International Journal of Mathematics Trends and Technology
ISSN: 2231 – 5373 /doi:10.14445/22315373/IJMTT-V67I1P521
Volume 67 Issue 1, 159-165, January, 2021

A Solution to the Problem of Dispersion of Air Pollutants

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Abstract *- In this paper, an analytical solution to the problem of dispersion of air pollutants with constant removal rate and constant wind velocity emitted from a point source in unsteady state condition is studied where eddy diffusivity coefficients are also taken as constants. The methods of separation of variables and integral transform technique have been used for the solution of the problem. It is found that the value of concentration of pollutants first increases with some increasing value of vertical and downwind distance and becomes maximum at some height and then decreases and finally tends to zero.*

Keywords - *Dispersion of air pollutants, Point source, Constant wind velocity, Removal rate, Unsteady condition*

I. INTRODUCTION

Air pollutants released from various sources affect directly or indirectly man and his environment. The resulted ground level concentration patterns have to be estimated for a wide variety of air quality analysis for social planning and industrial growth. Air pollutants emitted from different sources are transported, dispersed or deposited by meteorological and topographical conditions. The prediction for the air quality is based on various modeling approaches. Air pollution models are routinely used in environmental impact assessments, risk analysis and emergency planning, and source apportionment studies. In highly polluted cities such as Athens, Los Angeles and Mexico, regional scale air quality models are used to forecast air pollution episodes-the results from these models may initiate compulsory shutdown of industries or vehicle restrictions. The various roles served by the air pollution models, which cover a broad range of scales from local to global, lead to distinct modeling requirements [6]. Atmospheric dispersion modeling refers to the mathematical description of contaminant transport in the atmosphere. The term dispersion is used to describe the combination of diffusion (clean air mixing with contaminated air through the process of molecular motion) and advection (transport of pollutants by the wind) that occurs within the air near the Earth's surface.

The importance and the need of mathematical modeling are well known in the scientific community. There are various modeling approaches that have been used effectively in the past to deal with air pollution dispersion. Also, persistent efforts are being made to improve the accuracy of predictions using latest advancements in the computing technology and improvement in the observational and modeling framework [4].

Sharan et al. made an attempt to review the major research concerning atmospheric dispersion modeling in the last few decades and in another paper Sharan et al. have formulated a mathematical model for low wind conditions by taking into account the diffusion in the downwind direction [4, 5]. Essa S.M.et al. have solved the two dimensional diffusion equation to obtain the concentration by using separation of variables under the variation of eddy diffusivity which depend on the vertical height in unstable case and in another paper they have given an analytical solution of the three dimensional advection-diffusion equations taking the vertical eddy diffusivity and the wind speed as the dependent variables to the vertical height z with assuming that the concentration distribution of pollutants in the crosswind direction has a Gaussian shape[9,8]. Kumar and Goyal have formulated the analytical model for point, line and area sources. They have obtained an analytical solution of an advection- diffusion equation with the Neumann (total reflection) boundary conditions for a bounded domain for point sources using the separation of variable and wind speed as a power law profile of vertical height above the ground [1]. Essa S.M.and Mubarak have estimated a short range model calculating ground level concentration from elevated sources which realized a Fickian-type formula, taking the source and mixing height as the functions of the wind velocity and eddy diffusivity profiles[7].

Srivastava et al. have presented a three dimensional atmospheric diffusion model with variable removal rate and variable wind velocity using power law profile [11]. Costa et al. have given a three-dimensional solution of the steady state advection diffusion equation considering a vertically inhomogeneous planetary boundary layer. They used the generalized integral transform technique [2]. Verma,V.S. et al. have solved a problem of dispersion equation in steady state condition using variable wind velocity as a wave function[14]. Verma,V.S. et al. have given an analytical solution to the dispersion equation of air pollutants with taking wind velocity as a power law profile emitted from a point source[15].

Essa S.M. has solved the advection diffusion equation in two directions to obtain the crosswind integrated concentration using Laplace transformation technique and considering the wind speed depends on the vertical height and eddy diffusivity depends on downwind and vertical distances and compared between the two predicted concentrations and observed concentration data taken in the Copenhagen in Denmark [10]. Verma has given an analytical approach to the solution of the problem of dispersion of an air pollutant in steady state condition with constant wind velocity and constant removal rate taking eddy diffusivities as constant applying Fourier transform technique [12].

Agarwal et al. have solved an unsteady state three- dimensional atmospheric diffusion equation with a point source assuming that the wind velocity vary with downwind distance in the form wave function and removal rate as constant [3]. In this paper, the dispersion of an air pollutant emitted from a point source is studied taking wind velocity, removal rate and eddy diffusivities as constants under an unstable state condition

II. Mathematical Model

The partial differential equation describing an unsteady state of dispersion of an air pollutant from a point source located at height h_s from the ground is given by

$$\frac{\partial c}{\partial t} + U\frac{\partial c}{\partial X} = \frac{\partial}{\partial y}\left(K_y\frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z\frac{\partial c}{\partial z}\right) - \alpha c \tag{1}$$

where x, y, z are the Cartesian co-ordinates, t is the time period,α is the rate of removal of pollutant due to some natural mechanism present in the atmosphere which is also constant,$k_y and\, k_z$ are the eddy diffusivity in y and z directions respectively which are also constants. U is the constant wind velocity. C is the concentration of the air pollutant.

The initial and boundary conditions to solve the equation (1) are given as:

$$C=0 \quad , \quad t=0 \tag{2}$$

$$C = \frac{Q\delta(y)\delta(z-h_s)}{U}, \quad x = 0,\ 0 \le h_s \le H,\ t \ge 0 \tag{3}$$

$$C = 0 \quad , \quad y \to \pm\infty, \quad t \ge 0 \tag{4}$$

$$k_z \frac{\partial C}{\partial z} = V_d C \quad , \quad z = 0, \quad t \ge 0 \tag{5}$$

$$\frac{\partial C}{\partial z} = 0 \quad , \quad z = H, \quad t \ge 0 \tag{6}$$

where δ is the Dirac-delta function ,H is the height of the inversion layer and V_d is the deposition velocity of concentration of pollutant at ground $z=0$.

III. Method of Solution

To solve the problem, the dispersion equation (1) and the boundary conditions are made dimensionless by introducing the following dimensionless quantities:

$$N^*=\frac{V_d H}{K_z}, \quad t^*=\frac{K_z t}{H^2}, \quad c^*=\frac{U_H H^2 c}{Q}, \quad x^*=\frac{K_z x}{U_H H^2}, \quad \alpha^*=\frac{\alpha H^2}{K_z}, \quad y^*=\frac{y}{H}$$

On dropping the asterisk (*), the dispersion equation (1) and conditions (2)-(6) become:

$$\frac{\partial C}{\partial t} + U\frac{\partial C}{\partial x} = \beta\frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2} - \alpha C, \quad \text{where } \beta = \frac{K_y}{K_z} \tag{7}$$

$$C=0 \quad , \quad t=0 \tag{8}$$

$$C = \frac{Q\delta(y)\delta(z-h_s)}{U}, \quad x = 0, t \ge 0 \tag{9}$$

$$C = 0 \quad , \quad y \to \pm\infty \tag{10}$$

$$\frac{\partial C}{\partial z} = NC \quad , \quad z = 0 \quad , \quad t \geq 0 \tag{11}$$

$$\frac{\partial C}{\partial z} = 0 \quad , \quad z = 1 \quad , \quad t \geq 0 \tag{12}$$

Taking Laplace transform of equation (7), we get

$$S\bar{C} + U\frac{\partial \bar{C}}{\partial x} = \beta\frac{\partial^2 \bar{C}}{\partial y^2} + \frac{\partial^2 \bar{C}}{\partial z^2} - \alpha\bar{C} \tag{13}$$

where bar (-) denotes the Laplace transform of the function and '*S*' is the corresponding Laplace transformparameter.

Again, taking Fourier transform of equation (13), we get

$$U\frac{\partial \hat{\bar{C}}}{\partial x} + (\alpha + s + \beta p^2)\,\hat{\bar{C}} = \frac{\partial^2 \hat{\bar{C}}}{\partial z^2} \tag{14}$$

where cap (^) represents the Fourier transform of the function with the corresponding Fourier transform parameter p.

Using the Laplace and Fourier transforms on the boundary conditions (8)-(12), we get

$$\hat{\bar{C}} = \frac{Q\delta(z-h_S)}{SU} , \quad x = 0 \tag{15}$$

$$\frac{\partial \hat{\bar{C}}}{\partial z} = 0 \quad , \quad z = 1 \tag{16}$$

$$\frac{\partial \hat{\bar{C}}}{\partial z} = N\hat{\bar{C}} \quad , \quad z = 0 \tag{17}$$

To solve the equation (14), we assume a trial solution as:

$$\hat{\bar{C}} = X(x)\,Z\,(z) \tag{18}$$

where *X(x)* is a function of *x* and*Z (z)* is a function of *z* .

Using equation (18) in (14), we get

$$U\frac{\partial}{\partial x}\{X(x)Z\,(z)\} + (\alpha + s + \beta p^2)X(x)Z\,(z) = \frac{\partial^2}{\partial z^2}\{X(x)Z\,(z)\}$$

$$or \quad X'Z + (\alpha + s + \beta p^2)X\,Z = XZ''$$

$$or \quad U\frac{X'Z}{XZ} + (\alpha + s + \beta p^2) = \frac{XZ''}{XZ}$$

$or \quad U\frac{X'}{X} + (\alpha + s + \beta p^2) = \frac{Z''}{Z} = -k^2$ (say), where k^2 is a separation constant.

Therefore, we have two differential equations as:

$$\frac{Z''}{Z} = -k^2$$

$$or \quad Z'' + k^2 Z = 0 \qquad or \quad \frac{d^2Z}{dz^2} + k^2 Z = 0 \tag{19}$$

and $$U\frac{X'}{X} + (\alpha + s + \beta p^2) = -k^2 \tag{20}$$

The solution of the equation (19) is given by

$$Z\,(z) = A\cos kz + B\sin kz \tag{21}$$

Again from (20), we get

$$U\frac{dX}{X}\{(\alpha+s+\beta p^2)+k^2\}dx = 0$$

$$or\ \frac{dX}{X}+\frac{(\alpha+s+\beta p^2+k^2)}{U}dx = 0$$

Integrating, we get

$$log\, X = -\frac{(\alpha+s+\beta p^2+k^2)}{U}x + log\, M$$

$$or\ X = M\, e^{-\frac{(\alpha+s+\beta p^2+k^2)}{U}x}$$

Therefore $X(x) = M\, e^{-\frac{(\alpha+s+\beta p^2+k^2)}{U}x}$ (22)

Now, using conditions (16), (17) in (18), we get the Eigen value equation as below:

$k_n tan k_n = N$, where n =1, 2, 3… where $k_n = n\pi$.

Using the values from (21), (22) in (18), we get

$$\hat{\bar{C}} = M_n e^{-\{\frac{(\alpha+s+\beta p^2+k^2)}{U}\}x}(Acoskz + Bsinkz).$$

$$or\ \hat{\bar{C}} = \sum_{n=1}^{\infty}\frac{M_n}{cosk_n}cos\, k_n\,(z-1)\, e^{-\{\frac{(\alpha+s+\beta p^2+k^2)}{U}\}x} \quad (23)$$

Using condition (15) in equation (23), we get

$$\frac{Q\delta(z-h_s)}{SU} = \sum_{n=1}^{\infty}\frac{M_n}{cosk_n}\, cosk_n\,(z-1) \quad (24)$$

Now, multiplying (24) by $cosk_m(z-1)$ and integrate with respect to z from 0 to 1 ,we get

$$\int_0^1\frac{Q\delta(z-h_s)}{SU}cosk_m(z-1)dz = \sum_{n=1}^{\infty}\frac{M_n}{cosk_n}\int_0^1 cos\, k_n(z-1)\, cos\, k_m(z-1)\, dz \quad (25)$$

Using $\int_0^1\delta(z-h_s)\, f_n(z)dz = f_n(h_s)$, $\int_0^1 Z^p f_m(z)f_n(z)dz = 0\, , m \neq n$

And $\int_0^1\delta(z-h_s).\, cosk_m(z-1)dz = cos\, k_m(h_s-1)\ if\ m = n$

$$\int_0^1 cos\, k_n(z-1)\, cos\, k_m(z-1)\, dz = \int_0^1 cos^2 k_n(z-1)\, dz \quad if\ m = n$$

Equation (25) reduces to

$$\frac{Q}{SU}\, cosk_n(h_s-1) = \sum_{n=1}^{\infty}\frac{M_n}{cosk_n}\int_0^1 cos^2 k_n(z-1)\, dz$$

So, equation (23) becomes

$$\hat{\bar{C}} = \frac{Q}{SU}\sum_{n=1}^{\infty}\frac{cosk_n(h_s-1)cosk_n(z-1)}{\int_0^1 cos^2 k_n(z-1)\, dz}e^{-\{\frac{(\alpha+s+\beta p^2+k_n{}^2)}{U}\}x}$$

$$= \frac{Q}{SU}\sum_{n=1}^{\infty}\frac{cosk_n(h_s-1)cosk_n(z-1)}{p_n}\, .\, e^{\frac{-\beta p^2 x}{U}}.\, e^{-(\alpha+s+k_n{}^2)x}$$

Therefore, $\hat{\bar{C}} = \frac{Q}{SU} e^{\frac{-\beta p^2 x}{U}} \sum_{n=1}^{\infty} \frac{cosk_n(h_s-1)cosk_n(z-1)}{p_n} . e^{-\frac{(\alpha+s+k_n{}^2)}{U}x}$ (26)

where $p_n = \int_0^1 cos^2 k_n(z-1)\, dz$

Taking inverse Fourier transform of (26), we get

$$\bar{C} = \frac{Q}{2\sqrt{\pi}} \frac{1}{SU} \frac{1}{\sqrt{\beta x}} e^{\frac{-y^2}{4\beta x}} \sum_{n=1}^{\infty} \frac{cosk_n(h_s-1)cosk_n(z-1)}{p_n} e^{-\frac{(\alpha+s+k_n{}^2)}{U}x}$$

or $\bar{C} = \frac{Q}{2\sqrt{\pi}} \frac{1}{SU} \frac{1}{\sqrt{\beta x}} e^{\frac{-y^2}{4\beta x}} \sum_{n=1}^{\infty} \frac{cosk_n(h_s-1)cosk_n(z-1)}{p_n} . e^{-\frac{(\alpha+k_n{}^2)}{U}x} . e^{-\frac{Sx}{U}}$

Now, taking inverse Laplace's transform, we get

$C = \frac{Q}{2\sqrt{\pi}} \frac{1}{\sqrt{\beta x}} e^{\frac{-y^2}{4\beta x}} \sum_{n=1}^{\infty} \frac{cosk_n(h_s-1)cosk_n(z-1)}{p_n} e^{-(\alpha+k_n{}^2)x} H(t-x)$ (27)

where *H (t-x)* is a heavy side function.

IV. RESULTS AND DISCUSSION

The dispersion of pollutants from a continuous point source is studied. It is assumed that the wind velocity is constant.Using the above equation (27), the concentration of pollutants in non-dimensional form is calculated. The analysis of concentration profile is done taking the following meteorological conditions and dispersion parameters:

H=1, h_s=0.2, U=1, $\beta = 10$, $\alpha = 2$, Q=1

For different parametric conditions, the behavior of the non-dimensional concentration profile is shown graphically.

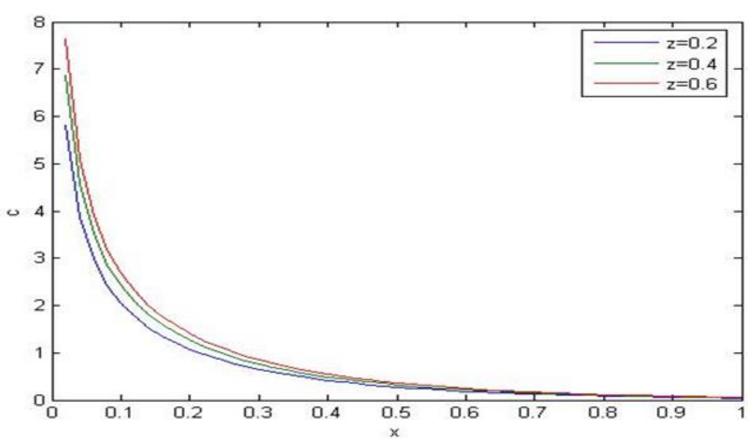

Fig.1-variation of concentration C(x,0,z) with downwind distance x for various values of vertical distance z

The concentration profile with respect to downwind distance (0≤x≤1) is shown in figure 1 for different values of vertical distances (z=0.2, 0.4, 0.6) and crosswind distance y=0. It is seen that the value of c first increases with increasing the value of z or x and becomes maximum at some height and then decreases with increasing the value of z or x and finally tends zero for large value of z or x, which fairly matches with the natural behavior of pollution dispersion.

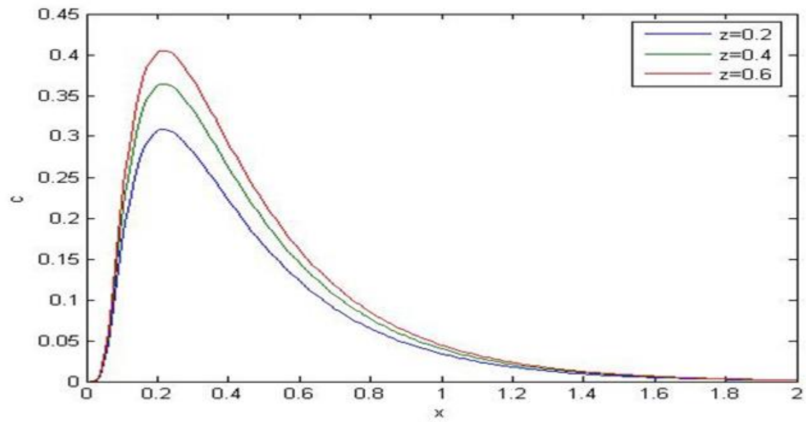

Fig.2-variation of concentration C(x, 1, z) with downwind distance x for various values of vertical distance z

The concentration profile with respect to downwind distance ($0 \le x \le 2$) is shown in figure 2 for different values of vertical distances (z=0.2, 0.4, 0.6) and crosswind distance y=1. It is seen that the value of C first increases with increasing the value of z or x and becomes maximum at some height and then decreases with increasing the value of z or x and finally tends zero for large value of z or x, which fairly matches with the natural behavior of pollution dispersion.

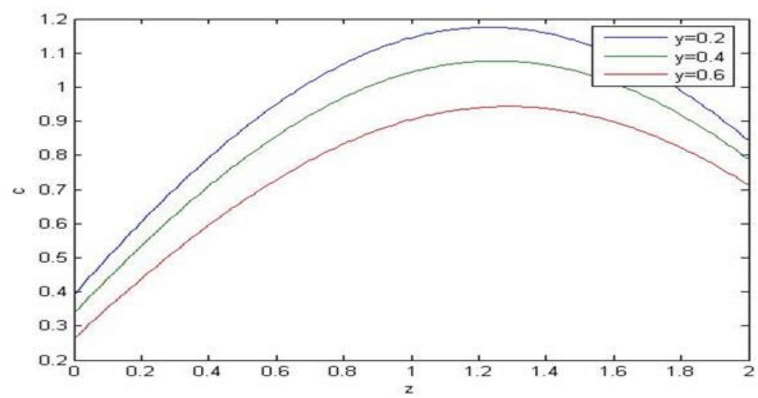

Fig.3 variation of concentration C(0.2,y,z) with vertical distance z for various values of cross-wind distance y

The concentration profile with respect to vertical height is shown in figure 3 for different values of crosswind distances (y= 0.2, 0.4, 0.6) and downwind distance x=0.2 It is seen that the concentration reaches a maximum along the centre line of the plume and then followed by extended spreading. It is also seen that concentration decreases and it is marked lateral spreading with increasing vertical distance.

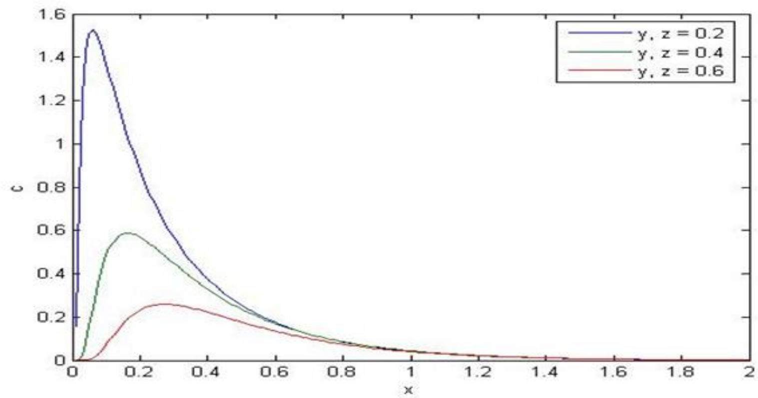

Fig.4-variation of concentration C(x,y,z) with fixed y and z for various values of down-wind distance x

The concentration with respect to downwin distance ($0 \le x \le 1$) is shown in figure 4 for different values of y and z. It is seen that the value of concentration attains maximum at some downwind value and then decreases with increasing the value of x.

V. CONCLUSION

In this model, a solution to an unsteady state dispersion equation from a point source is developed using variable separation and integral transform techniques with suitable boundary conditions. In figures 1 and 2, the concentration profile with respect to downwind distance using crosswind distance *y=0 and y=1* are shown. In both the figures, we have found that the concentration first increases with increasing z or x and becomes maximum at some height and then decreases with increasing z or x and finally tends to zero for large value of z or x. From figure 3, we have found that the concentration against the vertical distance reaches a maximum along the centre line of the plume and then extended spreading. From figure 4, we have obtained that the concentration attains maximum at some downwind value and then decreases with increasing the downwind distance.The model can be used to predict effects of pollutants from the point source and its removal by the natural or artificial sinks present in the atmosphere.

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