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μ- Graph of a Finite Group

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Abstract: *μ - graph of a finite group G is a graph whose vertex set is same as G itself and two vertices x, y are adjacent if and only if μ (|x| |y|) =μ (| x|) μ (|y|) . The objective of this paper is to introduce $\mu -$ graph of a finite group and discuss some of its properties.*

Keywords - *Adjacency matrix, Eigen value, Energy, Mobius function, $\mu -$ graph of a finite group*

I. INTRODUCTION

Studies in group theory and graph theory play a crucial role in modern mathematics. Associating a group to a graph can be done in different ways and it is available in literature. Graph theory in mathematics is the most explored research field mainly because of its applications in different areas like chemistry, biology, physics, engineering and computer science. One of the important mathematical tool used for learning symmetries comprises group theory. They are usually connected to automorphisms of graphs. Studies in both the branches act a crucial role in modern mathematics.

A deep analysis of algebraic structure by graphs may unfold interesting results in the area of algebraic graph theory. We can associate a group to graph in many ways. One such method is by using order of the elements of the group. In [9], M. Sattananathan and R. Kala defined the Order Prime Graphs of a finite groups and studied some properties of order prime graphs. Further Ma et al [6] presented the order prime graph of finite group, they are known as Coprime Graphs. Also in [1], R. H. Aravinth and R. Vignesh introduce the Mobius function graph M_n (G).

The aim of this paper is to introduce the concept of mobius function in connection to order of elements of a group. Here we try to define a new graph namely $\mu -$ graph of finite group and investigate some of its properties. In section 2 we try to remember some of the basic concepts. Section 3 we try to introduce a new graph namely $\mu -$ graph of finite group with an example. Then in section 4 we deal with some properties of $\mu -$ graph. In section 5 we discuss about the eigen values of μ - graph.

II. PRELIMINARIES

Here we are going to give a fast out look to the definitions and theorems which we are useful in the upcoming sections.

Definition 2.1:[10] The Mobius function μ is defined as follows:

$$\mu\ (1) = 1$$

If n > 1, write n = $p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$. Then

$$\mu(n) = \begin{cases} (-1)^k & \text{if } a_1 = a_2 = \cdots = a_k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Definition 2.2:[7] For a pair of vertices u and v of a graph G, the length of any shortest path between u and v of a connected graph G is called the distance between u and v and is denoted by d(u, v).

The diameter of G is defined as max {d (u ,v)| u ,v $\in$ V(G)} and is denoted by diam(G).

Definition 2.3:[3,4] Let G be a graph with n vertices v_1 ,v_2 ,… , v_n. Then the adjacency matrix, A = A (G) is a square matrix of order n whose (i,j)- entry is defined as

$$A_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

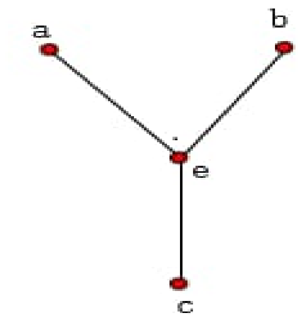
The eigenvalues of A(G) are said to be the eigen values of the graph G. We denote largest and smallest eigenvalues of a graph G by λ_{max} and λ_{min} respectively. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a graph G then the energy of G is defined as, E(G) $=\sum_{i=1}^{n} \left| \lambda_i \right|$

III. μ - GRAPH OF A FINITE GROUP

In this section we define the concept of μ -graph of a finite group and explain it using an example. Throughout this paper G represents a group and p denotes the prime number.

Definition 3.1: Let G be a finite group. The $\mu -$ graph of a finite group denoted by $\mu_g(G)$ is a graph whose vertex set is elements of G itself and any two distinct elements x, y $\in V(\mu_g(G))$ are adjacent if and only if $\mu(|x||y|) = \mu(|x|)\mu(|y|)$

Example 3.2: Consider the Klein-4 group, V. The graph $\mu_g(V)$ has vertex set $V(\mu_g(G)) = \{e, a, b, c\}$.
Here |e| = 1, |a| = |b| = |c| =2.



Observations

From definition 3.1, it is clear that for any finite group G, $\mu_g(G)$ is a simple graph .

Theorem 3.3: For any group G of order n, $\mu_g(G)$ is connected and its maximum degree $\Delta(\mu_g(G)) =$ n-1.

Proof: Let G be a group of order n .

The only element in G of order 1 is identity element.

Hence we can conclude that $\mu(|e||x|) = \mu(|e|)\mu(|x|) \, \forall \; x \neq e \in V(\mu_g(G))$.

Therefore the vertex associated with identity element is adjacent to all other (n-1) vertices.

$\therefore$ $\mu_g(G)$ is always connected and its maximum degree $\Delta(\mu_g(G))$ = n-1.

IV. PROPERTIES OF μ - GRAPH OF A FINITE GROUP

In this section we shall discuss and prove some properties of μ - graph of a finite group.

Theorem 4.1: Let G be a group of prime order then $\mu_g(G)$ is a star graph.

Proof: Given |G | = p, a prime number.

Then $\mu_g(G)$ is a graph with p vertices.

Let $V\left(\mu_g(G)\right)$ = {e,a_1,a_2,…,a_{p-1}}, where e is the identity element of G.

Clearly |e| = 1, | a_i | = p for i = 1,2,3,…,p-1.

μ(|e|| a_i |) = μ (p) = -1

μ (|e|) μ (| a_i |) = μ (1) μ (p) = -1

Hence that $\mu(|e||a_i|) = \mu(|e|)\mu(|a_i|)$

Thus by using the definition of μ -graph of a finite group, vertex e is adjacent the other vertices a_i, i=1,2,…,p-1.

Also for any two vertices $a_i, a_j \in V\left(\mu_g(G)\right)$

μ (| a_i || a_j |)= μ (p.p)= 0

μ (| a_i |) μ (| a_j|) = μ (p) μ (p)= 1 for $i \neq j$= 1,2,…, p-1

Hence that $\mu(|a_i||a_j|) \neq \mu(|a_i|)\mu(|a_j|)$

Therefore by using the definition no two a_i' s ,i= 1,2,…,p-1 are adjacent.

Thus $\mu_g(G)$ is a star graph.

Conversely,

Given that $\mu_g(G)$ is a star graph. Let e be the identity element of group G. Then $\mu_g(G)$ -{e} is totally disconnected. That is no two non- identity elements are adjacent.

Suppose |G|= n, a composite number.

Case 1: n= p^k

Let $x \neq e, y \neq e \in V\left(\mu_g(G)\right)$ such that |x| = p^i and |y| = p^j , $1 < i, j \leq k$

μ (| x || y |) = μ ($p^i.p^j$) = 0

μ (| x |) μ (| y|) = μ (p^i) μ (p^j) = 0 .

Hence $\mu(|x||y|) = \mu(|x|)\mu(|y|)$

$\therefore$ x and y are adjacent, which is a contradiction.

Case 2: n = $p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$

Let $x \neq e, y \neq e \in V\left(\mu_g(G)\right)$ such that |x| = p_i and |y| = p_j, $i \neq j$= 1,2,…, p-1

μ (| x || y |) = μ ($p_i.p_j$) = 1

μ (| x |) μ (| y|) = μ (p_i) μ (p_j) = 1.

Hence $\mu(|x||y|) = \mu(|x|)\mu(|y|)$

Clearly x and y are adjacent, which is a contradiction.

From case 1 and 2 we can conclude that |G| = p, a prime number.

Corollary 4.2: $\mu_g(G)$ is a tree if and only if G is a group of prime order.

Proof: We have for any group of order p, $\mu_g(G)$ is always connected and $\Delta(\mu_g(G))$ = p-1.

By the Theorem 4.1 , $G \approx Z_p \Leftrightarrow \mu_g(G) \approx K_{1,p-1}$

Hence $\mu_g(G)$ is a tree

Theorem 4.3: If $G \approx Z_{2^n}$ then $\mu_g(G)$ is a complete graph.

Proof : Given $G \approx Z_{2^n}$, then $|G| = 2^n$

$V\left(\mu_g(G)\right) = \{a^1, a^2, \ldots, a^{2^n}\}$

For any x $\in V\left(\mu_g(G)\right)$

$$|a^m| = \begin{cases} 1 & if \quad m = 2^n \\ 2 & if \quad m = 2^{n-1} \\ 2^k & if \quad 3 \leq k \leq n \end{cases}$$

As we know that identity element is adjacent to all other vertices.

From the above definition of $|a^m|$ it is obvious that for any x, y $\in V\left(\mu_g(G)\right)$, $\mu(|x||y|) = \mu(|x|)\mu(|y|)$

Hence $\mu_g(G)$ is complete.

Observation

Above theorem is true only when the group is cyclic.

Consider Klien 4 Group V = {e ,a ,b ,c}, with |e| = 1, |a| = |b| = |c| = 2.

Here $\mu_g(V)$ is a star graph.

Theorem 4.4: For any group with $|G| = p_1p_2\ldots p_r$, where p_i's are primes , then $\mu_g(G)$ has at least 2 pendent vertices.

Proof: Given $|G| = p_1p_2\ldots p_r$

Clearly there exist an element a $\in V\left(\mu_g(G)\right)$ such that $|a| = p_1p_2\ldots p_r$.

Also we can find $a^{-1} \in V\left(\mu_g(G)\right)$ such that $|a^{-1}| = p_1p_2\ldots p_r$.

Hence $\mu(|a||a^{-1}|) \neq \mu(|a|)\mu(|a^{-1}|)$

Now consider a vertex, b $\in V\left(\mu_g(G)\right) - \{e, a, a^{-1}\}$

As we know that |b| divides $p_1p_2\ldots p_r$.

$\mu(|a||b|) = 0$ and $\mu(|a|)\mu(|b|) = 1$.

Therefore, $\mu(|a||b|) \neq \mu(|a|)\mu(|b|)$.

Similarly the case of a^{-1}.

Thus we can conclude that identity element e is the only vertex adjacent to both a and a^{-1}.

Hence $\mu_g(G)$ has at least 2 pendent vertices.

Theorem 4.5: For any finite group G, the girth of $\mu_g(G)$ is either 3 or infinity

Proof: In $\mu_g(G)$ vertex associated with identity element is adjacent to all other vertices.

Let $x \neq e$, $y \neq e \in V\left(\mu_g(G)\right)$ such that $\mu(|x||y|) = \mu(|x|)\,\mu(|y|)$.

Thus { x, y, e } form a 3- cycle. Hence the girth of $\mu_g(G)$ is 3.

Theorem 4.6: Let G be a finite group, then diam $(\mu_g(G)) \leq 2$.

Proof: Given a group G with |G| = n

Let $\mu_g(G)$ be the μ – graph of G. Let x and y be two distinct vertices of $\mu_g(G)$.

If $\mu(|x||y|) = \mu(|x|)\,\mu(|y|)$, then x and y are adjacent and hence d(x,y)=1.

Suppose we assume that x and y are not adjacent.

Then $\mu(|x||y|) \neq \mu(|x|)\,\mu(|y|)$

By theorem it is clear that identity element e is adjacent to all other vertices of $V\left(\mu_g(G)\right)$.

In this case d(x,y) = 2

Thus we can conclude that diam$(\mu_g(G)) \leq 2$

Theorem 4.7: For any two finite groups G_1 and G_2, if $G_1 \approx G_2$, then $\mu_g(G_1) \approx \mu_g(G_2)$.
Proof : Given $G_1 \approx G_2$, Let ϕ be an isomorphism from G_1 to G_2.
Consider any two vertices x, y in $\mu_g(G_1)$, then ϕ(x), ϕ(y) are the corresponding vertices of $\mu_g(G_2)$.
Now two vertices x and y of $\mu_g(G_1)$ are adjacent.
$\Leftrightarrow \mu(|x||y|) = \mu(|x|)\mu(|y|)$

$$\Leftrightarrow \mu\left(\left|\phi(x)\right|\left|\phi(y)\right|\right) = \mu\left(\left|\phi(x)\right|\right)\mu\left(\left|\phi(y)\right|\right)$$

$\Leftrightarrow$ ϕ(x) and ϕ(y) are adjacent in $\mu_g(G_2)$.
Thus $\mu_g(G_1) \approx \mu_g(G_2)$.

V. ADJACENCY MATRIX OF μ – GRAPH OF FINITE GROUP

In this section we shall discuss and prove some properties of adjacency matrix of μ- graph of a finite group

Theorem 4.1: Let G be a group of prime order p, then the adjacency matrix of μ - graph of G has eigen value 0 of multiplicity (p-2) and $\pm\sqrt{(p-1)}$.

Proof: Adjacency matrix of μ -graph of a group of prime order p is a $p \times p$ matrix and is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}_{p\times p}$$

To obtain eigen values of A we solve det($\lambda I - A$)= 0.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 & -1 & \dots & -1 \\ -1 & \lambda & 0 & \dots & 0 \\ -1 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -1 & 0 & 0 & \dots & \lambda \end{bmatrix}_{p\times p}$$

$$= \det \begin{bmatrix} A_{1\times 1} & B_{1\times p-1} \\ B^{T}{}_{p-1\times 1} & D_{p-1\times p-1} \end{bmatrix}_{p\times p}$$

= det (A) det(D-CA^{-1}B)

$$= \det[\lambda] \det \begin{bmatrix} \lambda - \frac{1}{\lambda} & -\frac{1}{\lambda} & \dots & -\frac{1}{\lambda} \\ -\frac{1}{\lambda} & \lambda - \frac{1}{\lambda} & \dots & -\frac{1}{\lambda} \\ \vdots & \vdots & \dots & \vdots \\ -\frac{1}{\lambda} & -\frac{1}{\lambda} & \dots & \lambda - \frac{1}{\lambda} \end{bmatrix}_{p-1\times p-1}$$

$$= \lambda^{p-2}(\lambda^2 - (p-1))$$

Hence det($\lambda I - A$) = 0 $\Rightarrow \lambda^{p-2}(\lambda^2 - (p-1)) = 0$

$\Rightarrow \lambda$ = 0 of multiplicity p-2 and $\pm\sqrt{p-1}$

Corollary 5.2: For a group G of prime order, $\lambda_{min}\left(\mu_g(G)\right) = -\lambda_{max}\left(\mu_g(G)\right)$

Proof: From theorem 5.1 the adjacency matrix of $\mu_g(G)$ for a group of prime order p has eigen values 0 of multiplicity p-2 and $\pm\sqrt{(p-1)}$

Hence we get $\lambda_{min}\left(\mu_g(G)\right) = -\lambda_{max}\left(\mu_g(G)\right)$.

Corollary 5.3: For a group G of prime order, E($\mu_g(G)$)= $2\sqrt{(p-1)}$.

Proof: From theorem 5.1 the adjacency matrix of $\mu_g(G)$ for a group of prime order p has eigen values 0 of multiplicity p-2 and $\pm\sqrt{(p-1)}$

Also by using Definition 2.3, it is clear that energy of an adjacency matrix is the sum of its eigen values. Hence the theorem

Theorem 5.4: Let G be a finite group of order n then its $\mu_g(G)$ has at least three eigen values.
Proof: A connected graph with diameter d, has at least d+1 distinct eigen values.
By theorem 4.6, we have for a finite group of order n, diam $(\mu_g(G)) \leq 2$.
Hence $\mu_g(G)$ have at least three distinct eigen values.

VI. CONCLUSION

In this paper we introduce the new concept of μ - graph of a finite group and explain using examples. Also discuss some main theorems and some of its properties connecting adjacency matrix.

VII. REFERENCES

[1] R. H Aravinth, R. Vignesh., Mobius Function Graph M_n (G), IJITEE , 8(10)(2019).
[2] R.Balakrishnan and K. Ranganathan , A Textbook of Graph Theory, Springer.
[3] R. B. Bapat, Graph And Matrices, Springer, Hindustan book agency.
[4] I. Gutman., The energy of regular graph, Ber.Math-Statist.Sekt.Forchungsz.Graz,103(1978) 1-22.
[5] I. N. Herstein, Topics in Algebra, Second Edition, John Wiley and Sons, (2003).
[6] Ma.X, Wei.H and Yang.L, The Coprime Graph of a Group, Int.J.Group Theory.,3(3)(2014) 13-23.
[7] K. R. Parthasarathy , Basic Graph Theory, Tata McGraw- Hill publishing company limited, New Delhi.
[8] Rodriguez Luke, Automorphism Groups of Simple Graphs.
[9] M. Sattanathan and R. Kala. An Introduction to order prime graphs, Int.J.Contemp.Math.Science, 4(10)(2009) 467-474.
[10] Tom M. Apostol, Introduction to analytic Number theory, Springer International Student Editor.