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Frame operator for K-frame in 2-inner product space

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Abstract

In this paper we discuss a few properties of 2-frame in the aspect of 2-inner product spaces. We also give a relationship between 2-K-frames and quotient operators in 2-inner product spaces.

Keywords: *Frame, K-frame, quotient operator, 2-inner product space, 2-normed space.*
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1 Introduction

Duffin and Schaeffer were introduced frame in Hilbert space in their fundamental paper [10], they used frame as a tool in the study of nonharmonic Fourier series. After some decades, frame theory was popularized by Daubechies, Grossman, Meyer [6]. Every separable Hilbert space has a countable orthonormal basis i.e., every elements in this space can be represented by its Fourier series expansion with respect to the basis elements. A frame for a separable Hilbert space is a generalization of such an orthonormal basis and this is such a tool that also allows each vector in the space to be written as a linear combination of elements from the frame but, linear independence among the frame elements is not required. Several generalizations of frames namely, K-frame [12], Fusion frame [4], K-fusion frame [2], G-frame [13], etc. have been introduced in recent times.

K-frames for a separable Hilbert space were introduced by Lara Gavruta to study the basic notions about atomic system for a bounded linear operator K. K-frame is more generalization than the ordinary frame and many properties of ordinary frame may not hold for such generalization of frame.
After the introduction of 2-inner product space [5, 8, 9] 2-norm was introduced by S. Gahler [11]. The concepts of 2-inner product and 2-inner product spaces are closely related to the concepts of 2-norm and 2-normed space.

The notion of a frame in a 2-inner product space has been introduced by A. Arefijamaal and G. Sadeghi [1] and they also established some fundamental properties of 2-frames for 2-inner product space. The concept of 2-atomic systems which is a generalization of families of local 2-atoms in a 2-inner product spaces was introduced by B. Dastourian and M. Janfada [7] and they also defined 2-K-frame as the generalization of 2-frame.

In this paper, we shall discuss some properties of 2-frame and establish some relationship between 2-K-frame and quotient operators.

Throughout this paper, X will denote a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{B}(X)$ denote the space of all bounded linear operator on X. We also denote $\mathcal{R}(T)$ for range set of T, $\mathcal{N}(T)$ for null space of T where $T \in \mathcal{B}(X)$ and $l^2(\mathbb{N})$ denote the space of square summable scalar-valued sequences with index set $\mathbb{N}$.

2 Preliminaries

Definition 2.1. *[3] A sequence $\{f_i\}_{i=1}^{\infty}$ of elements in X is said to a frame for X if there exist constants $A, B > 0$ such that*

$$A \| f \|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq B \| f \|^2 \;\forall\, f \in X.$$

The constants A and B are called frame bounds.If the collection $\{f_i\}_{i=1}^{\infty}$ satisfies

$$\sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq B \| f \|^2 \;\forall\, f \in X$$

then it is called a Bessel sequence.

Definition 2.2. *[3] Let $\{f_i\}_{i=1}^{\infty}$ be a frame for X then operator defined by*

$$T : l^2(\mathbb{N}) \to X,\, T(\{c_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} c_i f_i$$

is called pre-frame operator and its adjoint operator given by

$$T^* : X \to l^2(\mathbb{N}),\, T^*(f) = \{\langle f, f_i \rangle\}_{i=1}^{\infty}$$

is called the analysis operator. The frame operator is given by

$$S : X \to X,\, S f = T T^* f = \sum_{i=1}^{\infty} \langle f, f_i \rangle f_i.$$

Definition 2.3. *[12] Let $K : X \to X$ be a bounded linear operator. Then a sequence $\{f_i\}_{i=1}^{\infty}$ in X is said to be K-frame for X if there exist constants $A, B > 0$ such that*

$$A \| K^* f \|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq B \| f \|^2 \;\forall\, f \in X.$$

Definition 2.4. *[12] Let $U, V : X \to X$ be two bounded linear operator with $\mathcal{N}(U) \subset \mathcal{N}(V)$. The quotient operator $T = [U / V]$ is a linear operator which is defined as*

$$T = [U / V] : \mathcal{R}(V) \to \mathcal{R}(U),\, T(V x) = U x.$$

Definition 2.5. *[5, 8] Let* X *be a linear space of dimension greater than 1 over the field* $\mathbb{K}$*, where* $\mathbb{K}$ *is the real or complex numbers field. A function* $\langle \cdot, \cdot | \cdot \rangle : X \times X \times X \to \mathbb{K}$ *is said to be an 2-inner product on* X *if*
(I1) $\langle x, x | z \rangle \geq 0$ *and* $\langle x, x | z \rangle = 0$ *if and only if* x, z *are linearly dependent,*
(I2) $\langle x, x | z \rangle = \langle z, z | x \rangle$,
(I3) $\langle x, y | z \rangle = \overline{\langle y, x | z \rangle}$,
(I4) $\langle \alpha x, y | z \rangle = \alpha \langle x, y | z \rangle$, *for all* $\alpha \in \mathbb{K}$,
(I5) $\langle x_1 + x_2, y | z \rangle = \langle x_1, y | z \rangle + \langle x_2, y | z \rangle$.
A linear space X *equipped with an 2-inner product* $\langle \cdot, \cdot | \cdot \rangle$ *on* X *is called an 2-inner product space [8].*

Definition 2.6. *[11] A 2-norm* $\| \cdot, \cdot \|$ *on a linear space* X *is a real valued function defined on* $X \times X$ *satisfying the following conditions:*
(N1) $\| x, y \| = 0$ *if and only if* x, y *are linearly dependent,*
(N2) $\| x, y \| = \| y, x \|$,
(N3) $\| \alpha x, y \| = |\alpha| \| x, y \| \, \forall \alpha \in \mathbb{R}$,
(N4) $\| x, y + z \| \leq \| x, y \| + \| x, z \|$.
Then the pair $(X, \| \cdot, \cdot \|)$ *is called a linear 2-normed space.*

Theorem 2.7. *[8] Let* $(X, \langle \cdot, \cdot | \cdot \rangle)$ *be a 2-inner product space then*

$$|\langle x, y | z \rangle| \leq \| x, z \| \| y, z \|$$

hold for all $x, y, z \in X$.

Theorem 2.8. *[8] For every 2-inner product space* $(X, \langle \cdot, \cdot | \cdot \rangle)$,

$$\| x, y \| = \sqrt{\langle x, x | y \rangle}$$

defines a 2-norm for which

$$\langle x, y | z \rangle = \frac{1}{4} \left(\| x + y, z \|^2 - \| x - y, z \|^2 \right), \&$$

$$\| x + y, z \|^2 + \| x - y, z \|^2 = 2 \left(\| x, z \|^2 + \| y, z \|^2 \right)$$

hold for all $x, y, z \in X$.

Definition 2.9. *[11] Let* $(X, \| \cdot, \cdot \|)$ *be a linear 2-normed space. A sequence* $\{x_n\}$ *in* X *is said to be converges to some* $x \in X$ *if*

$$\lim_{n \to \infty} \| x_n - x, y \| = 0$$

for every $y \in X$ *and it is called Cauchy sequence if*

$$\lim_{n, m \to \infty} \| x_n - x_m, z \| = 0$$

for every $z \in X$*. The space* X *is said to be complete if every Cauchy sequence in this space is convergent in* X*. A 2-inner product space is called 2-Hilbert space if it is complete with respect to its induce norm.*

Definition 2.10. *[1] Let* $(X, \langle \cdot, \cdot | \cdot \rangle)$ *be a* 2*-Hilbert space and* $h \in X$. *A sequence* $\{f_i\}_{i=1}^{\infty} \subseteq X$ *is called a 2-frame associated to* h *if there exist constants* $A, B > 0$ *such that*

$$A \| f, h \|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i | h \rangle|^2 \leq B \| f, h \|^2 \ \forall f \in X.$$

A sequence $\{f_i\}_{i=1}^{\infty}$ *which satisfies the inequality*

$$\sum_{i=1}^{\infty} |\langle f, f_i | h \rangle|^2 \leq B \| f, h \|^2 \ \forall f \in X.$$

is called a 2*-Bessel sequence associated to* h.

Theorem 2.11. *[1] Let* $(X, \langle \cdot, \cdot | \cdot \rangle)$ *be a* 2*-Hilbert space and* L_h *denote the* 1*-dimensional linear subspace generated by a fixed* $h \in X$. *Let* M_h *be the algebraic complement of* L_h. *Define* $\langle x, y \rangle_h = \langle x, y | h \rangle$ *on* X. *Then* $\langle \cdot, \cdot \rangle_h$ *is a semi-inner product on* X *and this semi-inner product induces an inner product on the quotient space* X / L_h *which is given by*

$$\langle x + L_h, y + L_h \rangle_h = \langle x, y \rangle_h = \langle x, y | h \rangle \ \ \forall x, y \in X.$$

By identifying X / L_h *with* M_h *in an obvious way, we obtain an inner product on* M_h. *Now, for* $x \in M_h$, *define* $\| x \|_h = \sqrt{\langle x, x \rangle_h}$. *Then* $(M_h, \| \cdot \|_h)$ *is a norm space.*

Let X_h be the completion of the inner product space M_h.

Theorem 2.12. *[1] Let* $(X, \langle \cdot, \cdot | \cdot \rangle)$ *be a* 2*-Hilbert space and* $h \in X$. *Then a sequence* $\{f_i\}_{i=1}^{\infty}$ *in* X *is a* 2*-frame associated to* h *with bounds* A & B *if and only if it is a frame for the Hilbert space* X_h *with bounds* A & B.

Theorem 2.13. *[1] Let* $\{f_i\}_{i=1}^{\infty}$ *be a* 2*-Bessel sequence associated to* h *then the* 2*-pre frame operator*

$$T_h : l^2(\mathbb{N}) \to X_h, \ T_h(\{c_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} c_i f_i$$

is well-defined and bounded and its adjoint operator given by,

$$T_h^* : X_h \to l^2(\mathbb{N}), \ T_h^*(f) = \{\langle f, f_i | h \rangle\}_{i=1}^{\infty}$$

is also well-defined and bounded.

Definition 2.14. *[1] Let* $\{f_i\}_{i=1}^{\infty}$ *be a* 2*-frame associated to* h. *The operator*

$$S_h : X_h \to X_h, \ S_h f = T_h T_h^* f = \sum_{i=1}^{\infty} \langle f, f_i | h \rangle f_i$$

is called the frame operator for $\{f_i\}_{i=1}^{\infty}$.

Theorem 2.15. *[1] The frame operator S_h is bounded, invertible, self-adjoint, and positive.*

Definition 2.16. *[7]) Let $K_h : X_h \to X_h$ be a bounded linear operator. Then a sequence $\{ f_i \}_{i=1}^{\infty} \subseteq X$ is called a 2-K-frame associated to h if there exist constants $A, B > 0$ such that*

$$A \| K_h^* f \|_h^2 \leq \sum_{i=1}^{\infty} | \langle f , f_i | h \rangle |^2 \leq B \| f \|_h^2 \;\; \forall \; f \in X_h.$$

3 Some properties of 2-frame

Theorem 3.1. *Let Y be a closed subspace of X_h and P_Y be the orthogonal projection on Y. Then for a sequence $\{ f_i \}_{i=1}^{\infty} \subseteq X_h$ the following hold:*

(i) If $\{ f_i \}_{i=1}^{\infty}$ is a 2-frame associated to h for X with frame bounds A, B then $\{ P_Y f_i \}_{i=1}^{\infty}$ is a frame for Y with the same bounds.

(ii) If $\{ f_i \}_{i=1}^{\infty}$ is a frame for Y with frame operator S_h, then the orthogonal projection on Y is given by,

$$P_Y f = \sum_{i=1}^{\infty} \left\langle f , S_h^{-1} f_i | h \right\rangle f_i \;\; \forall \; f \in X_h.$$

Proof. Using the definition of an orthogonal projection of X_h onto Y, we get

$$P_Y f = \begin{cases} f & \text{if} \quad f \in Y \\ 0 & \text{if} \quad f \in Y^{\perp}. \end{cases} \tag{1}$$

(i) Suppose $\{ f_i \}_{i=1}^{\infty}$ is a 2-frame associated to h for X with frame bounds $A, B \Rightarrow \{ f_i \}_{i=1}^{\infty}$ is a frame for X_h with frame bounds A, B. So we can write,

$$A \| f \|_h^2 \leq \sum_{i=1}^{\infty} | \langle f , f_i \rangle_h |^2 \leq B \| f \|_h^2 \;\; \forall \; f \in X_h$$

Using (1), the above can be write as

$$A \| f \|_h^2 \leq \sum_{i=1}^{\infty} | \langle f , P_Y f_i \rangle_h |^2 \leq B \| f \|_h^2 \;\; \forall \; f \in Y.$$

This shows that $\{ P_Y f_i \}_{i=1}^{\infty}$ is a frame for Y with the same bounds.

(ii) Let $\{ f_i \}_{i=1}^{\infty}$ is a frame for Y with frame operator S_h. Then

$$f = \sum_{i=1}^{\infty} \left\langle f , S_h^{-1} f_i | h \right\rangle f_i \;\; \forall \; f \in Y.$$

Therefore,

$$P_Y f = \sum_{i=1}^{\infty} \left\langle f, S_h^{-1} f_i \,|\, h \right\rangle f_i \;\; \forall f \in Y \; [\text{ using } (1) \,]$$

Since S_h is a bijection on Y, then $S_h^{-1} f_i \in Y$. Now, if $f \in Y^{\perp}$ then $\left\langle f, S_h^{-1} f_i \,|\, h \right\rangle = 0$ and $P_Y f = 0$ if $f \in Y^{\perp}$. Therefore

$$P_Y f = \sum_{i=1}^{\infty} \left\langle f, S_h^{-1} f_i \,|\, h \right\rangle f_i \;\; \forall f \in X_h.$$

□

Note 3.2. *Let* $\{f_i\}_{i=1}^{\infty}$ *be a 2-frame associated to* h *for* X. *If for* $f \in X_h$, $f = \sum\limits_{i=1}^{\infty} c_i f_i$ *for some* $\{c_i\}_{i=1}^{\infty} \in l^2(\mathbb{N})$, *then*

$$\sum_{i=1}^{\infty} |c_i|^2 = \sum_{i=1}^{\infty} \left| \left\langle f, S_h^{-1} f_i \,|\, h \right\rangle \right|^2 + \sum_{i=1}^{\infty} \left| c_i - \left\langle f, S_h^{-1} f_i \,|\, h \right\rangle \right|^2.$$

Theorem 3.3. *Let* $\{f_i\}_{i=1}^{\infty}$ *be a 2-frame associated to* h *for* X *with pre frame operator* T_h. *Then the pseudo-inverse of* T_h *is given by*

$$T_h^{\dagger} : X_h \to l^2(\mathbb{N}),\; T_h^{\dagger} f = \left\{ \left\langle f, S_h^{-1} f_i \,|\, h \right\rangle \right\}_{i=1}^{\infty},$$

where S_h *be the corresponding frame operator.*

Proof. By the Theorem (2.15), $\{f_i\}_{i=1}^{\infty}$ is a frame for X_h. Then for $f \in X_h$ has a representation $f = \sum\limits_{i=1}^{\infty} c_i f_i$ for some $\{c_i\}_{i=1}^{\infty} \in l^2(\mathbb{N})$ and this can be written as $T_h \{c_i\}_{i=1}^{\infty} = f$. By note (3.2), the frame coefficient $\left\{ \left\langle f, S_h^{-1} f_i \,|\, h \right\rangle \right\}_{i=1}^{\infty}$ have minimal l^2-norm among all sequences representing f. Hence, the above equation has a unique solution of minimal norm namely, $T_h^{\dagger} f = \left\{ \left\langle f, S_h^{-1} f_i \,|\, h \right\rangle \right\}_{i=1}^{\infty}$. □

Theorem 3.4. *Let* $\{f_i\}_{i=1}^{\infty}$ *be a 2-frame associated to* h *for* X, *then the optimal frame bounds* A, B *are given by*

$$A = \left\| S_h^{-1} \right\|^{-1} = \left\| T_h^{\dagger} \right\|^{-2},\; B = \| S_h \| = \| T_h \|^2,$$

where T_h *is the pre frame operator,* $T_h^{\dagger}$ *is the pseudo inverse of* T_h *and* S_h *is the frame operator.*

Proof. By the definition, the optimal upper frame bound is given by

$$B = \sup_{\|f, h\| = 1} \sum_{i=1}^{\infty} \left| \left\langle f, f_i \,|\, h \right\rangle \right|^2 = \sup_{\|f, h\| = 1} \left\langle S_h f, f \,|\, h \right\rangle = \| S_h \|$$

Therefore, $B = \| S_h \| = \| T_h T_h^* \| = \| T_h \|^2$. We know that the dual frame $\{ S_h^{-1} f_i \}_{i=1}^{\infty}$ has frame operator S_h^{-1} and the optimal upper bound is A^{-1}. So by the above similar process $A^{-1} = \| S_h^{-1} \| \Rightarrow A = \| S_h^{-1} \|^{-1}$. Now, from the Theorem (3.3), we obtain

$$\| S_h^{-1} \| = \sup_{\| f , h \| = 1} \sum_{k=1}^{\infty} | \langle f , S_h^{-1} f_k | h \rangle |^2 = \sup_{\| f \|_h = 1} \left\| T_h^{\dagger} f \right\|_h^2 = \left\| T_h^{\dagger} \right\|^2.$$

Thus, $A = \| S_h^{-1} \|^{-1} = \left\| T_h^{\dagger} \right\|^{-2}$. This completes the proof of the Theorem. □

4 Frame operator for 2-K-frame

Theorem 4.1. *Let $\{ f_i \}_{i=1}^{\infty}$ be a 2-Bessel sequence in X with the frame operator S_h and $K_h \in \mathcal{B}(X_h)$. Then $\{ f_i \}_{i=1}^{\infty}$ is a 2-K-frame if and only if the quotient operator $\left[K_h^* / S_h^{\frac{1}{2}} \right]$ is bounded.*

Proof. Let $\{ f_i \}_{i=1}^{\infty}$ be a 2-K-frame for X. Then there exists positive constants A, B such that

$$A \| K_h^* f \|_h^2 \leq \sum_{i=1}^{\infty} | \langle f , f_i | h \rangle |^2 \leq B \| f \|_h^2 \quad \forall f \in X_h. \tag{2}$$

Using the definition of frame operator S_h, we can write

$$\langle S_h f , f | h \rangle = \sum_{i=1}^{\infty} | \langle f , f_i | h \rangle |^2 , \quad \forall f \in X_h. \tag{3}$$

Using (3), The inequality (2) can be written as

$$A \| K_h^* f \|_h^2 \leq \langle S_h f , f | h \rangle \leq B \| f \|_h^2 \quad \forall f \in X_h$$

$$\Rightarrow A \| K_h^* f \|_h^2 \leq \left\langle S_h^{\frac{1}{2}} f , S_h^{\frac{1}{2}} f | h \right\rangle \leq B \| f \|_h^2 \quad \forall f \in X_h$$

$$\Rightarrow A \| K_h^* f \|_h^2 \leq \left\| S_h^{\frac{1}{2}} f \right\|_h^2 \leq B \| f \|_h^2 \quad \forall f \in X_h \tag{4}$$

Let us now define the operator,

$$T = \left[K_h^* / S_h^{\frac{1}{2}} \right] : \mathcal{R}\left(S_h^{\frac{1}{2}} \right) \to \mathcal{R}(K_h^*) , \text{ by } T\left(S_h^{\frac{1}{2}} f \right) = K_h^* f \quad \forall f \in X_h.$$

Now, let $f \in \mathcal{N}\left(S_h^{\frac{1}{2}}\right)$. Then $S_h^{\frac{1}{2}} f = \theta \Rightarrow \left\|S_h^{\frac{1}{2}} f\right\|_h^2 = 0$, so by (4),

$A \| K_h^* f \|_h^2 = 0 \Rightarrow K_h^* f = \theta \Rightarrow f \in \mathcal{N}(K_h^*) \Rightarrow \mathcal{N}\left(S_h^{\frac{1}{2}}\right) \subseteq \mathcal{N}(K_h^*)$.

This shows that the quotient operator T is well-defined. Also for all $f \in X_h$,

$$\left\|T\left(S_h^{\frac{1}{2}} f\right)\right\|_h = \| K_h^* f \|_h \leq \frac{1}{\sqrt{A}} \left\|S_h^{\frac{1}{2}} f\right\|_h$$

Hence, T is bounded.

Conversely, suppose that the quotient operator $\left[K_h^* / S_h^{\frac{1}{2}}\right]$ is bounded. Then there exists $B > 0$ such that,

$$\left\|T\left(S_h^{\frac{1}{2}} f\right)\right\|_h^2 \leq B \left\|S_h^{\frac{1}{2}} f\right\|_h^2 \quad \forall f \in X_h$$

$$\Rightarrow \| K_h^* f \|_h^2 \leq B \left\|S_h^{\frac{1}{2}} f\right\|_h^2 = B \left\langle S_h^{\frac{1}{2}} f , S_h^{\frac{1}{2}} f \,|\, h \right\rangle$$

$$= B \langle S_h f , f \,|\, h \rangle \quad \left[\text{since } S_h^{\frac{1}{2}} \text{ is also self-adjoint}\right]$$

$$= B \sum_{i=1}^{\infty} |\langle f , f_i \,|\, h \rangle|^2 \tag{5}$$

Also, $\{f_i\}_{i=1}^{\infty}$ be a 2-Bessel sequence associated to h in X,so there exists $C > 0$ such that

$$\sum_{i=1}^{\infty} |\langle f , f_i \,|\, h \rangle|^2 \leq C \| f \|_h^2 , \quad \forall f \in X_h \tag{6}$$

Hence, from (5) and (6), $\{f_i\}_{i=1}^{\infty}$ is a 2-K-frame associated to h for X. □

Theorem 4.2. *Let $\{f_i\}_{i=1}^{\infty}$ be a 2-K-frame sequence in X with the frame operator S_h and $T \in \mathcal{B}(X_h)$. Then the following are equivalent:*

(1) $\{T f_i\}_{i=1}^{\infty}$ is a 2-TK-frame associated to h for X.

(2) $\left[(T K_h)^* \,/\, S_h^{\frac{1}{2}} T^* \right]$ *is bounded.*

(3) $\left[(T K_h)^* \,/\, (T S_h T^*)^{\frac{1}{2}} \right]$ *is bounded.*

Proof. $(1) \Rightarrow (2)$ Suppose that $\{T f_i\}_{i=1}^{\infty}$ is a 2-TK-frame associated to h for X. Then there exists constants $A, B > 0$ such that

$$A \, \| (T K_h)^* f \|_h^2 \leq \sum_{i=1}^{\infty} | \langle f, T f_i \,|\, h \rangle |^2 \leq B \, \| f \|_h^2, \;\; \forall \, f \in X_h. \tag{7}$$

Using the definition of frame operator S_h, we can write

$$\langle S_h f, f \,|\, h \rangle = \sum_{i=1}^{\infty} | \langle f, f_i \,|\, h \rangle |^2, \;\; \forall \, f \in X_h.$$

Now,

$$\sum_{i=1}^{\infty} | \langle f, T f_i \,|\, h \rangle |^2 = \sum_{i=1}^{\infty} | \langle T^* f, f_i \,|\, h \rangle |^2 = \langle S_h (T^* f), T^* f \,|\, h \rangle$$

$$= \left\langle S_h^{\frac{1}{2}} (T^* f), S_h^{\frac{1}{2}} (T^* f) \,|\, h \right\rangle = \left\| S_h^{\frac{1}{2}} (T^* f) \right\|_h^2$$

Let us now consider the quotient operator,

$$\left[(T K_h)^* \,/\, S_h^{\frac{1}{2}} T^* \right] : \mathcal{R}\left(S_h^{\frac{1}{2}} T^* \right) \to \mathcal{R}\left((T K_h)^* \right) \text{ by}$$

$$\left(S_h^{\frac{1}{2}} T^* \right) f \mapsto (T K_h)^* f \;\; \forall \, f \in X_h.$$

From (7), we can write

$$A \, \| (T K_h)^* f \|_h^2 \leq \left\| S_h^{\frac{1}{2}} (T^* f) \right\|_h^2 \;\; \forall \, f \in X_h.$$

$$\Rightarrow \; \| (T K_h)^* f \|_h^2 \leq \frac{1}{A} \left\| S_h^{\frac{1}{2}} (T^* f) \right\|_h^2 \;\; \forall \, f \in X_h.$$

This shows that the quotient operator $\left[(T K_h)^* / S_h^{\frac{1}{2}} T^*\right]$ is bounded.

$(2) \Rightarrow (3)$ Suppose that the quotient operator $\left[(T K_h)^* / S_h^{\frac{1}{2}} T^*\right]$ is bounded.

Then there exists constant $B > 0$ such that

$$\|(T K_h)^* f\|_h^2 \leq B \left\|S_h^{\frac{1}{2}} (T^* f)\right\|_h^2 \quad \forall f \in X_h. \tag{8}$$

Now, for each $f \in X_h$, we have

$$\left\|S_h^{\frac{1}{2}} (T^* f)\right\|_h^2 = \langle S_h(T^* f), T^* f \,|\, h\rangle = \langle T S_h(T^* f), f \,|\, h\rangle$$

$$= \left\langle (T S_h T^*)^{\frac{1}{2}} f, (T S_h T^*)^{\frac{1}{2}} f \,|\, h \right\rangle = \left\|(T S_h T^*)^{\frac{1}{2}} f\right\|_h^2. \tag{9}$$

From (8) and (9), we get

$$\|(T K_h)^* f\|_h^2 \leq B \left\|(T S_h T^*)^{\frac{1}{2}} f\right\|_h^2 \quad \forall f \in X_h.$$

Hence, the quotient operator $\left[(T K_h)^* / (T S_h T^*)^{\frac{1}{2}}\right]$ is bounded.

$(3) \Rightarrow (1)$ Suppose the quotient operator $\left[(T K_h)^* / (T S_h T^*)^{\frac{1}{2}}\right]$ is bounded.

Then there exists constant $B > 0$ such that

$$\|(T K_h)^* f\|_h^2 \leq B \left\|(T S_h T^*)^{\frac{1}{2}} f\right\|_h^2 \quad \forall f \in X_h. \tag{10}$$

It is easy to verify that $T S_h T^*$ is self-adjoint and positive and hence the square root of $T S_h T^*$ exists. Now, for each $f \in X_h$, we have

$$\sum_{i=1}^{\infty} |\langle f, T f_i \,|\, h\rangle|^2 = \sum_{i=1}^{\infty} |\langle T^* f, f_i \,|\, h\rangle|^2 = \langle S_h(T^* f), T^* f \,|\, h\rangle$$

$$= \left\langle S_h^{\frac{1}{2}} T^* f , S_h^{\frac{1}{2}} T^* f \,|\, h \right\rangle = \left\langle \left(S_h^{\frac{1}{2}} T^* \right)^* S_h^{\frac{1}{2}} T^* f , , f \,|\, h \right\rangle$$

$$= \left\langle T S_h^{\frac{1}{2}} S_h^{\frac{1}{2}} T^* f , f \,|\, h \right\rangle = \langle T S_h T^* f , f \,|\, h \rangle$$

$$= \left\langle T S_h^{\frac{1}{2}} S_h^{\frac{1}{2}} T^* f , f \,|\, h \right\rangle = \langle T S_h T^* f , f \,|\, h \rangle = \left\| (T S_h T^*)^{\frac{1}{2}} f \right\|_h^2 . \quad (11)$$

From (10) and (11),

$$\frac{1}{B} \| (T K_h)^* f \|_h^2 \leq \sum_{i=1}^{\infty} | \langle f , T f_i \,|\, h \rangle |^2 \quad \forall f \in X_h.$$

On the other hand, since $\{ f_i \}_{i=1}^{\infty}$ is a 2-K-frame associated to h for X,

$$\sum_{i=1}^{\infty} | \langle f , T f_i \,|\, h \rangle |^2 = \sum_{i=1}^{\infty} | \langle T^* f , f_i \,|\, h \rangle |^2 \leq C \| T \|^2 \| f \|_h^2$$

Hence, $\{ T f_i \}_{i=1}^{\infty}$ is a 2-K-frame associated to h for X. □

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