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Logistic Chen Distribution with Properties and Applications

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Abstract *- In this study, we have introduced a three-parameter univariate continuous distribution called Logistic Chen distribution. Some distributional properties of the distribution such as the shapes of the probability density, cumulative distribution and hazard rate functions, quantile function, survival function, the skewness, and kurtosis measures are derived and established. To estimate the model parameters, we have employed three well-known estimation methods namely maximum likelihood estimation (MLE), least-square estimation (LSE), and Cramer-Von-Mises estimation (CVME) methods. A real data set is considered to explore the applicability and capability of the proposed distribution also AIC, BIC, CAIC and HQIC are calculated to assess the potentiality of the Logistic Chen distribution.*

Keywords - *Chen distribution, CVME, Logistic distribution, LSE and MLE.*

I. INTRODUCTION

In most of the literature of probability distributions and applied statistics, it is observed that the study of reliability and survival analysis in various fields of applied statistics and life sciences, the probability distributions are often used. In modeling survival data, existing models do not always reveal a better fit. Hence most of the researchers are interested to generalizing standard distributions and investigating their flexibility and applicability. Usually, these new compounded models produces an improved fit as compared to usual classical survival models and are obtained by introducing one or more additional shape parameter(s) to the parent distribution.

A compounded survival model that includes the different shapes like bathtub-shaped, increasing, decreasing, and inverted bathtub-Shaped failure rate in a single model would be beneficial in survival analysis. Such a model would provide considerable flexibility and goodness of fit for fitting a broad variety of lifetime data sets. Such a survival model might also be taken to determine the distribution class from which the data is selected, by constructing confidence interval over its parameters. The proposed distribution introduced here satisfies these criteria.

Reference [1] has proposed a new two-parameter lifetime distribution with bathtub shaped or increasing failure rate (IFR) function. The cumulative distribution function (CDF) of Chen distribution is

$$G(x) = 1 - \exp[\lambda(1 - e^{x^{\alpha}})]; \ \alpha, \lambda > 0, \ x > 0 \qquad (1.1)$$

And its probability density function (PDF) is

$$f(x) = \alpha\lambda x^{\alpha-1} e^{x^{\alpha}} \exp[\lambda(1 - e^{x^{\alpha}})]; \ \alpha, \lambda > 0, \ x > 0 \qquad (1.2)$$

The motivation to extend the Chen distribution is to introduce a flexible model that has revealed the various shapes of the hazard and density functions. The Markov Chain Monte Carlo methods for Bayesian inference of the Chen model has introduced by [2]. [3] have introduced the extended Chen (EC) distribution is derived from the generalized Burr-Hatke differential equation and nexus between the exponential and gamma variables. [4] was introduced a new lifetime distribution with increasing, decreasing and bathtub-shaped hazard rate function which is constructed by the compounding of the Weibull and Chen distributions and is called Weibull–Chen (WC) distribution. [5] was presented the Lindley-Chen Distribution which is more flexible lifetime distribution having increasing, decreasing and bathtub-shaped hazard rate function.

The logistic distribution is a univariate continuous distribution and both its PDF and CDF functions have been used in many different areas such as logistic regression, logit models and neural networks. It has been used in the physical sciences, demography, sports modeling, and recently in finance. The logistic distribution has wider tails than a normal distribution so it is more consistent with the underlying data and provides better insight into the likelihood of extreme events. If X follows the logistic random variable with shape parameter $\lambda > 0$, its cumulative distribution function is given by

$$F(x;\lambda) = \frac{1}{1+e^{-\lambda x}}; \quad \lambda > 0, x \in \Re \qquad (1.3)$$

and its corresponding PDF is

$$f(x;\lambda) = \frac{\lambda e^{-\lambda x}}{\left(1+e^{-\lambda x}\right)^2}; \quad \lambda > 0, x \in \Re \qquad (1.4)$$

[6] have presented logistic modified exponential distribution. [7] introduce a new family of continuous distributions generated from a logistic random variable called the *logistic-X family*. Its density function can be

symmetrical, left-skewed, right-skewed and reversed-J shaped, and can have increasing, decreasing, bathtub and upside-down bathtub hazard rates shaped. [8] have introduced the logistic exponential power distribution having flexible hazard rate function. [9] have presented the logistic inverse Weibull distribution. [10] have introduced an approach to define the logistic compounded model and introduced the logistic–exponential survival distribution. This has several useful probabilistic properties for lifetime modeling. Unlike most distributions in the bathtub and upside down bathtub classes, the logistic–exponential distribution exibit closed-form density, hazard, cumulative hazard, and survival functions. The survival function of the logistic–exponential distribution is

$$S\left(x;\lambda\right)=\frac{1}{1+\left(e^{\lambda x}-1\right)^{\alpha}};\quad \alpha>0,\lambda>0,x\geq 0 \qquad (1.5)$$

Using the same approach used by [10] we have defined the new distribution called logistic Chen (LC) distribution. The main aim of this study is to present a more flexible distribution by adding just one extra parameter to the Chen distribution to attain a better fit to the lifetime data sets. We have discussed some distributional properties and its applicability. The remaining sections of the proposed study are arranged as follows. In Section 2 we present the new logistic Chen exponential (LC) distribution and its various mathematical and statistical properties. We have make use of three well-known estimation methods to estimate the model parameters namely the maximum likelihood estimation (MLE), least-square estimation (LSE) and Cramer-Von-Mises estimation (CVME) methods. For the maximum likelihood (ML) estimate, we have constructed the asymptotic confidence intervals using the observed information matrix are presented in Section 3. In Section 4, a real data set has been analyzed to explore the applications and capability of the proposed distribution. In this section, we present the estimated value of the parameters and log-likelihood, AIC, BIC and CAIC criterion for ML, LSE, and CVME. Finally, in Section 5 we present some concluding remarks.

II. THE LOGISTIC CHEN DISTRIBUTION

Let X be a positive random variable with a positive shape parameter α and a positive scale parameter λ then CDF of logistic Chen distribution can be defined using (1.1) and (1.2) and we can write

$$F\left(x\right)=1-\frac{1}{1+\left[\exp\left\{\lambda\left(e^{x^{\beta}}-1\right)\right\}-1\right]^{\alpha}};\ (\alpha,\beta,\lambda)>0,\ x>0 \qquad (2.1)$$

And its PDF is

$$f\left(x\right)=\frac{\alpha\beta\lambda\exp\left\{\lambda\left(e^{x^{\beta}}-1\right)+x^{\beta}\right\}\left[\exp\left\{\lambda\left(e^{x^{\beta}}-1\right)\right\}-1\right]^{\alpha-1}}{\left\{1+\left[\exp\left\{\lambda\left(e^{x^{\beta}}-1\right)\right\}-1\right]^{\alpha}\right\}^{2}};\ x>0 \qquad (2.2)$$

This CDF function be similar to the log logistic CDF function with the second term of the denominator being changed in its base to Chen function, hence we called it Logistic Chen distribution.

A. Reliability function

The reliability function of Logistic Chen (LC) distribution is

$$R(x)=1-F(x)$$
$$=\frac{1}{1+\left[\exp\left\{\lambda\left(e^{x^{\beta}}-1\right)\right\}-1\right]^{\alpha}};\ (\alpha,\beta,\lambda)>0,\ x>0 \qquad (2.3)$$

B. Hazard function

The failure rate function of LC distribution can be defined as,

$$h(x)=\frac{f(x)}{R(x)}$$
$$=\frac{\alpha\beta\lambda\exp\left\{\lambda\left(e^{x^{\beta}}-1\right)+x^{\beta}\right\}\left[\exp\left\{\lambda\left(e^{x^{\beta}}-1\right)\right\}-1\right]^{\alpha-1}}{\left\{1+\left[\exp\left\{\lambda\left(e^{x^{\beta}}-1\right)\right\}-1\right]^{\alpha}\right\}};\ (\alpha,\beta,\lambda)>0,\ x>0 \qquad (2.4)$$

In Fig. 1, we have displayed the plots of the PDF and hazard rate function of LC distribution for different values of α, β and λ.

C. Quantile function:

The Quantile function of Logistic Chen distribution is

$$Q\left(p\right)=\left[\ln\left\{\frac{1}{\lambda}\ln\left[\left(\frac{p}{1-p}\right)^{1/\alpha}+1\right]+1\right\}\right]^{1/\beta};0<p<1 \qquad (2.5)$$

D. Skewness and Kurtosis:

The Skewness and Kurtosis based on quantile function are, Bowley's coefficient of skewness is

$$\Im_{Sk}=\frac{Q\left(0.75\right)+Q\left(0.25\right)-2Q\left(0.5\right)}{Q\left(0.75\right)-Q\left(0.25\right)},\ \text{and} \qquad (2.6)$$

Coefficient of kurtosis based on octiles which was defined by [11] is

$$K_{u}\left(M\right)=\frac{Q\left(0.875\right)-Q\left(0.625\right)+Q\left(0.375\right)-Q\left(0.125\right)}{Q\left(3/4\right)-Q\left(1/4\right)},$$

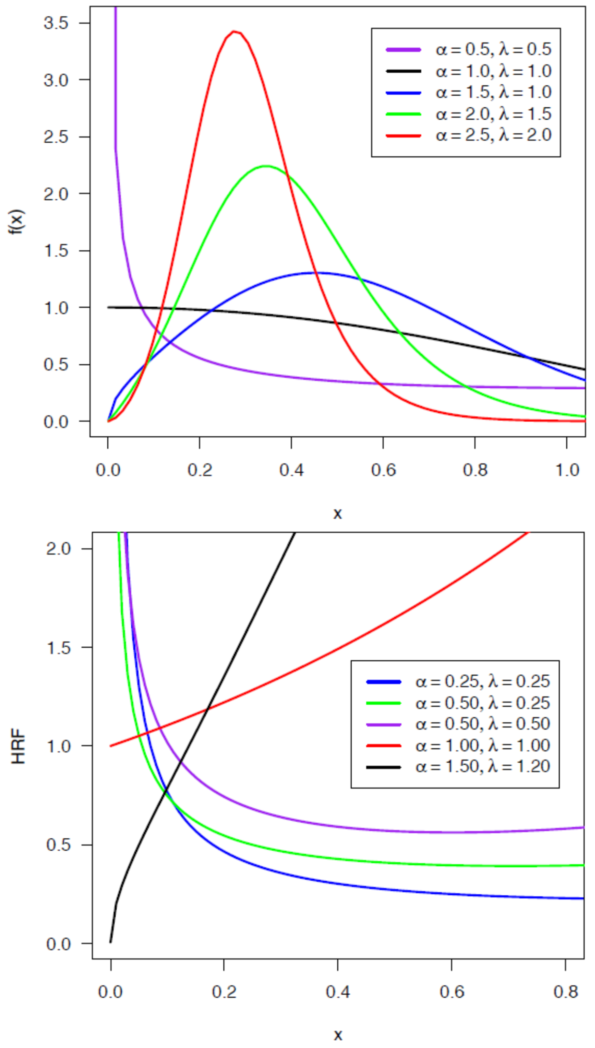

Fig. 1. Plots of PDF (left panel) and hazard function (right panel) for different values of α, β and λ.

III. METHODS OF ESTIMATION

In this section, we have presented some well-known estimation methods for estimating parameters of the proposed model, which are as follows

A. Maximum Likelihood Estimates

For the estimation of the parameter, the maximum likelihood method is the most commonly used method [12]. Let, $x_1, x_2, \ldots x_n$ is a random sample from $LC(\alpha,\beta,\lambda)$ and the likelihood function, $L(\alpha,\beta,\lambda)$ is given by

$$L(\eta; x_1, x_2 \ldots x_n) = f(x_1, x_2, \ldots x_n / \eta) = \prod_{i=1}^{n} f(x_i / \eta)$$

$$L(\alpha,\beta,\lambda) = \alpha\beta\lambda \prod_{i=1}^{n} \frac{\exp\left\{\lambda\left(e^{x_i^\beta}-1\right)+x_i^\beta\right\}\left[\exp\left\{\lambda\left(e^{x_i^\beta}-1\right)\right\}-1\right]^{\alpha-1}}{\left\{1+\left[\exp\left\{\lambda\left(e^{x_i^\beta}-1\right)\right\}-1\right]^{\alpha}\right\}^2}; \quad x>0$$

Now log-likelihood density is

$$\ell(\alpha,\beta,\lambda \mid \underline{x}) = n\ln(\alpha\beta\lambda) + (\beta-1)\sum_{i=1}^{n}\ln x_i + \lambda\sum_{i=1}^{n}\left(e^{x_i^\beta}-1\right) + \sum_{i=1}^{n} x_i^\beta + (\alpha-1)\sum_{i=1}^{n}\ln\left[\exp\left\{\lambda\left(e^{x_i^\beta}-1\right)\right\}-1\right] - 2\sum_{i=1}^{n}\ln\left\{1+\left[\exp\left\{\lambda\left(e^{x_i^\beta}-1\right)\right\}-1\right]^{\alpha}\right\} \tag{3.1.1}$$

Differentiating (3.1.1) with respect to α, β and λ we get,

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left[A(x_i)-1\right] - 2\sum_{i=1}^{n}\frac{\left[A(x_i)-1\right]^{\alpha}\ln\left[A(x_i)-1\right]}{1+\left[A(x_i)-1\right]^{\alpha}}$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\ln x_i + \lambda\sum_{i=1}^{n} x_i^\beta e^{x_i^\beta}\ln x_i + \beta\sum_{i=1}^{n} x_i^{\beta-1} + (\alpha-1)\lambda\sum_{i=1}^{n}\frac{x_i^\beta e^{x_i^\beta}\ln x_i A(x_i)}{\left[A(x_i)-1\right]} - 2\alpha\lambda\sum_{i=1}^{n}\frac{x_i^\beta e^{x_i^\beta}\ln x_i A(x_i)\left[A(x_i)-1\right]^{\alpha-1}}{1+\left[A(x_i)-1\right]^{\alpha}}$$

$$\frac{\partial l}{\partial \lambda} = -n + \frac{n}{\lambda} + \sum_{i=1}^{n} e^{x_i^\beta} + (\alpha-1)\sum_{i=1}^{n}\frac{(e^{x_i^\beta}-1)A(x_i)}{\left[A(x_i)-1\right]} - 2\alpha\sum_{i=1}^{n}(e^{x_i^\beta}-1)\frac{A(x_i)\left[A(x_i)-1\right]^{\alpha-1}}{1+\left[A(x_i)-1\right]^{\alpha}}$$

Where $A(x_i) = \exp\left\{\lambda\left(e^{x_i^\beta}-1\right)\right\}$

Equating above three non linear equations to zero and solving simultaneously for α, β and λ, we get the maximum likelihood estimate $\hat{\alpha},\ \hat{\beta}\ and\ \hat{\lambda}$ of the parameters α, β and λ. By using computer software like R, Matlab, Mathematica etc for maximization of (3.1.1) we can obtain the estimated value of α, β and λ. For the confidence interval estimation of α, β and λ and testing of the hypothesis, we have to calculate the observed information matrix. The observed information matrix for α, β and λ can be obtained as,

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix}$$

Where

$$J_{11} = \frac{\partial^2 l}{\partial \alpha^2},\ J_{12} = \frac{\partial^2 l}{\partial \alpha \partial \beta},\ J_{13} = \frac{\partial^2 l}{\partial \alpha \lambda}$$

$$J_{21} = \frac{\partial^2 l}{\partial \beta \partial \alpha},\ J_{22} = \frac{\partial^2 l}{\partial \beta^2},\ J_{23} = \frac{\partial^2 l}{\partial \beta \partial \lambda}$$

$$J_{31} = \frac{\partial^2 l}{\partial \lambda \partial \alpha},\ J_{32} = \frac{\partial^2 l}{\partial \beta \partial \lambda},\ J_{33} = \frac{\partial^2 l}{\partial \lambda^2}$$

Let $\Omega=(\alpha,\beta,\lambda)$ denote the parameter space and the corresponding MLE of Ω as $\hat{\Omega}=(\hat{\alpha},\hat{\beta},\hat{\lambda})$, then $\left(\hat{\Omega}-\Omega\right)\rightarrow N_3\left[0,\left(J(\Omega)\right)^{-1}\right]$ where $J(\Omega)$ is the Fisher's information matrix. Using the Newton-Raphson algorithm to maximize the likelihood creates the observed information matrix and hence the variance-covariance matrix is obtained as,

$$\left[J(\Omega)\right]^{-1}=\begin{pmatrix} \mathrm{var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha},\hat{\beta}) & \mathrm{cov}(\hat{\alpha},\hat{\lambda}) \\ \mathrm{cov}(\hat{\alpha},\hat{\beta}) & \mathrm{var}(\hat{\beta}) & \mathrm{cov}(\hat{\beta},\hat{\lambda}) \\ \mathrm{cov}(\hat{\alpha},\hat{\lambda}) & \mathrm{cov}(\hat{\beta},\hat{\lambda}) & \mathrm{var}(\hat{\lambda}) \end{pmatrix}$$

Hence from the asymptotic normality of MLEs, approximate 100(1-α) % confidence intervals for α, β and λ can be constructed as,

$$\hat{\alpha}\pm Z_{\alpha/2}SE(\hat{\alpha}),\ \hat{\beta}\pm Z_{\alpha/2}SE(\hat{\beta})\text{ and, }\hat{\lambda}\pm Z_{\alpha/2}SE(\hat{\lambda})$$

where $Z_{\alpha/2}$ is the upper percentile of standard normal variate

B. Method of Least-Square Estimation (LSE)

The ordinary least square estimators and weighted least square estimators are proposed by [13] to estimate the parameters of Beta distributions. Here we have employed the same method for the LC distribution. The least-square estimators of the unknown parameters α, β and λ of LC distribution can be obtained by minimizing

$$T\left(X;\alpha,\beta,\lambda\right)=\sum_{i=1}^{n}\left[G(X_i)-\frac{i}{n+1}\right]^2$$

with respect to unknown parameters α, β and λ.

Consider $G(X_i)$ represents the distribution function of the ordered random variables $X_{(1)}<X_{(2)}<\ldots<X_{(n)}$ where $\{X_1,X_2,\ldots,X_n\}$ is a random sample of size n from a distribution function G(.). The least-square estimators of α, β and λ say $\hat{\alpha},\hat{\beta}$ *and* $\hat{\lambda}$ respectively, can be obtained by minimizing

$$T\left(X;\alpha,\beta,\lambda\right)=\sum_{i=1}^{n}\left[1-\frac{1}{1+\left[\exp\left\{\lambda\left(e^{x_i^{\beta}}-1\right)\right\}-1\right]^{\alpha}}-\frac{i}{n+1}\right]^2;x\geq 0,(\alpha,\beta,\lambda>0).$$

(3.2.1)

with respect to α, β and λ.

Differentiating (3.2.1) with respect to α, β and λ we get,

$$\frac{\partial T}{\partial\alpha}=-2\sum_{i=1}^{n}\left[1-\frac{1}{1+\left(B(x_i)-1\right)^{\alpha}}-\frac{i}{n+1}\right]\frac{\left[B(x_i)-1\right]^{\alpha}\ln\left[B(x_i)-1\right]}{\left\{1+\left[B(x_i)-1\right]^{\alpha}\right\}^2}$$

$$\frac{\partial T}{\partial\beta}=-2\alpha\lambda\sum_{i=1}^{n}\ln(x_i)\left[1-\frac{1}{1+\left[B(x_i)-1\right]^{\alpha}}-\frac{i}{n+1}\right]\frac{\left[B(x_i)-1\right]^{\alpha-1}B(x_i)e^{x_i^{\beta}}x_i^{\beta}}{\left\{1+\left[B(x_i)-1\right]^{\alpha}\right\}^2}$$

$$\frac{\partial T}{\partial\lambda}=-2\alpha\sum_{i=1}^{n}\left[1-\frac{1}{1+\left(B(x_i)-1\right)^{\alpha}}-\frac{i}{n+1}\right]\frac{\left[B(x_i)-1\right]^{\alpha-1}B(x_i)\left(e^{x_i^{\beta}}-1\right)}{\left\{1+\left[B(x_i)-1\right]^{\alpha}\right\}^2}$$

Where $B(x_i)=\exp\left\{\lambda\left(e^{x_i^{\beta}}-1\right)\right\}$

Similarly, the weighted least square estimators can be obtained by minimizing

$$T\left(X;\alpha,\beta,\lambda\right)=\sum_{i=1}^{n}w_i\left[G(X_{(i)})-\frac{i}{n+1}\right]^2$$

with respect to α, β and λ. The weights w_i are

$$w_i=\frac{1}{Var(X_{(i)})}=\frac{\left(n+1\right)^2\left(n+2\right)}{i\left(n-i+1\right)} \qquad (3.1.2)$$

Hence, the weighted least square estimators of α, β and λ respectively can be obtained by minimizing,

$$T\left(X;\alpha,\beta,\lambda\right)=\sum_{i=1}^{n}\frac{\left(n+1\right)^2\left(n+2\right)}{i\left(n-i+1\right)}\left[1-\frac{1}{1+\left[\exp\left\{\lambda\left(e^{x_i^{\beta}}-1\right)\right\}-1\right]^{\alpha}}-\frac{i}{n+1}\right]^2$$

with respect to α, β and λ.

C. Method of Cramer-Von-Mises estimation (CVME)

The CVME estimators of α, β and λ of LC distribution are obtained by minimizing the function

$$U=\frac{1}{12n}+\sum_{i=1}^{n}\left[G\left(x_{i:n}\mid\alpha,\beta,\lambda\right)-\frac{2i-1}{2n}\right]^2;x\geq 0,(\alpha,\beta,\lambda>0)$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[1-\frac{1}{1+\left[\exp\left\{\lambda\left(e^{x_i^{\beta}}-1\right)\right\}-1\right]^{\alpha}}-\frac{2i-1}{2n}\right]^2$$

(3.2.1)

(3.3.1)

Differentiating (3.3.1) with respect to α, β and λ we get,

$$\frac{\partial U}{\partial\alpha}=-2\sum_{i=1}^{n}\left[1-\frac{1}{1+\left(B(x_i)-1\right)^{\alpha}}-\frac{2i-1}{2n}\right]\frac{\left[B(x_i)-1\right]^{\alpha}\ln\left[B(x_i)-1\right]}{\left\{1+\left[B(x_i)-1\right]^{\alpha}\right\}^2}$$

$$\frac{\partial U}{\partial\beta}=-2\alpha\lambda\sum_{i=1}^{n}\ln(x_i)\left[1-\frac{1}{1+\left[B(x_i)-1\right]^{\alpha}}-\frac{2i-1}{2n}\right]\frac{\left[B(x_i)-1\right]^{\alpha-1}B(x_i)e^{x_i^{\beta}}x_i^{\beta}}{\left\{1+\left[B(x_i)-1\right]^{\alpha}\right\}^2}$$

(3.2.2)

$$\frac{\partial U}{\partial\lambda}=-2\alpha\sum_{i=1}^{n}\left[1-\frac{1}{1+\left(B(x_i)-1\right)^{\alpha}}-\frac{2i-1}{2n}\right]\frac{\left[B(x_i)-1\right]^{\alpha-1}B(x_i)\left(e^{x_i^{\beta}}-1\right)}{\left\{1+\left[B(x_i)-1\right]^{\alpha}\right\}^2}$$

Here we use $B(x_i)=\exp\left\{\lambda\left(e^{x_i^{\beta}}-1\right)\right\}$

By solving $\frac{\partial U}{\partial\alpha}=0,\frac{\partial U}{\partial\beta}=0$ *and* $\frac{\partial U}{\partial\lambda}=0$

simultaneously we will get the CVM estimators.

IV. ILLUSTRATION WITH TWO REAL DATASETS

For the illustration we have used two real data sets used by previous researchers, which are as follows

Dataset-I (NP data)

We illustrate the applicability of the LC model using a real dataset used by former researchers. We have taken 100 observations on breaking the stress of carbon fibers (in Gba) used by [14].

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

The MLEs are calculated directly by using the optim() function in R software [15] and [16] by maximizing the likelihood function (3.1). By maximizing the likelihood function in (3.1) we have obtained $\hat{\alpha}$ = 2.3155, $\hat{\beta}$ = 0.6160, $\hat{\lambda}$ = 0.1387 and corresponding Log-Likelihood value is l = -141.4061. In Table I, we have demonstrated the MLE's with their standard errors (SE) and 95% confidence interval for α, β, and λ.

Table I
MLE, SE AND 95% CONFIDENCE INTERVAL FOR α, β AND λ

Parameter	MLE	SE	95% ACI
alpha	2.3155	0.7103	(0.9233, 3.7077)
beta	0.6160	0.1277	(0.3657, 0.8663)
lambda	0.1387	0.0388	(0.0627, 0.2147)

.

We have displayed the graph of the profile log-likelihood function of α, β, and λ in Fig. 2 and observed that the MLEs are unique.

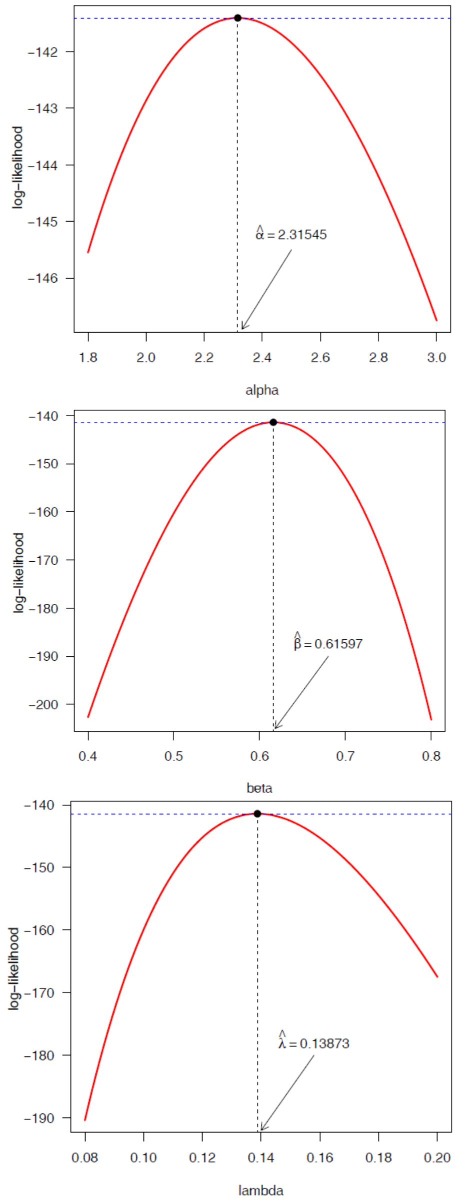

Fig. 2. Graph of profile log-likelihood function of α, β, and λ.

In Fig. 3 we have presented the Q-Q plot (empirical quantile against theoretical quantile) and CDF plot (empirical distribution function against theoretical distribution function).

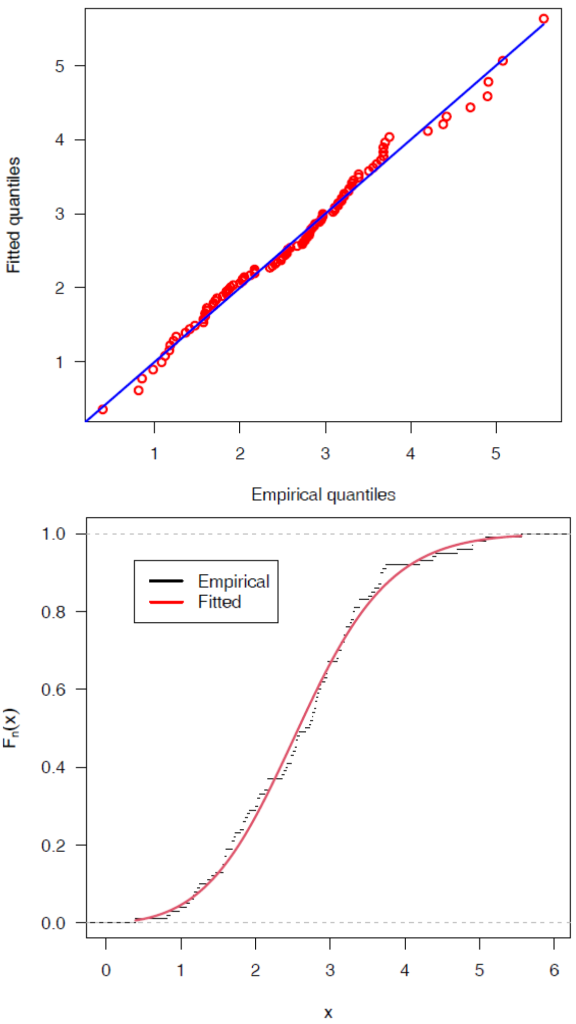

Fig 3. The Q-Q plot (left panel) and CDF plot (right panel) of LEE distribution

Dataset-II (Lee)

The second real data set represents the remission times (in months) of a random sample of 128 bladder cancer patients [17], sorted data

0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05

By maximizing the likelihood function in (3.1) we have obtained $\hat{\alpha}$ = 4.4642, $\hat{\beta}$ = 0.1551, $\hat{\lambda}$ = 0.2490 and corresponding Log-Likelihood value is l = -409.421. In Table II we have demonstrated the MLE's with their standard errors (SE) and 95% confidence interval for α, β, and λ.

TABLE II

MLE AND 95% CONFIDENCE INTERVAL FOR α, β AND λ

Parameter	MLE	SE	95% ACI
alpha	4.4642	1.8973	(0.7455, 8.1829)
beta	0.1551	0.0559	(0.0455, 0.2647)
lambda	0.2490	0.0479	(0.1551, 0.3429)

We have displayed the graph of the profile log-likelihood function of α, β, and λ in Fig. 2 and observed that the MLEs are unique.

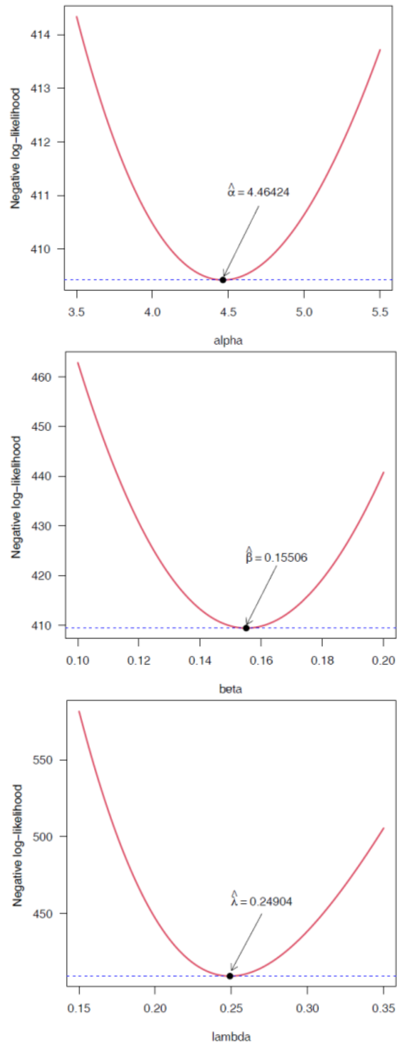

Fig. 2. Graph of profile log-likelihood function of α, β, and λ.

In Fig. 3 we have presented the Q-Q plot (empirical quantile against theoretical quantile) and CDF plot (empirical distribution function against theoretical distribution function).

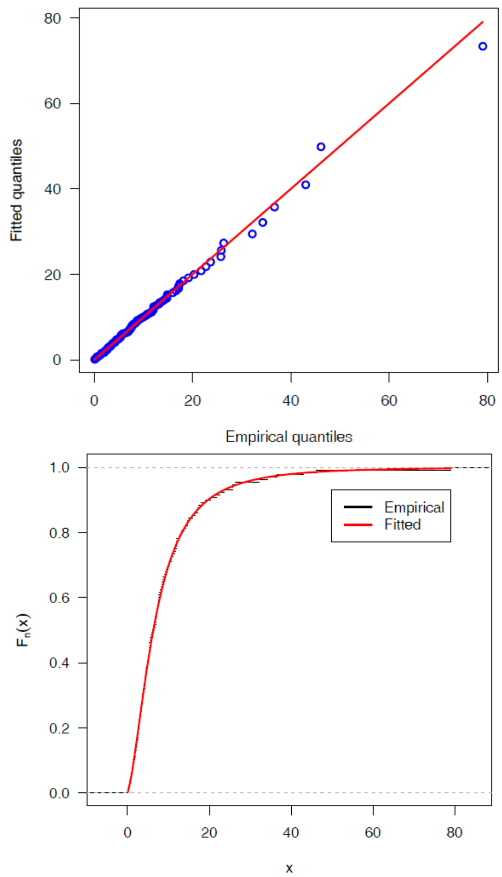

Fig 3. The Q-Q plot (left panel) and CDF plot (right panel) of LEE distribution

By using MLE method we estimate the parameter of each of these distributions. For the goodness of fit purpose we use negative log-likelihood (-LL), Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike Information criterion (CAIC) and Hannan-Quinn information criterion (HQIC), statistic to select the best model among selected models. The expressions to calculate AIC, BIC, CAIC and HQIC are listed below:

a) $AIC = -2l(\hat{\theta}) + 2k$

b) $BIC = -2l(\hat{\theta}) + k\log(n)$

c) $CAIC = AIC + \dfrac{2k(k+1)}{n-k-1}$

d) $HQIC = -2l(\hat{\theta}) + 2k\log\left[\log(n)\right]$

where k is the number of parameters and n is the size of the sample in the model under consideration. Further, in order to evaluate the fits of the LHC distribution with some selected distributions we have taken the Kolmogorov-Simnorov (KS), the Anderson-Darling (W) and the Cramer-Von Mises (A^2) statistic. These statistics are widely used to compare non-nested models and to illustrate how closely a specific CDF fits the empirical distribution of a given data set. These statistics are calculated as

$$KS = \max_{1\le i\le n}\left(d_i - \frac{i-1}{n}, \frac{i}{n} - d_i\right)$$

$$W = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln d_i + \ln\left(1-d_{n+1-i}\right)\right]$$

$$A^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[\frac{(2i-1)}{2n} - d_i\right]^2$$

where $d_i = CDF(x_i)$; the x_i's being the ordered observations. In Table III and Table IV we have displayed the estimated value of the parameters of Logistic Chen distribution using MLE, LSE and CVME method and their corresponding KS, W and A^2 statistic with p-value.

TABLE III
(Dataset-I)
ESTIMATED PARAMETERS, KS, W AND A^2 STATISTIC WITH P-VALUE OF MLE, LSE AND CVME METHOD

Estimation Method	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	*KS(p-value)*	*W(p-value)*	*A^2(p-value)*
MLE	2.31545	0.61597	0.13873	0.0628(0.8246)	0.0649(0.7846)	0.3849(0.8633)
LSE	1.29409	0.89243	0.07217	0.0472(0.9793)	0.0448(0.9078)	0.7447(0.5223)
CVME	1.28886	0.9032	0.07012	0.0496(0.9663)	0.0442(0.9115)	0.8032(0.4784)

TABLE IV
(Dataset-II)
ESTIMATED PARAMETERS, KS, W AND A^2 STATISTIC WITH P-VALUE OF MLE, LSE AND CVME METHOD

Estimation Method	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	*KS(p-value)*	*W(p-value)*	*A^2(p-value)*
MLE	4.46424	0.15506	0.24904	0.0309(0.9997)	0.0144(0.9997)	0.0930(0.9999)
LSE	6.93816	0.10434	0.29433	0.0301(0.9998)	0.0127(0.9999)	0.0987(0.9999)
CVME	6.98329	0.10491	0.29382	0.0307(0.9997)	0.0122(0.9999)	0.0968(0.9999)

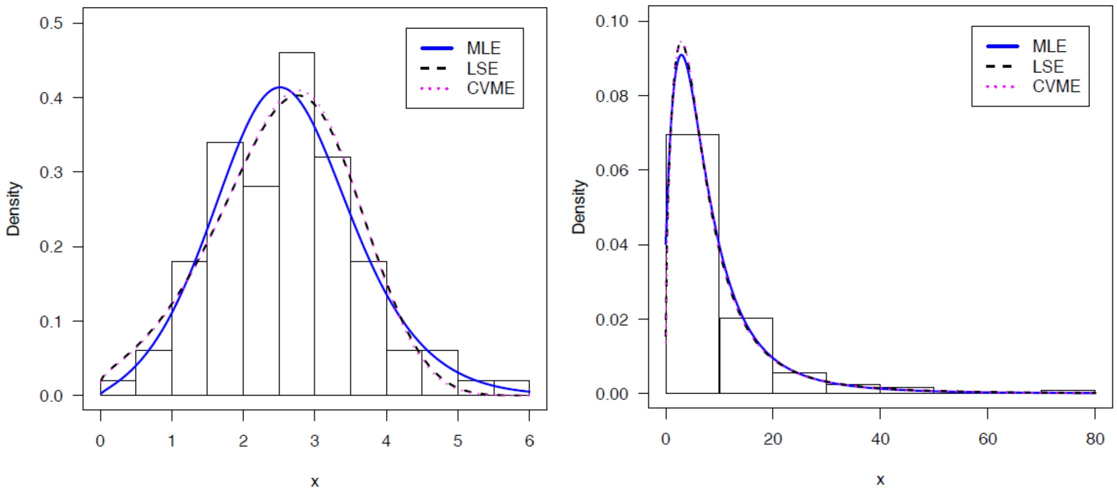

Fig 4. **The Histogram and the density function of fitted distributions of dataset-I (left panel) and dataset-II (right panel) of MLE, LSE and CVME.**

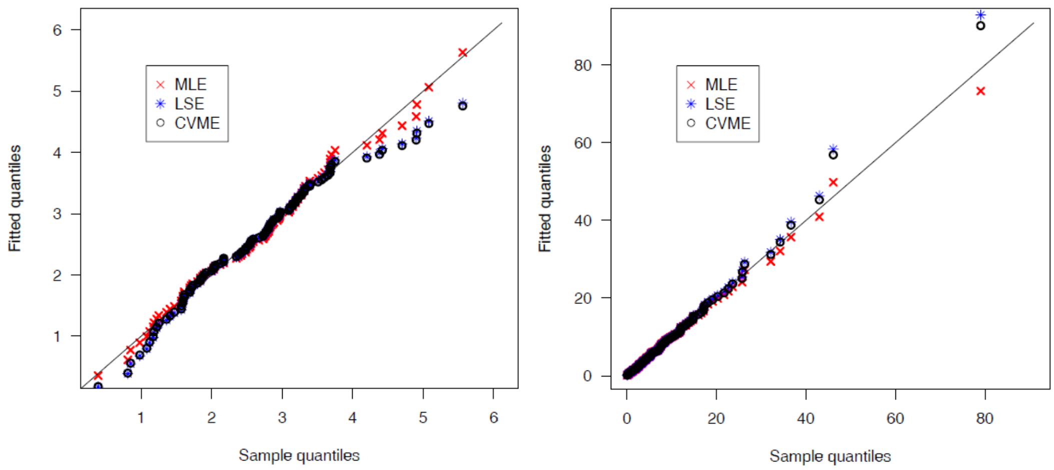

Fig 5. Sample quantiles verses fitted quantiles of MLE, LSE and CVME of dataset-I (left panel) and dataset-II (right panel).

To illustrate the goodness of fit of the Lindley inverse exponential distribution, we have taken some well known distribution for comparison purpose which are listed below,

A. Generalized Exponential Extension (GEE) distribution:

The probability density function of GEE introduced by [18] having upside down bathtub-shaped hazard function distribution with parameters α , β and λ is

$$f_{GEE}(x;\alpha,\beta,\lambda)=\alpha\beta\lambda\,(1+\lambda x)^{\alpha-1}\,exp\left\{1-(1+\lambda x)^{\alpha}\right\}\left[1-exp\left\{1-(1+\lambda x)^{\alpha}\right\}\right]^{\beta-1}\quad;\ x\geq 0.$$

B. Lindley-Exponential (LE) distribution:

The probability density function of LE [19] can be expressed as

$$f_{LE}(x)=\lambda\left(\frac{\theta^2}{1+\theta}\right)e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\theta-1}\left\{1-\ln\left(1-e^{-\lambda x}\right)\right\};\ \lambda,\theta>0,\ x>0$$

C. Generalized Exponential (GE) distribution

The probability density function of generalized exponential distribution defined by [20]

$$f_{GE}(x;\alpha,\lambda)=\alpha\,\lambda\,e^{-\lambda x}\left\{1-e^{-\lambda x}\right\}^{\alpha-1};(\alpha,\lambda)>0,x>0$$

D. Chen distribution

The probability density function of Chen distribution

$$f_{CN}(x;\lambda,\beta)=\lambda\,\beta\;x^{\beta-1}\,e^{x^{\beta}}\,exp\left\{\lambda\left(1-e^{x^{\beta}}\right)\right\}\quad;(\lambda,\beta)>0,x>0$$

E. Exponential power (EP) distribution

The probability density function Exponential power (EP) distribution defined by [21] is

$$f_{EP}(x)=\alpha\,\lambda^{\alpha}\;x^{\alpha-1}\,e^{(\lambda\,x)^{\alpha}}\;\exp\left\{1-e^{(\lambda\,x)^{\alpha}}\right\}\quad;(\alpha,\lambda)>0,\quad x\geq 0$$

where α and λ are the shape and scale parameters, respectively.

For the judgment of potentiality of the proposed model we have presented the value of Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) which are presented in Table V and Table VI for dataset I and II.

TABLE V

LOG-LIKELIHOOD (LL), AIC, BIC, CAIC AND HQIC (DATASET-I)

Model	-LL	AIC	BIC	CAIC	HQIC
LCD	141.4061	288.8121	296.6276	289.0621	291.9752
GEE	141.3708	288.7416	296.5571	288.9916	291.9047
LE	143.2473	290.4946	295.7049	290.6183	292.6033
EP	145.9589	295.9179	301.1282	296.0391	298.0266
GE	146.1823	296.3646	301.5749	296.4883	298.4733
Chen	148.9044	301.8089	307.0192	301.9326	303.9176

TABLE VI

LOG-LIKELIHOOD (LL), AIC, BIC, CAIC AND HQIC (DATASET-II)

Model	-LL	AIC	BIC	CAIC	HQIC
LCD	409.4744	824.9487	833.5048	825.1423	828.4251
GEE	410.6013	827.2026	835.7586	827.3961	830.6789
LE	412.6254	829.2507	834.9548	829.3467	831.5683
EP	413.0776	830.1552	835.8592	830.2512	832.4728
GE	426.6474	857.2948	862.9989	857.3893	859.6124
Chen	431.1625	866.3251	872.0291	866.4211	868.6427

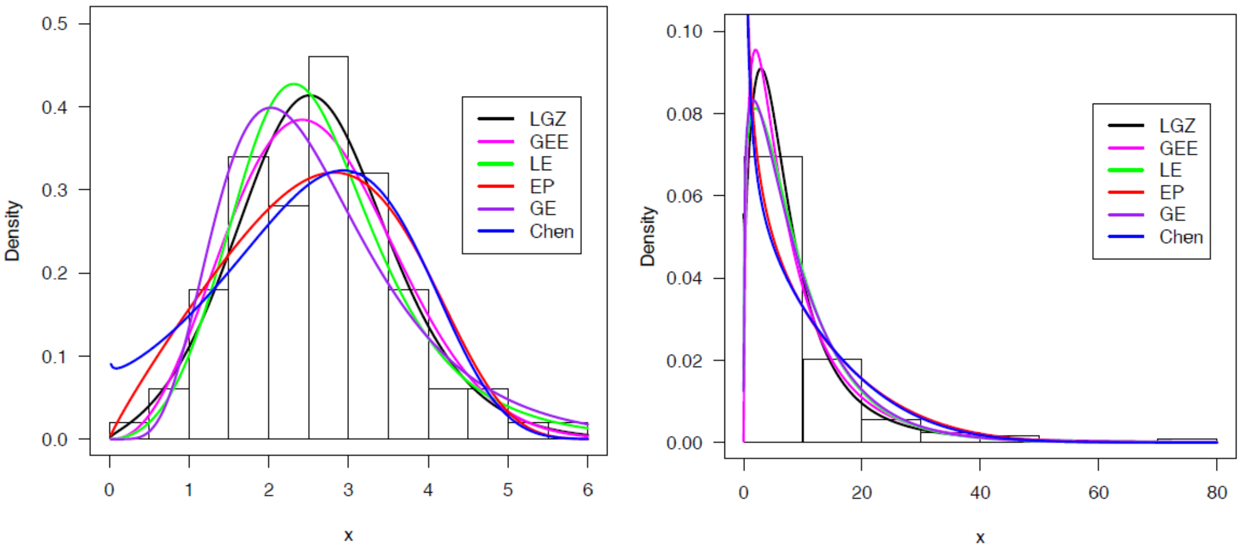

Fig. 6. The Histogram and the density function of fitted distributions for the dataset-I (left panel) and dataset-II (right panel). Empirical distribution function with estimated distribution function for the dataset-I (left panel) and dataset-II (right panel).

The Histogram and the density function of fitted distributions and Empirical distribution function with estimated distribution function of LIE and some selected distributions are presented in Fig. 6.

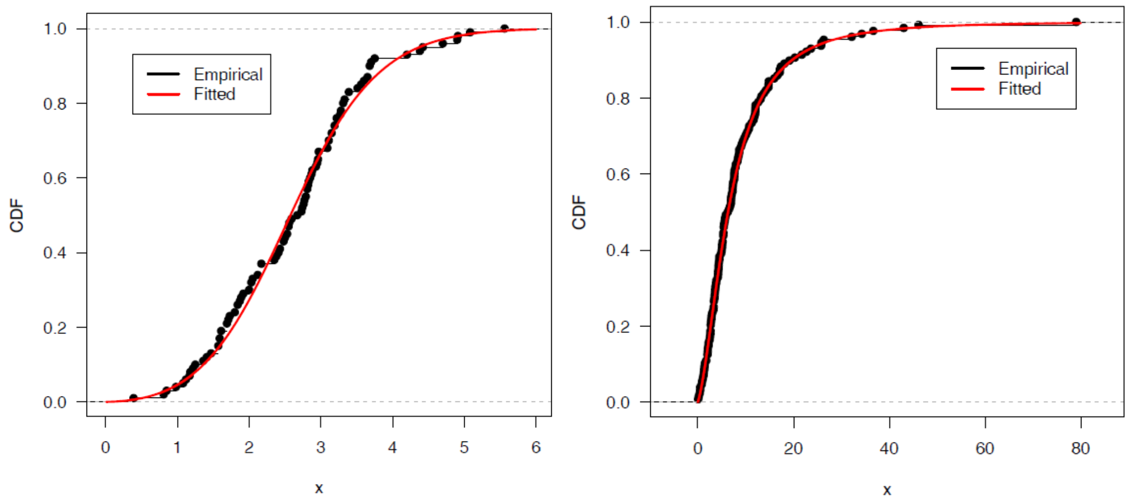

Fig. 7. Empirical distribution function with estimated distribution function for the dataset-I (left panel) and dataset-II (right panel).

To compare the goodness-of-fit of the LC distribution with other competing distributions we have presented the value of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistics in Table VII and Table VIII. It is observed that the LC distribution has the minimum value of the test statistic and higher *p*-value thus we conclude that the LC distribution gets quite better fit and more consistent and reliable results from others taken for comparison.

TABLE VII
THE GOODNESS-OF-FIT STATISTICS AND THEIR CORRESPONDING P-VALUE (DATASET-I)

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
LCD	0.0628(0.8246)	0.0649(0.7846)	0.3849(0.8633)
GEE	0.0654(0.7862)	0.0723(0.7385)	0.4202(0.8281)
LE	0.0838(0.4836)	0.1225(0.4860)	0.7042(0.5549)
EP	0.0993(0.2771)	0.1861(0.2963)	1.3081(0.2297)
GE	0.1078(0.1959)	0.2293(0.2174)	1.2250(0.2581)
Chen	0.0945(0.3336)	0.2180(0.2353)	1.6938(0.1364)

TABLE VIII
THE GOODNESS-OF-FIT STATISTICS AND THEIR CORRESPONDING P-VALUE (DATASET-II)

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
LCD	0.0321(0.9994)	0.0149(0.9997)	0.1010(0.9999)
GEE	0.0442(0.9636)	0.0394(0.9367)	0.2630(0.9631)
LE	0.0691(0.5740)	0.1131(0.5252)	0.6276(0.6219)
EP	0.0725(0.5115)	0.1279(0.4652)	0.7137(0.5472)
GE	0.1199(0.0503)	0.5993(0.0223)	3.6745(0.0126)
Chen	0.1426(0.0108)	0.6879(0.0135)	4.3878(0.0057)

V. CONCLUSIONS

In this study, we have introduced a three-parameter univariate continuous Logistic Chen distribution. Some distributional and statistical properties of the LC distribution are presented such as the shapes of the probability density, cumulative density and hazard rate functions, survival function, hazard function quantile function, the skewness, and kurtosis measures are derived and established and found that the proposed model is flexible and inverted bathtub shaped hazard function. The model parameters are estimated by using three well-known estimation methods namely maximum likelihood estimation (MLE), least-square estimation (LSE), and Cramer-Von-Mises estimation (CVME) methods and we concluded that the MLEs are quite better than LSE, and CVM. A real data set is considered to explore the applicability and suitability of the proposed distribution and found that the proposed model is quite better than other lifetime model taken into consideration.

REFERENCES

[1] Z. Chen, A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function. Statistics & Probability Letters, 49(2000) 155-161.

[2] A. K. Srivastava, and V. Kumar, Markov Chain Monte Carlo methods for Bayesian inference of the Chen model. International Journal of Computer Information Systems, 2(2) (2011) 07-14.

[3] F. A. Bhatti, G. G. Hamedani, S. M. Najibi, and M. Ahmad, On the Extended Chen Distribution: Development, Properties, Characterizations and Applications. Annals of Data Science. (2019) 1-22.

[4] B. Tarvirdizade, and M. Ahmadpour, A New Extension of Chen Distribution with Applications to Lifetime Data. Communications in Mathematics and Statistics, (2019) 1-16.

[5] R. K. Joshi, and V. Kumar, Lindley-Chen Distribution with Applications. International Journals of Engineering, Science & Mathematics (IJESM), 9(10)(2020) 12-22.

[6] A.K. Chaudhary and V. Kumar, A Study on Properties and Applications of Logistic Modified Exponential Distribution. International Journal of Latest Trends in Engineering and Technology (IJLTET), 17(5) (2020).

[7] M. H. Tahir, G. M. Cordeiro, A. Alzaatreh, M. Mansoor, and M. Zubair, The logistic-X family of distributions and its applications. Communications in Statistics-Theory and Methods, 45(24)(2016) 7326-7349.

[8] R. K. Joshi, L. P. Sapkota, and V. Kumar, The Logistic-Exponential Power Distribution with Statistical Properties and Applications, International Journal of Emerging Technologies and Innovative Research, 7(12)(2020) 629-641.

[9] A.K. Chaudhary and V. Kumar, A Study on Properties and Goodness-of- Fit of the Logistic Inverse Weibull Distribution. Global Journal of Pure and Applied Mathematics (GJPAM), 16(6)(2020) 871-889.

[10] Y. Lan, and L. M. Leemis, The logistic–exponential survival distribution. Naval Research Logistics (NRL), 55(3)(2008) 252-264.

[11] J. J. A. Moors, A quantile alternative for kurtosis. Journal of the Royal Statistical Society: Series D (The Statistician), 37(1) (1988) 25-32.

[12] G. Casella, and R. L. Berger, Statistical Inference. Brooks/ Cole Publishing Company(1990) California.

[13] J. J. Swain, S. Venkatraman, and J. R. Wilson, Least-squares estimation of distribution functions in johnson's translation system. Journal of Statistical Computation and Simulation, 29(4)(1988) 271–297.

[14] M. D. Nichols, and W. J. Padgett, A bootstrap control chart for Weibull percentiles. Quality and reliability engineering international, 22(2)(2006) 141-151.

[15] R Core Team, R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria (2020). URL https://www.R-project.org/.

[16] E. G. Ming Hui, Learn R for applied statistics. Springer, (2019) New York.

[17] E. T. Lee, and J. Wang, Statistical methods for survival data analysis 476(2003) John Wiley & Sons.

[18] A. J. Lemonte, A new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function. Computational Statistics & Data Analysis, 62(2013) 149-170.

[19] D. Bhati, M.A. Malik, and H.J. Vaman, Lindley–Exponential distribution: properties and applications. METRON. 73(2015) 335–357.

[20] R. D. Gupta, and D. Kundu, Generalized exponential distributions, Australian and New Zealand Journal of Statistics, 41(2)(1999) 173 - 188.

[21] R.M. Smith, and L.J. Bain, An exponential power life-test distribution, Communications in Statistics, 4(1975) 469-481.