z|^2, \qquad z \in U,$$

with

$$f(z) = z - \sum_{k=2}^{\infty} \frac{(1-\rho)}{(4-\rho)\{(1-\delta)B_k(m) + \delta A_k(\alpha,\beta,m)^m\}} z^2, \qquad z \in U, (|z| = r)$$

for equality. This corresponds Theorem 4.2 (iii) in Swamy [10].

Remark 3.3.10 Some interesting consequences of Theorem 3.3.5 are the following:

(i) When $\lambda = \mu = t = 0$ in Theorem 3.3.5, we get the Theorem 4.3 (i**)** in Swamy [11]

(ii) When $\lambda = \mu = \frac{1}{2}, t = 0$ in Theorem 3.3.5, we get the Theorem 4.3 (ii) in Swamy [11]

(iii) When $\lambda = \mu = 1, t = 0$ in Theorem 3.3.5, we get the Theorem 4.3 (iii) in Swamy [11]

IV. CONCLUSION

In our present investigation, we have defined and studied a new subclass $\mathcal{M}_{\alpha,\beta,\delta}^{m,\lambda,\mu,\sigma,\varphi,t}(\rho)$ of analytic functions in an open disk. We have pointed out relevant connections of the results presented here with previous known results in the literature and in particular, the ones in the work of Swamy [13]. It is hoped that the introduced new operators and subclasses pointed out can inspire further research by other researchers.

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