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A Simple Model for Calculating Concentration Levels of Pollution in River: A Case Study on Subarnarekha River

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Abstract - *Water is essential part of human and ecology on earth. The quantity and quality of water now a days a very serious problem in many region* especially *in industrial and .mining area of India. Since ecology of river depends upon the quantity of dissolved oxygen in the water and due to disposal of waste produced in river Subarnarekha, organic pollution increase and disturbed ecological balance. Here we make a steady state model which is a pair of coupled reaction diffusion advection equation of pollution and dissolved oxygen concentration. Such a model and its solution will aid on a decision support restriction to be imposed on forming an urban practices and simulation enable scenario to be tested for fish survival which is usually taken as above 30% of the saturated dissolved oxygen concentration.*

Keywords: *Dissolved oxygen, Dispersion, Diffusion, Ecology of River, Steady state.*

I. INTRODUCTION

Subarnarekha means 'streak of gold'. With a drainage area of 1.93 million ha this smallest of India's major inter-state river basins is a mute host to effluents from various uranium mining and processing units. While most rivers in the country are classified -- depending on the pollution load -- on a 'best designated use' basis, the Subarnarekha defies any classification, as the existing parameters do not include radioactivity. The rain-fed Subarnarekha originates 15 kms south of Ranchi on the Chhotanagpur plateau draining the states of Jharkhand, Orissa and West Bengal before entering the Bay of Bengal. The total length of the river is 450 kms and its important tributaries include the Raru, Kanchi, Karkari, Kharkai, Garra and Sankh rivers. Between Mayurbhanj and Singhbhum districts, on the right banks of the Subarnarekha, are the country's richest copper deposits. The proliferation of unplanned and unregulated mining and mineral processing industries has led to a devastating environmental degradation of the region. Improper mining practices have led to uncontrolled dumping of overburden (rock and soil extracted while mining) and mine tailings. During monsoons, this exposed earth flows into the river, increasing suspended solid and heavy metal load in the water, silting the dams and reservoirs..Chapra [1] and Simachaya [3] described water quality model. Murphy [4] gave same information about Dissolved Oxygen. Loukes [7] developed some mathematical model on water pollution. Exploitation of and related industries in the area has exerted a great impact on the water pollution[8] Mibagheri et. al [8] developed mathematical model of water quality in river system in Jajroad River Shukla et. al [9] developed mathematical model and studied its analysis of the depletion of dissolved oxygen. Benedini [10] developed for Mathematical model on water quality in river and stream.

The organic pollution load in urban area is largely due to domestic effluent and survey shows 69% of total BOD load is contributed by domestic effluent and the remaining from industries effluents [2,5,6] As per C.P.C.B, New Delhi "Water Quality in India". The following findings have been observed. Dissolved Oxygen is 1.0 mg/ liter at D/s Mango, Jamshedpur .B.O.D is 0.2-70 mg/ liter with highest at D/s Mango, Jamshedpur. Total Coliform is 140 MPN/ 100ml to 2.7 X 105 MPN/ 100ml.

Figure 1: Subarnarekha River

Nomenclatures:

- t is time (day)
- x is the position (m)
- P is the pollutant concentration (kg m^{-3})
- X is the dissolved oxygen concentration (kg m^{-3})
- L is the length of river (m)
- D_p is the dispersion coefficient of pollutant in the x direction (m_2 day^{-1})
- D_x is the dispersion coefficient of dissolved oxygen in the x direction (m_2 day^{-1}), taken as the same as D_p
- v is the water velocity in the x direction (m day^{-1}) A is the cross-section area (m_2)
- Q is the added pollutant rate along the river (kg $m^{-1}day^{-1}$)
- K_1 is the degradation rate coefficient at 20 °C for pollutant (day^{-1})
- K_2 is the de-aeration rate coefficient at 20°C for dissolved oxygen(day^{-1})
- k is the half-saturated oxygen demand concentration for pollutant decay (kg m^{-3})
- α is the mass transfer of oxygen from air to water(m_2 day-1) ; α = re-aeration rate ,re-aeration rate = 0.055 day^{-1}. A′ = width of 300 unit length of 1)
- S is the saturated oxygen concentration (kg m^{-3}), S = 10 mg L^{-1} 0.01

Mathematical Model formulation

Consider the coupled equations for the pollutant and dissolved oxygen concentrations. The coupling occurs because the oxygen reacts with the pollutant producing harmless compounds. For simplicity, we assume that diffusion is in one dimension along the river and is accompanied by forced convection and so the concentration $P(x, t)$ (of pollutant) and $X(x, t)$ (of dissolved oxygen) satisfies reaction-diffusion-advection equations.

The system of equations which describes the rate of change of the concentration with position x and time t can be expressed in one dimension as

$$\frac{\partial(AP)}{\partial t} = D_P \frac{\partial^2(AP)}{\partial x^2} - \frac{\partial(vAP)}{\partial x} - K_1 \frac{X}{X+k} AP + q\,H(x) \quad , \quad (x < L < \infty, t > 0) \tag{1}$$

$$\frac{\partial(AX)}{\partial t} = D_x \frac{\partial^2(AX)}{\partial x^2} - \frac{\partial(vAX)}{\partial x} - K_2 \frac{X}{X+k} AP + \alpha(S-X) \quad , \quad (x < L < \infty, t > 0) \tag{2}$$

These equations are standard (see Chapra 1997).The first equation includes both addition of pollutant (q), and removal by aeration, and the second equation is a mass balance for oxygen, with addition at the surface, and consumption by pollutant. Here, $H(x)$ is the Heaviside function

$$\mathrm{H}(x) = \begin{cases} 1 \ , \ 0 < x < L \\ 0 \ , \ otherwise \end{cases} \tag{3}$$

This is used to capture the fact that pollutant is discharged for $x > 0$ only. We consider a river where pollutants are discharges in the form of wastes. It is assumed that these pollutants use dissolved oxygen for various biochemical and biodegradation processes. The discharge of pollutants into the river is at the constant rate q and the rate of depletion of concentration P due to biochemical embodying a "Michaelis-Menten" model is given by interaction involving the concentration of dissolved oxygen as well as the concentration P. For dissolved oxygen, it is assumed that the rate of growth of concentration by movement from the air into the river is proportional to the saturated concentration S less the concentration X, that is $\alpha(S–X)$. Interaction involves the concentration of dissolved oxygen as well as the pollutant concentration P. We consider cases with and without dispersion k negligible ($k \approx 0$) and k nonzero. To simplify the equations, we set the values A, v, q, α and S to be constant.

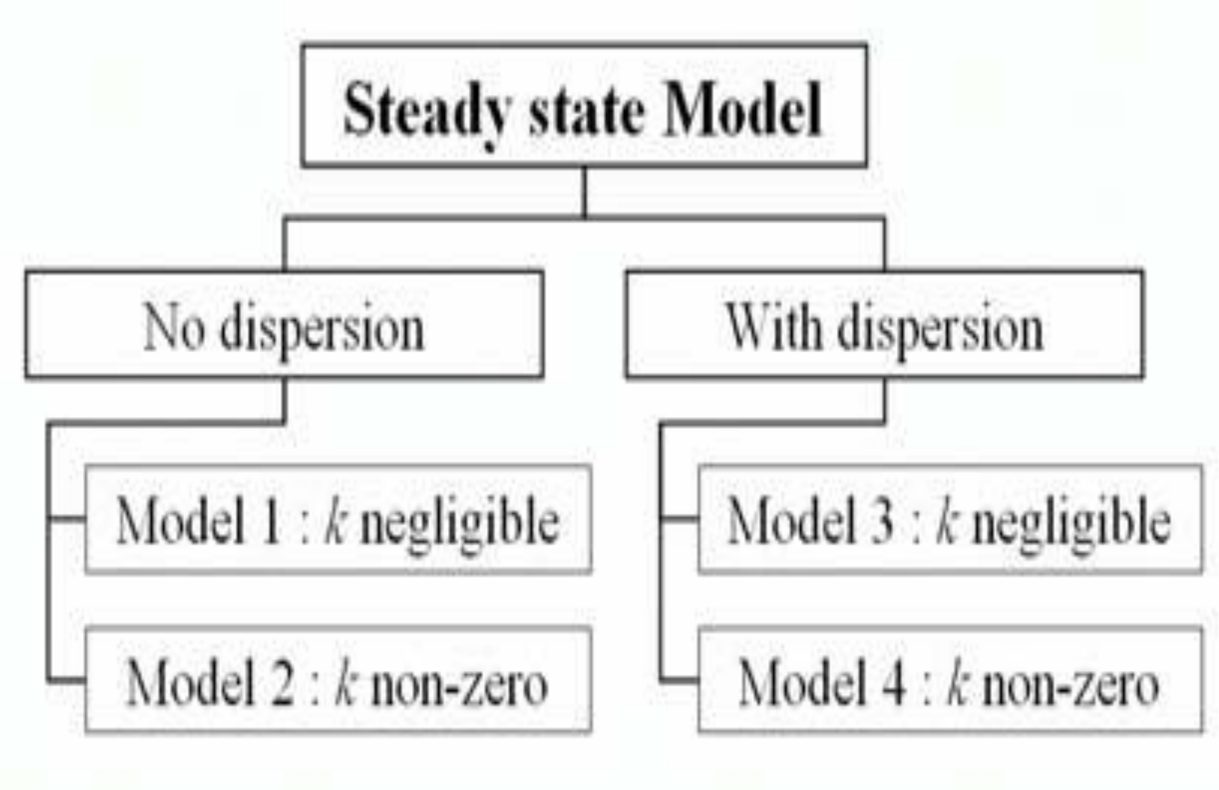

Figure 2: Model special cases

Special Cases of the Mathematical Model

We now consider various special cases from the scheme shown in Figure 2, gradually building up towards the full model. In this paper we consider steady-states, presuming conditions are constant.

Mathematical Model 1

This model is used for the steady state analysis. In this model we have zero dispersion

($D_p = 0$, $D_x = 0$).

$$\frac{d\left(vAP_s(x)\right)}{dx} = -K_1\, AP_s(x) + q\ ,\ (x>0, t>0) \tag{4}$$

$$\frac{d\left(vAX_s(x)\right)}{dx} = -K_2\, AP_s(x) + \alpha\left(S - X_s(x)\right),\ (x>0, t>0) \tag{5}$$

We consider k negligible ($k \approx 0$) and boundary conditions $P_s(0) = 0$ and $X_s(0) = S$. The far downstream pollutant concentration is $Ps(x) = (q/K_1A)(1\text{-}\exp(-K_1x/v))$, and so the downstream Limit is given by q/K_1A. This is shown in Figure 3.

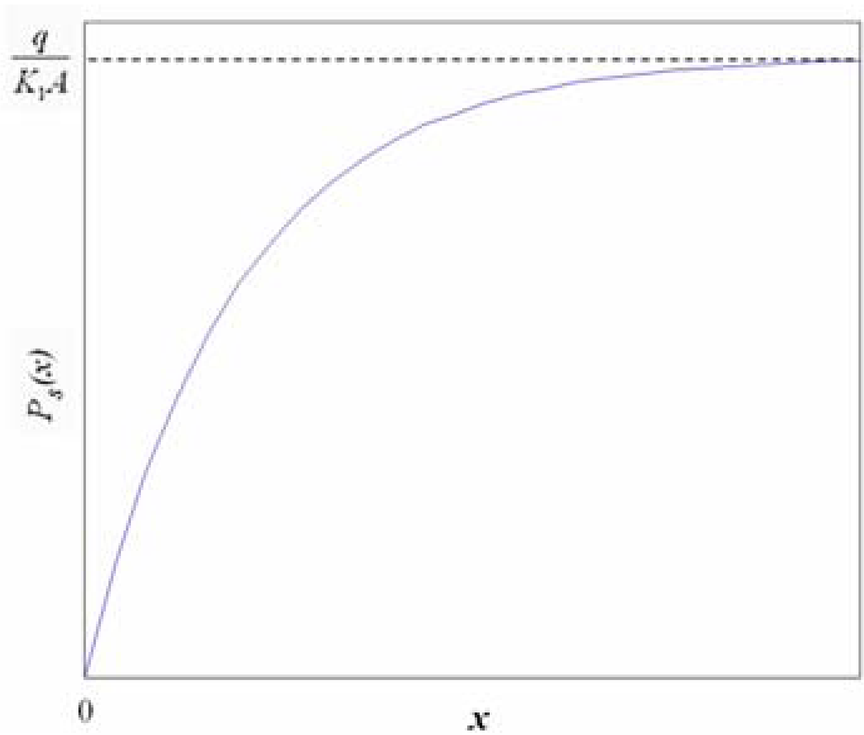

Figure 3. The steady state solution for P with no dispersion in the case k negligible.

Upstream there is no pollution as there is no dispersion. For the dissolved oxygen concentration, the solution is

$$X_s(x) = S - \frac{K_2 q}{K_1}\left[\frac{1}{\alpha} - \left(\frac{1}{\alpha - K_1 A}\right)e^{\frac{-K_1 x}{v}}\right] - \left(\frac{K_2 qA}{\alpha(\alpha - K_1 A)}\right)e^{\frac{-\alpha x}{vA}}, \tag{6}$$

$$\lim_{x\to\infty} X_s(x) = S - \frac{K_2 q}{\alpha K_1} \tag{7}$$

This is shown in Figure 4. If the discharge from inhabitants residing and farming along the river q is such that X is less than 30% of the saturated values S, the fish do not survive (PCD 2000).
So the requirement for q is $q < 0.7\alpha K_1 S/K_2$

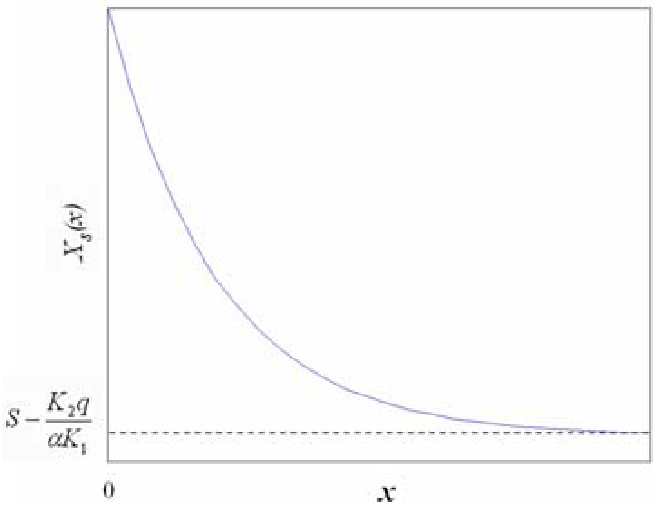

Figure 4. The steady state solution for X with no dispersion in the case k negligible.

Model 2

The previous model simplified because the half-saturated oxygen demand concentration for pollutant decay is negligible ($k \approx 0$). If instead it is significant then

$$\frac{d\left(vAP_s(x)\right)}{dx} = -K_1 \frac{X_s(x)}{X_s(x)+k} AP_s(x) + q\ ,\ (x>0, t>0) \tag{8}$$

$$\frac{d\left(vAX_s(x)\right)}{dx} = -K_2 \frac{X_s(x)}{X_s(x)+k} AP_s(x) + \alpha\left(S - X_s(x)\right),\ (x>0, t>0) \tag{9}$$

With the same boundary conditions $P_s(0) = 0$ and $X_s(0) = S$, the far downstream solutions for pollutant and dissolved oxygen concentration are given, respectively.

$$\left(P_s(x), X_s(x)\right)\Big/_{\text{large } x} = \left(\frac{q}{K_1 A} + \frac{\alpha k q}{\alpha K_1 S - q K_2}, S - \frac{q K_2}{\alpha K_1}\right) \tag{10}$$

The steady far-downstream solution depends therefore on the parameters k and q.
We note the downstream solution above does not exist if $q \geq \alpha K_1 S/K_2$ and in that case,
$X_s(\infty) = 0$.

Mathematical Model 3

We now consider the steady state case with dispersion terms (let $D_p \neq 0$, $D_x \neq 0$).

$$D_p \frac{d^2\left(AP_s(x)\right)}{dx^2} - \frac{d\left(vAP_s(x)\right)}{dx} - K_1 AP_s(x) + q\,H(x) = 0 \quad ,(x > L, t > 0) \tag{11}$$

$$D_x \frac{d^2(AX)}{dx^2} - \frac{d\,(vAX_s(x))}{dx} - K_2 AP_s(x) + \alpha\left(S - X_s(x)\right) = 0 \quad ,(x > L, t > 0) \tag{12}$$

In this model, k is assumed to be negligible ($k \approx 0$) and the solution below is obtained.

$$P_s(x) = \begin{cases} \dfrac{q}{K_1 A}\left(1 - \left(\dfrac{\delta+\beta}{2\beta}\right)e^{(\delta-\beta)x}\right), x \geq 0 \\ \dfrac{q}{K_1 A}\left(\dfrac{\delta-\beta}{2\beta}\right)e^{(\delta+\beta)x}, x < 0 \end{cases} \tag{13}$$

Where

$$\delta = v/2D_p \;\; and \;\; \beta = \left(\sqrt{v^2 + 4D_p K_1}\right)\Big/ 2D_p \tag{14}$$

In this model we used the conditions $P_s(\infty) < \infty$ and $P_s(-\infty) < \infty$. We also require $P'_s(x)$ and $P_s(x)$ to be continuous at $x = 0$. There are no point sources of pollutant (only distributed sources),which makes $P_s(x)$ continuous. Since the dispersive flux $DP'_s(x) - v\,P_s(x)$ is also continuous this implies that $P'_s(x)$ is also continuous. For dissolved oxygen, we find

$$X_s(x) = \begin{cases} S - \dfrac{K_2 q}{K_1 \alpha} + \dfrac{K_2 q}{K_1}\left[\left(\dfrac{\gamma+\eta}{2\eta\alpha} - \dfrac{\delta+\beta}{4\beta\eta A^*} + \dfrac{\delta+\beta}{4\beta\eta B^*}\right)e^{(\gamma-\eta)x} - \dfrac{\delta+\beta}{2\beta\eta A^*}\,x\,e^{(\delta-\beta)x}\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , x \geq 0 \\ S + \dfrac{K_2 q}{K_1}\left[\left(\dfrac{\gamma-\eta}{2\eta\alpha} - \dfrac{\delta+\beta}{4\beta\eta A^*} + \dfrac{\delta-\beta}{4\beta\eta B^*}\right)e^{(\gamma+\eta)x} - \dfrac{\delta-\beta}{2\beta\eta B^*}\,x\,e^{(\delta+\beta)x}\right], x < 0 \end{cases} \tag{15}$$

$$\gamma = \frac{v}{2D_x} \;, \eta = \frac{\sqrt{v^2 + \dfrac{4\alpha D_x}{A}}}{2D_x} \;, A^* = 2AD_x(\delta-\beta) - vA \,, B^* = 2AD_x(\delta+\beta) - vA \tag{16}$$

Again we use the initial conditions $X_s(\infty) < \infty$ and $X_s(-\infty) = S$. Also $X_s(x)$ and $X'_s(x)$ are continuous at $x = 0$. The pollutant concentration $P_s(x)$ is relatively smooth with a discontinuity of q/D_pA in the second derivative at $x = 0$. The dissolved oxygen concentration $X_s(x)$ has a discontinuity in the fourth derivative at $x = 0$. To test our model we set the parameters A, v, q, D_p, D_x and K_2 to be 1; α, S and K_1 to be 2. For the above set of parameters the graph of P versus X is shown in Figure as below.

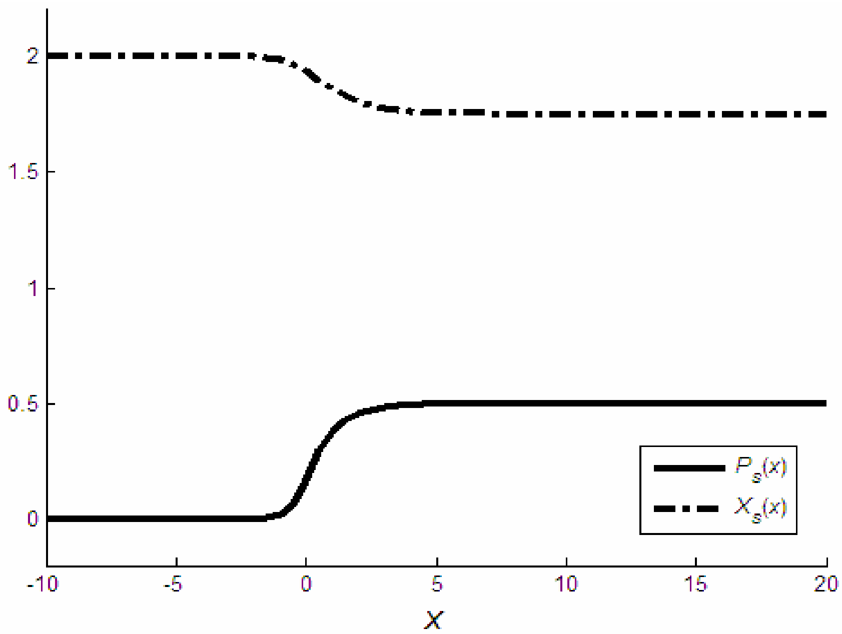

Figure 5: The analytical steady state solution with dispersion for P and X

Mathematical Model 4

The last model includes dispersion terms (let $D_p \neq 0$, $D_x \neq 0$). In this case k is non zero.
The following system of nonlinear second order differential equations is obtained:

$$D_p \frac{d^2\left(AP_s(x)\right)}{dx^2} - \frac{d\left(vAP_s(x)\right)}{dx} - K_1 \frac{X_s(x)}{X_s(x)+k} AP_s(x) + q\,H(x) = 0 \quad ,(x > L, t > 0) \tag{17}$$

$$D_x \frac{d^2(AX_s(x))}{dx^2} - \frac{d\,(vAX_s(x))}{dx} - K_2 \frac{X_s(x)}{X_s(x)+k} AP_s(x) + \alpha\left(S - X_s(x)\right) = 0 \quad ,(x > L, t > 0) \tag{18}$$

Boundary concentrations for P and X are still given by $P_s(-\infty) = 0$ and $X_s(-\infty) = S$ far upstream and far downstream, respectively.

And

$$P_s(\infty) = \frac{q}{K_1 A}\left(1 + \frac{k}{X_s(\infty)}\right) \tag{19}$$

$$X_s(\infty) = S - \frac{K_2 q}{K_1 \alpha} \tag{20}$$

Furthermore, we obtain flux conditions by mathematical analysis directly,

$$P_0'(0) \leq P_k'(0) \leq q/Av \text{ and } X_0'(0) \leq X_k'(0) \leq 0$$

Discussion and Conclusions

We have proposed a mathematical model for river pollution comprising a coupled pair of nonlinear equations and investigated the effect of aeration on the degradation of pollutant. The results from numerical calculations (Figure 6) agree with the analytical solution under the conditions of no pollution and saturated dissolved oxygen far upstream, tending to a steady state far downstream for a long (considered infinite) river. Using this technique we will be able to obtain the steady state solution for the nonlinear model. The actual rate of pollutant insertion is q = 0.06 kg m^{-1} day^{-1}. This makes $X_s = 0$ for large x, that is, the river is ecologically dead. From Model 1, the fish survival constraint on q for the Subarnarekha river is $q < 0.015$. However, with the value of q = 0.06 kg m^{-1} day^{-1} , subsequent investigations (to be published later) show that for a river of length of that in this study (the lower 325 km of the Subarnarekha), the oxygen level fortunately remains above the critical value of 30% of the saturated oxygen concentration and reaches zero far beyond 325 km.

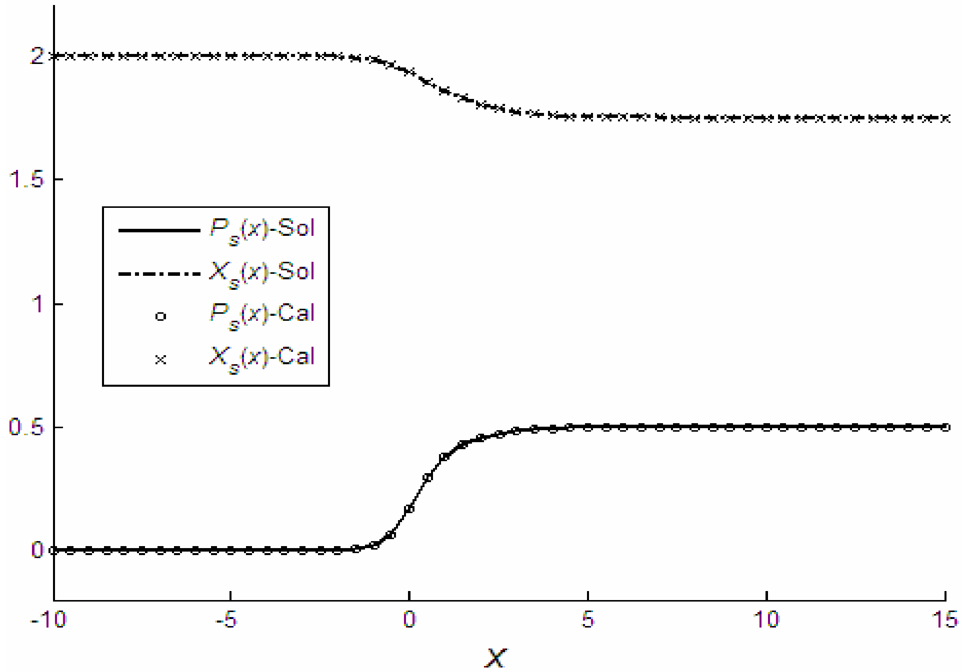

Figure 6. The numerical steady state solutions of $P_s(0)$ and $X_s(0)$ for negligible *k*.

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