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GENERALIZED HYERS-ULAM-RASSIAS TYPE STABILITY OF ADDITIVE TYPE FUNCTIONAL EQUATIONS WITH $2k$-VARIABLE IN (l, β)-NORMED SPACES

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Abstract. *In this paper, we study to solve the Hyers − Ulam − Rassias stability* type *of* the Cauchy *functional equation* and then Jensen *functional equation in non − Archimdean* (l, β)*-normed space.* and that of the pexiderized Cauchy functional equation in (l, β)*-normed space Then I will show that the solutions of equation are additive mapping. These are the main results of this paper.*

Keywords:Hyers-Ulam-Rassias stability, (l,β)-normed space, non-Archimdean (l,β)-normed space, complete non-Archimdean (l,β)-normed space, Cauchy functional equation with 2k-variable, Jensen functional equation with 2k-variable, pexiderized Cauchy functional equation.
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1. Introduction

Let $\mathbf{X}$ and $\mathbf{Y}$ be a normed spaces on the same field $\mathbb{K}$, and $f : \mathbf{X} \to \mathbf{Y}$ be a mapping. We use the notation $\|\cdot\|_{\beta_1}$ $\big(\|\cdot\|_{\beta}\big)$ for corresponding the norms on $\mathbf{X}$ and $\mathbf{Y}$. In this paper, we investigate the stability of the Cauchy *functional equation* and then Jensen *functional equation in Non − Archimdean* (l, β)*-normed space.* In fact, when $\mathbf{X}$ is a non-Archimedean (n,β)-normed space with norm $\|\cdot\|_{\beta_1}$ and that $\mathbf{Y}$ is a Banach non-Archimedean (n,β)-normed space with norm with norm $\|\cdot\|_{\beta}$.
We solve and prove the Hyers-Ulam-Rassias type stability of the *functional equation in non−Archimdean* (l, β)*-normed space*, associated to the Cauchy type additive functional equation and Jensen type additive functional equation with $2k$ variable:

$$f\Big(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\Big) = \sum_{j=1}^{k} f\big(x_j\big) + \sum_{j=1}^{k} f\Big(\frac{x_{k+j}}{k}\Big) \tag{1.1}$$

and

$$2kf\Big(\frac{1}{2k}\sum_{j=1}^{k} x_j + \frac{1}{2k}\sum_{j=1}^{k} \frac{x_{k+j}}{k}\Big) = \sum_{j=1}^{k} f\big(x_j\big) + \sum_{j=1}^{k} f\Big(\frac{x_{k+j}}{k}\Big) \tag{1.2}$$

The study of the functional equation stability originated from a question of S.M. Ulam [24], concerning the stability of group homomorphisms. Let $\big(\mathbf{G}, *\big)$ be a group and let $\big(\mathbf{G}', \circ, d\big)$ be a metric group with metric $d\big(\cdot,\cdot\big)$. Geven $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : \mathbf{G} \to \mathbf{G}'$ satisfies

$$d\big(f\big(x * y\big), f\big(x\big) \circ f\big(y\big)\big) < \delta$$

for all $x, y \in \mathbf{G}$ then there is a homomorphism $h : \mathbf{G} \to \mathbf{G}'$ with

$$d\big(f\big(x\big), h\big(x\big)\big) < \epsilon$$

for all $x \in \mathbf{G}$?, if the answer, is affirmative, we would say that equation of homomophism $h(x * y) = h(y) \circ h(y)$ is stable. The concept of stability for a functional equation arises when we replace functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given function equation?
The Hyers [8] gave firts affirmative partial answer to the equation of *Ulam* in *Banach* spaces.

. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Rassias [17] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [7] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the Cauchy equation.
The functional equation

$$f\Big(\frac{x+y}{2}\Big) = \frac{1}{2}f(x) + \frac{1}{2}f(y)$$

is called the Jensen additive functional equation.

The first work on the stability problem for functional equations in non-Archimedean spaces was started by Moslehian and Rassias [11]. Moslehian and Sadeghi [10] investigated the stability of cubi functional equations in non-Archimedean normed space. Next the mathematicians Xiuzhong Yang, Lidan Chang, Guofen Liu and Guannan Shen stabilty of functional equation in non-Archimedean (n,β)-normed space

concerning to the following Cauchy functional equation and Jensen functional equation

$$f(x+y) = f(x) + f(y)$$

$$2f\Big(\frac{x+y}{2}\Big) = f(x) + f(y)$$

. Recently, in [9, 10, 11, 25] the authors studied the on Hyers-Ulam-Rassias type stability the stability of the $functional\ equation\ in\ non-Archimdean\ (l,\beta)$*-normed space*, associated to the Cauchy type following additive functional equation and Jensen type additive functional equation.

$$f\Big(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\Big) = \sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f\Big(\frac{x_{k+j}}{k}\Big)$$

and

$$2kf\Big(\frac{1}{2k}\sum_{j=1}^{k} x_j + \frac{1}{2k}\sum_{j=1}^{k} \frac{x_{k+j}}{k}\Big) = \sum_{j=1}^{k} f(x_j) + \sum_{j=1}^{k} f\Big(\frac{x_{k+j}}{k}\Big)$$

,

ie the functional equation with $2k$-variables. Under suitable assumptions on spaces $\mathbf{X}$ and $\mathbf{Y}$, we will prove that the mappings satisfying the functional (1.1) and (1.2). Thus, the results in this paper are generalization of those in [10, 11, 25] for functional equation with $2k$-variables.

The paper is organized as followns:

In section preliminarie we remind some basic notations in [10,11,25] such as Banach space, Banach non-Archimdean space, non-Archimdean (l,β)-normed space, Banach non-Archimdean and solutions of the Cauchy function equation and Jensen function equation.

Section 3: is devoted to prove the Hyers-Ulam-Rassias type stability of the Cauchy type additive functional equations in non-Archimdean (l,β)-normed space when $\mathbf{X}$ is a non-Archimdean (l,β)-normed space with norm $\|\cdot\|_{\beta_1}$ and $\mathbf{Y}$ is a complete non-Archimdean (l,β)-normed space with norm $\|\cdot\|_{\beta}$.

Section 4: is devoted to prove the Hyers-Ulam-Rassias type stability of the Jensen type additive functional equations in non-Archimdean (l,β)-normed space when $\mathbf{X}$ is a vector and $\mathbf{Y}$ is a complete non-Archimdean (l,β)-normed space with norm $\|\cdot\|_{\beta}$.

Section 5: is devoted to prove the Hyers-Ulam-Rassias type stability of the pexiderized Cauchy type functional equations in non-Archimdean (l,β)-normed space when $\mathbf{X}$ is a vector and $\mathbf{Y}$ is a complete non-Archimdean (l,β)-normed space with norm $\|\cdot\|_{\beta}$.

2. PRELIMINARIES

2.1. (n,β)-normed spaces.

Definition 2.1.

Let $\{x_n\}$ be a sequence in a normed space $\mathbf{X}$.

(1) A sequence $\{x_n\}_{n=1}^{\infty}$ in a space $\mathbf{X}$ is a Cauchy sequence iff the sequence $\{x_{n+1} - x_n\}_{n=1}^{\infty}$ converges to zero.

(2) The sequence $\{x_n\}_{n=1}^{\infty}$ is said to be convergent if, for any $\epsilon > 0$, there are a positive integer N and $x \in \mathbf{X}$ such that

$$\Big\|x_n - x\Big\| \leq \epsilon . \forall n \geq N,$$

for all n,m $\geq N$. Then the point x$\in \mathbf{X}$ is called the limit of sequence x_n and denote $\lim_{n\to\infty} x_n = x$.

(3) If every sequence Cauchy in $\mathbf{X}$ converges, then the normed space $\mathbf{X}$ is called a Banach space.

Definition 2.2.

Let $\mathbf{X}$ be a linear space over $\mathbb{R}$ with dim$\mathbf{X} \geq n$, $n \in \mathbb{N}$ and $0 < \beta \leq 1$ let $\Big\|\cdot,...,\cdot\Big\| :$ $\mathbf{X^n} \to \mathbb{R}$. be a function satisfying the following properties:

(1) $\Big\|x_1,...,x_n\Big\|_{\beta} = 0$ if and only if $x_1,...,x_n$ are linearly dependent,

(2) $\Big\|x_1,...,x_n\Big\|_{\beta}$ is invariant under permutations of $x_1,...,x_n$

(3) $\Big\|\alpha x_1,...,x_n\Big\|_{\beta} = |\alpha|^{\beta}\Big\|x_1,...,x_n\Big\|$

(4) $\Big\|x_1,...,x_n,y+z\Big\|_{\beta} \leq \Big\|x_1,...,x_n,y\Big\|_{\beta} + \Big\|x_1,...,x_n,z\Big\|_{\beta}, \forall x_1,...,x_n,y,z \in \mathbf{X}$ and $\alpha \in \mathbb{R}$. Then the function $\Big\|\cdot,...,\cdot\Big\|$ is called an (n,β)-norm on $\mathbf{X}$ and the pair $\Big(\mathbf{X},\Big\|\cdot,...,\cdot\Big\|\Big)$ is called a linear (n,β)-normed space or an (n,β)-normed space.

* Note that the concept of a linear (n,β)-normed space is a generalization of a linear n-normed space $(\beta = 1)$ and of a linear n-normed space $(n = 1)$

Definition 2.3.

A sequence $\{x_n\}$ in a linear (n,β)-normed space $\mathbf{X}$ is called a convergent sequence if there is $x \in \mathbf{X}$ such that $\lim_{n\to\infty} \|x_n - x, z_1, z_2, ..., z_{n-1}\|_{\beta} = 0$ for all $z_1, z_2, ..., z_{n-1} \in \mathbf{X}$.

*Note we call that $\{x_n\}$ convergent to xor that x is the limit of $\{x_n\}$, witer $\{x_n\} \to x$ as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

Definition 2.4.

A sequence $\{x_n\}$ in a linear (n, β)-normed space $\mathbf{X}$ is called a Cauchy sequence if $\lim_{n,m\to\infty} \|x_n - x_m, z_1, z_2, ..., z_{n-1}\|_\beta = 0$ for all $z_1, z_2, ..., z_{n-1} \in \mathbf{X}$.

Definition 2.5.

A linear (n, β)-normed space in which every Cauchy sequence is convergent is called a complete (n, β)-normed space

2.2. **The properties of (n, β)-normed spaces.**

Lemma 2.6.

Let $\left(\mathbf{X}, \|\cdot, ..., \cdot\|_\beta\right)$ be a linear (n, β)-normed space, $k \geq 1$, $0 < \beta \leq 1$. If $x_1 \in \mathbf{X}$ and $\left\|x_1, z_1, z_2, ..., z_{n-1}\right\| = 0$ for all $z_1, z_2, ..., z_{n-1} \in \mathbf{X}$, then $x_1 = 0$.

Lemma 2.7.

For a convergent sequence $\{x_n\}$ in a linear (n, β)-normed space $\mathbf{X}$,

$$\lim_{n\to\infty} \|x_n, z_1, z_2, ..., z_{n-1}\|_\beta = \|\lim_{m\to\infty} x_m, z_1, z_2, ..., z_{n-1}\|_\beta = 0$$

for all $z_1, z_2, ..., z_{n-1} \in \mathbf{X}$.

2.3. **non-Archimedean (n, β)-normed spaces.** In this subsection we recall some basic notations fromsuch as non-Archimedean fields, non-Archimedean normed spaces and non-Archimedean Banach spaces.

A valuation is a function $|\cdot|$ from a field $\mathbb{K}$ into $[0, \infty)$ such that 0 is the unique element having the 0 valuation,

$$\left|r\right| = 0 \Leftrightarrow r = 0$$

$$|r \cdot s| := |r|\ |s|, \forall r, s \in \mathbb{K}$$

and the triangle inequality holds, i.e.,

$$|r + s| \leq |r| + |s|, \forall r, s \in \mathbb{K}.$$

A field $\mathbb{K}$ is called a valued field if $\mathbb{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuation. Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the strong triangle inequality is replaced by

$$|r + s| \leq max\{|r|, |s|\}, \forall r, s \in \mathbb{K},$$

then the function $|\cdot|$ is called a non-Archimedean valuation. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1, \forall n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ talking everything except for 0 into 1 and $|0| = 0$. In this paper, we assume that the base field is a non-Archimedean field with $|2| \neq 1$, hence call it simply a field.

.

Definition 2.8. Let be a vecter space over a filed K with a non -Archimedean $\left|\cdot\right|$. A function $\left\|\cdot\right\| : X \to \left[0, \infty\right)$ is said a non -Archimedean norm if it satisfies the follwing conditions:

(1) $\left\|x\right\| = 0$ if and only if $x = 0$;

(2) $\|rx\| = |r|\|x\|(r \in \mathbb{K}, x \in X)$;

(3) $\|x+y\| \leq max\{\|x\|, \|y\|\} x, y \in X$ hold.

Then $(X, \|\cdot\|)$ is called a norm -Archimedean norm space.

Definition 2.9.

A sequence $\{x_n\}$ in a norm -Archimedean (n, β)-normed space $\mathbf{X}$ is a Cauchy sequence if and only if $\{x_n - x_m\} \to 0$.

Definition 2.10. Let $\{x_n\}$ be a sequence in a norm -Archimedean normed space X.

(1) A sequence $\{x_n\}_{n=1}^{\infty}$ in a non -Archimedean space is a Cauchy sequence iff the sequence $\{x_{n+1} - x_n\}_{n=1}^{\infty}$ converges to zero.

(2) The sequence $\{x_n\}$ is said to be convergent if, for any $\epsilon > 0$, there are a positive integer N and $x \in X$ such that

$$\|x_n - x\| \leq \epsilon. \forall n \geq N,$$

for all n,m $\geq N$. The we call x$\in X$ a limit of sequence x_n and denote $\lim_{n\to\infty} x_n = x$.

(3) If every sequence Cauchy in X converger, then the norm -Archimedean normed space X is called a norm -Archimedean Bnanch space.

Definition 2.11.

Let $\mathbf{X}$ be a real space with dim$\mathbf{X} \geq n$ over a scalar filed $\mathbb{K}$ with a non -Archimedean nontrivial valuation $|\cdot|$, , where n is a positive integer and β is a constant with $0 < \beta \leq 1$. A real-valued function let $\|\cdot, ..., \cdot\| : \mathbf{X^n} \to \mathbb{R}$. is called an (n, β)-norn on $\mathbf{X}$ satisfying the following properties:

(1) $\|x_1, ..., x_n\|_\beta = 0$ if and only if $x_1, ..., x_n$ are linearly dependent,

(2) $\|x_1, ..., x_n\|_\beta$ is invariant under permutations of $x_1, ..., x_n$

(3) $\|\alpha x_1, ..., x_n\|_\beta = |\alpha|^\beta \|x_1, ..., x_n\|$

(4) $\|x_0 + x_1, ..., x_n\|_\beta \leq max\{\|x_0, ..., x_n\|_\beta, \|x_1, ..., x_n\|_\beta\}, \forall x_0, x_1, ..., x_n \in \mathbf{X}$ and $\alpha \in \mathbb{K}$. Then the function $\|\cdot, ..., \cdot\|$ is called an (n, β)-norm on $\mathbf{X}$ and the pair $(\mathbf{X}, \|\cdot, ..., \cdot\|)$ is called a non -Archimedean (n, β)-normed space or an (n, β)-normed space.

* Note that the concept of a non -Archimedean (n, β)-normed space is a non -Archimedean n-normed space if $(\beta = 1)$ and a a non -Archimedean β-normed space if n=1 respectively.

2.4. **Solutions of the equation.** The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the Cauchuy equation. In particular, every solution of the Cauchuy equation is said to be an *additive mapping.*

The functional equation $f\big(\frac{x+y}{2}\big) = \frac{1}{2}f\big(x\big) + \frac{1}{2}f\big(y\big)$ called the *Jensen equation.* . In particular, every solution of the Jensen equation is said to be a *Jensen additive mapping.*

Note: n is positive integer and $l \geq 2$.

3. Stability of the Cauchy type functional equation in non-Archimdean (l, β)-normed space

In section, we assumer that $\big|2k\big| \neq 1$. Under this condition we prove the Hyers-Ulam-Rassias type stability of the Cauchy type additive functional equations in non-Archimdean (l, β)-normed space when $\mathbf{X}$ is a non-Archimdean (l, β)-normed space with norm $\big\| \cdot \big\|_{\beta_1}$ and $\mathbf{Y}$ is a complete non-Archimdean (l, β)-normed space with norm $\big\| \cdot \big\|_{\beta}$.
or $\mathbf{X}$ is a vector space and $\mathbf{Y}$ is a complete non-Archimdean (l, β)-normed space with norm $\big\| \cdot \big\|_{\beta}$.

Under this setting, we can show that the mapping satisfying (1.1) is additive. These results are give in the following.

Theorem 3.1.

Suppose That $\mathbf{X}$ is a non-Archimedean β_1-normed space and that $\mathbf{Y}$ is a complete non-Archimedean $(l, \beta)-$normed space, where $l \geq 2$, $0 < \beta, \beta_1 \leq 1$. Let $\epsilon \in [0, \infty)$, $p, q \in \big(0, \infty\big)$ with $l\beta_1(p+q) > \beta$ and let

$$\varphi : \mathbf{Y^{l-1}} \to [0, \infty)$$

be a function. Suppose that a mapping

$$\mathbf{f}\colon \mathbf{X^{2k}} \to \mathbf{Y}$$

satisfying the inequality

$$\Big\|f\Big(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k} \frac{x_{k+j}}{k}\Big) - \sum_{j=1}^{k} f\Big(x_j\Big) - \sum_{j=1}^{k} f\Big(\frac{x_{k+j}}{k}\Big), z_1, z_2, ..., z_{l-1}\Big\|_{\beta} \leq \epsilon\Big(\prod_{j=1}^{k} \Big\|x_j\Big\|_{\beta_1}^{p} \cdot \prod_{j=1}^{k} \Big\|x_{k+j}\Big\|_{\beta_1}^{q}\Big)\varphi\Big(z_1, z_2, ..., z_{l-1}\Big) \tag{3.1}$$

for all $x_j, x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Then there exists a unique additive mapping

$$H : \mathbf{X} \to \mathbf{Y}$$

satisfying

$$\Big\|f\big(x\big) - H\big(x\big), z_1, z_2, ..., z_{l-1}\Big\|_{\beta} \leq \epsilon\Big|\big(2k\big)^{-\beta}\Big|\Big\|x\Big\|_{\beta_1}^{k\big(p+q\big)} \tag{3.2}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$.

Proof. Put $x_j = x, x_{k+j} = kx$ for all $j = 1 \to k$ in (3.1) and dividing both sides by $\left|(2k)^{-\beta}\right|$, we get

$$\left\|\frac{f\big(2kx\big)}{2k} - f\big(x\big), z_1, z_2, ..., z_{l-1}\right\|_\beta \leq \epsilon \cdot k^{kq\beta_1}\left|\big(2k\big)^{-\beta}\right|\left\|x\right\|_{\beta_1}^{k\big(p+q\big)} \varphi\big(z_1, z_2, ..., z_{l-1}\big) \quad (3.3)$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Replacing x by $\big(2k\big)^n x$ in (3.1)

and dividing both sides by $\left|(2k)^{n\beta}\right|$, we get

$$\begin{aligned}
&\left\|\frac{f\big((2k)^{n+1}x\big)}{(2k)^{n+1}} - \frac{f\big((2k)^n x\big)}{(2k)^n}, z_1, z_2, ..., z_{l-1}\right\|_\beta \\
&\leq \epsilon \cdot k^{kq\beta_1}\left|\frac{1}{\big(2k\big)^{n\beta}}\right|\left|\frac{1}{\big(2k\big)^{\beta}}\right|\left|(2k)^{nk\beta_1(p+q)}\right|\left\|x\right\|_{\beta_1}^{k\big(p+q\big)} \varphi\Big(z_1, z_2, ..., z_{l-1}\Big) \\
&= \epsilon \cdot k^{kq\beta_1}\left|\frac{1}{\big(2k\big)^{\beta}}\right|\left|(2k)^{k\beta_1(p+q)-\beta}\right|\left\|x\right\|_{\beta_1}^{k\big(p+q\big)} \varphi\Big(z_1, z_2, ..., z_{l-1}\Big) \quad (3.4)
\end{aligned}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{2k-1} \in \mathbf{Y}$. Since $k(p+q)\beta_1 > \beta$ and $\left|2k\right| \neq 1$, we get

$$\lim_{n\to\infty}\left\|\frac{f\big((2k)^{n+1}x\big)}{(2k)^{n+1}} - \frac{f\big((2k)^n x\big)}{(2k)^n}, z_1, z_2, ..., z_{l-1}\right\|_\beta = 0 \quad (3.5)$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$

. It follows from (3.5) that the sequence $\left\{\frac{f\big((2k)^n x\big)}{(2k)^n}\right\}$ is Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is completes space, the sequence $\left\{\frac{f\big((2k)^n x\big)}{(2k)^n}\right\}$ coverges. So one can define the mapping H: $\mathbf{X} \to \mathbf{Y}$ by

$$H\big(x\big) := \lim_{n\to\infty}\frac{f\big((2k)^n x\big)}{(2k)^n} \quad (3.6)$$

for all $x \in \mathbf{X}$.

It follows from (3.1) and (3.6) and lemma 2.7 that

$$\Big\|H\Big(\sum_{j=1}^{k}x_j+\frac{1}{k}\sum_{j=1}^{k}x_{k+j}\Big)-\sum_{j=1}^{k}H\big(x_j\big)-\sum_{j=1}^{k}H\Big(\frac{x_{k+j}}{k}\Big),z_1,z_2,...,z_{l-1}\Big\|_{\beta}$$

$$=\lim_{n\to\infty}\Big|\Big(2k\Big)^{-n\beta}\Big|\Big\|f\Big[\Big(2k\Big)^{n}\Big(\sum_{j=1}^{n}x_j+\frac{1}{k}\sum_{j=1}^{n}x_{k+j}\Big)\Big]-\sum_{j=1}^{k}f\Big(\Big(2k\Big)^{n}x_j\Big)$$

$$-\sum_{j=1}^{k}f\Big(\Big(2k\Big)^{n}\frac{x_{k+j}}{k}\Big),z_1,z_2,...,z_{l-1}\Big\|_{\beta}$$

$$\le\lim_{n\to\infty}\theta\Big|\Big(2k\Big)^{-n\beta}\Big|\Big(\prod_{j=1}^{k}\Big\|\Big(2k\Big)^{n}x_j\Big\|_{\beta_1}^{p}\cdot\prod_{j=1}^{k}\Big\|\Big(2k\Big)^{n}x_{k+j}\Big\|_{\beta_1}^{q}\Big)\varphi\big(z_1,z_2,...,z_{l-1}\big)$$

$$=\lim_{n\to\infty}\theta\Big|\Big(2k\Big)^{k\beta_1(p+q)-\beta}\Big|^{n}\Big(\prod_{j=1}^{k}\Big\|\Big(2k\Big)^{n}x_j\Big\|_{\beta_1}^{p}\cdot\prod_{j=1}^{k}\Big\|\Big(2k\Big)^{n}x_{k+j}\Big\|_{\beta_1}^{q}\Big)\varphi\big(z_1,z_2,...,z_{l-1}\big)$$

and so for all $x_j,\,x_{k+j}\in\mathbf{X}$ for all $j=1\to k$. and $z_1,z_2,...,z_{2k-1}\in\mathbf{Y}$. Since $k\beta_1(p+q)>\beta$ and $\big|2k\big|\neq1$, we get

$$\Big\|H\Big(\sum_{j=1}^{k}x_j+\frac{1}{k}\sum_{j=1}^{k}x_{k+j}\Big)-\sum_{j=1}^{k}H\Big(x_j\Big)-\sum_{j=1}^{k}H\Big(\frac{x_{k+j}}{k}\Big),z_1,z_2,...,z_{l-1}\Big\|=0$$

for all $x_j,\,x_{k+j}\in\mathbf{X}$ for all $j=1\to k$. and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. By lemma 2.6, we get

$$H\Big(\sum_{j=1}^{k}x_j+\frac{1}{k}\sum_{j=1}^{k}x_{k+j}\Big)-\sum_{j=1}^{k}H\Big(x_j\Big)-\sum_{j=1}^{k}H\Big(\frac{x_{k+j}}{k}\Big)=0$$

for all $x_j,\,x_{k+j}\in\mathbf{X}$ for all $j=1\to k$. So mapping H is additive. replace x by 2kx in (3.3) and dividing both sides by $\Big|(2k)^{\beta}\Big|$, we get

$$\Big\|\frac{f\big((2k)^2x\big)}{(2k)^2}-\frac{f\big(2kx\big)}{2k},z_1,z_2,...,z_{l-1}\Big\|_{\beta}$$

$$\le\epsilon\cdot k^{kq\beta_1}\Big|\Big(2k\Big)^{-2\beta}\Big|\Big\|2kx\Big\|_{\beta_1}^{k\big(p+q\big)}\varphi\big(z_1,z_2,...,z_{l-1}\big)\tag{3.7}$$

and keep replacing x by 2kx in (3.7)and dividing both sides by $\Big|(2k)^{\beta}\Big|$, we get

$$\Big\|\frac{f\big((2k)^3x\big)}{(2k)^3}-\frac{f\big((2k)^2x\big)}{(2k)^2},z_1,z_2,...,z_{l-1}\Big\|_{\beta}$$

$$\le\epsilon\cdot k^{kq\beta_1}\Big|\Big(2k\Big)^{-3\beta}\Big|\Big\|(2k)^2x\Big\|_{\beta_1}^{k\big(p+q\big)}\varphi\big(z_1,z_2,...,z_{l-1}\big)\tag{3.8}$$

...
and so on until

$$\Big\|\frac{f\big((2k)^{n+1}x\big)}{(2k)^{n+1}}-\frac{f\big((2k)^nx\big)}{(2k)^n},z_1,z_2,...,z_{l-1}\Big\|_{\beta}$$

$$\le\epsilon\cdot k^{kq\beta_1}\Big|\frac{1}{\Big(2k\Big)^{n\beta}}\Big|\Big\|(2k)^nx\Big\|_{\beta_1}^{k\big(p+q\big)}\varphi\Big(z_1,z_2,...,z_{l-1}\Big)\tag{3.9}$$

Thus by (3.7), (3.8) and (3.9) We get

$$\begin{aligned}
&\left\|f\big(x\big)-\frac{f\big((2k)^{n+1}x\big)}{(2k)^{n+1}},z_1,z_2,...,z_{l-1}\right\|_\beta \\
&\leq max\Bigg\{\left\|\frac{f\big((2k)x\big)}{2k}-f\big(x\big),z_1,z_2,...,z_{l-1}\right\|_\beta, \\
&\left\|\frac{f\big((2k)^2x\big)}{(2k)^2}-\frac{f\big(2kx\big)}{2k},z_1,z_2,...,z_{l-1}\right\|_\beta \\
&,...,\left\|\frac{f\big((2k)^{n+1}x\big)}{(2k)^{n+1}}-\frac{f\big((2k)^nx\big)}{(2k)^n},z_1,z_2,...,z_{l-1}\right\|_\beta\Bigg\} \\
&\leq max\Bigg\{\epsilon\cdot k^{kq\beta_1}\left|\big(2k\big)^{-\beta}\right|\left\|x\right\|_{\beta_1}^{k\left(p+q\right)}\varphi\big(z_1,z_2,...,z_{l-1}\big) \\
&,\epsilon\cdot k^{kq\beta_1}\left|\frac{1}{\big(2k\big)^{2\beta}}\right|\left\|(2k)^2x\right\|_{\beta_1}^{k\left(p+q\right)}\varphi\Big(z_1,z_2,...,z_{2k-1}\Big) \\
&,...,\epsilon\cdot k^{kq\beta_1}\left|\frac{1}{\big(2k\big)^{n\beta}}\right|\left\|(2k)^nx\right\|_{\beta_1}^{k\left(p+q\right)}\varphi\Big(z_1,z_2,...,z_{l-1}\Big)\Bigg\}
\end{aligned}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Since $k(p+q)\beta_1>\beta$ and $\left|2k\right|\neq 1$, we get

$$\left\|f\big(x\big)-\frac{f\big(2kx\big)}{2k},z_1,z_2,...,z_{2k-1}\right\|_\beta\leq\epsilon\cdot k^{kq\beta_1}\left|\big(2k\big)^{-\beta}\right|\left\|x\right\|_{\beta_1}^{k\left(p+q\right)}\varphi\big(z_1,z_2,...,z_{l-1}\big)\quad(3.10)$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$.

By induction on, n we can conclude that

$$\left\|f\big(x\big)-\frac{f\big((2k)^nx\big)}{(2k)^n},z_1,z_2,...,z_{l-1}\right\|_\beta\leq\epsilon\cdot k^{kq\beta_1}\left|\big(2k\big)^{-\beta}\right|\left\|x\right\|_{\beta_1}^{k\left(p+q\right)}\varphi\big(z_1,z_2,...,z_{l-1}\big)\quad(3.11)$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$

for all $n\in\mathbb{N}$, $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Replacing x with 2kx in (d) and dividing both sides by $\left|(2k)^\beta\right|$, we get

$$\begin{aligned}
&\left\|\frac{f\big(2kx\big)}{2k}-\frac{f\big((2k)^{n+1}x\big)}{(2k)^{n+1}},z_1,z_2,...,z_{l-1}\right\|_\beta \\
&\qquad\leq\epsilon\cdot k^{kq\beta_1}\left|\big(2k\big)^{-2\beta}\right|\left\|(2k)x\right\|_{\beta_1}^{k\left(p+q\right)}\varphi\big(z_1,z_2,...,z_{l-1}\big)\qquad(3.12)
\end{aligned}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$ and $n \in \mathbb{N}$. It follows from (3.3) and (3.12) that

$$\begin{aligned}&\left\|f\big(x\big) - \frac{f\big((2k)^{n+1}x\big)}{(2k)^{n+1}}, z_1, z_2, ..., z_{l-1}\right\|_\beta \\ &\qquad \le \epsilon \cdot k^{kq\beta_1}\left|\big(2k\big)^{-\beta}\right|\left\|(2k)x\right\|_{\beta_1}^{k\big(p+q\big)}\varphi\big(z_1, z_2, ..., z_{l-1}\big) \qquad (3.13)\end{aligned}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$ and $n \in \mathbb{N}$. This is completes the proof of (3.13). Taking the limit as $n \to \infty$ in (3.13) we can obtain (3.2) Now we prove the uniqueness of H. Assume that $H_1 : \mathbf{X} \to \mathbf{Y}$ is an additive mapping satisfing (3.2). Then we have

$$\begin{aligned}&\left\|H\big(x\big) - H_1\big(x\big), z_1, z_2, ..., z_{l-1}\right\|_\beta \\ &= \left|(2k)^{-n\beta}\right|\left\|H\big((2k)^n x\big) - H_1\big((2k)^n x\big), z_1, z_2, ..., z_{l-1}\right\|_\beta \\ &\le \left|(2k)^{-n\beta}\right| max\left\{\left\|H\big((2k)^n x\big) - f\big((2k)^n x\big), z_1, z_2, ..., z_{l-1}\right\|_\beta,\right. \\ &\left.\left\|f((2k)^n x\big) - H_1\big((2k)^n x\big), z_1, z_2, ..., z_{l-1}\right\|_\beta\right\} \\ &\le \epsilon \cdot k^{kq\beta_1}\left|(2k)^{-n\beta}\right|\left|\big(2k\big)^{-\beta}\right|\left\|(2k)^n x\right\|_{\beta_1}^{k\big(p+q\big)}\varphi\big(z_1, z_2, ..., z_{l-1}\big) \\ &= \epsilon \cdot k^{kq\beta_1}\left|(2k)^{k(p+q)\beta_1-\beta}\right|^n\left|\big(2k\big)^{-\beta}\right|\left\|x\right\|_{\beta_1}^{k\big(p+q\big)}\varphi\big(z_1, z_2, ..., z_{l-1}\big) \qquad (3.14)\end{aligned}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Taking the limit as $n \to \infty$, we have

$$\left\|H\big(x\big) - H_1\big(x\big), z_1, z_2, ..., z_{l-1}\right\|_\beta = 0$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. By lemma 2.6, we get $H\big(x\big) = H_1\big(x\big)$ for all $x \in \mathbf{X}$. So H is the unique additive mapping satisfying (3.2) □

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Theorem 3.2.

Suppose That $\mathbf{X}$ be a vector space and that $\mathbf{Y}$ is a complete non-Archimedean $(l, \beta)-$normed space, where $l \ge 2$, $0 < \beta \le 1$. Let

$$\varphi : \mathbf{X^{2k}} \to [0, \infty)$$

be a function such that

$$\lim_{n\to\infty}\left|(2k)^{n\beta}\right|\varphi\Big(\frac{(2k)^n x_1}{(2k)^n}, (2k)^n x_2, ..., (2k)^n x_k, (2k)^n k x_{k+1}, (2k)^n k x_{k+2}, ..., (2k)^n k x_{2k}\Big) = 0 \qquad (3.15)$$

for all $x_j, x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$, and

suppose that a mapping

$$\psi \colon \mathbf{Y^{l-1}} \to [0, \infty)$$

be a function. The limit

$$\lim_{n\to\infty} max\left\{\left|\frac{1}{(2k)^{i\beta}}\right|\varphi\Big((2k)^{i-1}x_1,(2k)^{i-1}x_2,...,(2k)^{i-1}x_{2k}\Big), 1\le i\le n\right\} \tag{3.16}$$

exists for $x\in\mathbf{X}$, and it is denoted by $\widetilde{\varphi}(x)$. Suppose that a mapping

$$f:\mathbf{X}\to\mathbf{Y}$$

satisfying the inequality

$$\left\|f\left(\sum_{j=1}^{k}x_j+\sum_{j=1}^{k}\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\Big(x_j\Big)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right),z_1,z_2,...,z_{l-1}\right\|_\beta$$
$$\le\varphi\Big(x_1,x_2,...,x_k,x_{k+1},x_{k+2},...,x_{2k}\Big)\cdot\psi\Big(z_1,z_2,...,z_{l-1}\Big) \tag{3.17}$$

for all $x_j,x_{k+j}\in\mathbf{X}$ for all $j=1\to k$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Then there exists a unique additive mapping

$$H:\mathbf{X}\to\mathbf{Y}$$

satisfying

$$\left\|f\Big(x\Big)-H\Big(x\Big),z_1,z_2,...,z_{l-1}\right\|_\beta\le\widetilde{\varphi}(x)\varphi\Big(z_1,z_2,...,z_{l-1}\Big) \tag{3.18}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Moreover, if

$$\lim_{h\to\infty}\lim_{n\to\infty} max\left\{\left|\frac{1}{(2k)^{i\beta}}\right|\varphi\Big((2k)^{i-1}x_1,(2k)^{i-1}x_2,...,(2k)^{i-1}x_k,\right.$$
$$\left.(2k)^{i-1}kx_{k+1},(2k)^{i-1}kx_{k+2},...,(2k)^{i-1}kx_{2k}\Big),1+h\le i\le n+h\right\}=0 \tag{3.19}$$

for all $x\in\mathbf{X}$, then H is a unique additive mapping satisfying (3.18).

Proof. Put $x_j=x,x_{k+j}=kx$ for all $j=1\to k$ in (3.17) and dividing both sides by $\Big|(2k)^\beta|$, we get

$$\left\|\frac{f\big(2kx\big)}{2k}-f\Big(x\Big),z_1,z_2,...,z_{l-1}\right\|_\beta$$
$$\le\left|\Big(2k\Big)^{-\beta}\right|\varphi\Big(x,x,...,x,kx,kx,...,kx\Big)\psi\Big(z_1,z_2,...,z_{l-1}\Big) \tag{3.20}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Replacing x by $\Big(2k\Big)^ix$ in (3.20)

and dividing both sides by $\left|(2k)^{i\beta}\right|$, we get

$$\left\|\frac{f\big((2k)^{i+1}x\big)}{(2k)^{i+1}}-\frac{f\big((2k)^{i}x\big)}{(2k)^{i}},z_1,z_2,...,z_{2k-1}\right\|_\beta$$
$$\le\left|\Big(2k\Big)^{-\beta}\right|\left|\Big(2k\Big)^{-i\beta}\right|\varphi\Big(\Big(2k\Big)^ix,\Big(2k\Big)^ix,...,\Big(2k\Big)^ix,$$
$$\Big(2k\Big)^ikx,\Big(2k\Big)^ikx,...,\Big(2k\Big)^ikx\Big)\psi\Big(z_1,z_2,...,z_{2k-1}\Big) \tag{3.21}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Taking the limit as $i \to \infty$ and considering (3.15)

$$\lim_{i\to\infty}\left\|\frac{f\Big((2k)^{i+1}x\Big)}{(2k)^{i+1}}-\frac{f\Big((2k)^{i}x\Big)}{(2k)^{i}}, z_1, z_2, ..., z_{l-1}\right\|_\beta = 0 \tag{3.22}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$

. It follows from (3.22) that the sequence $\left\{\frac{f\left((2k)^n x\right)}{(2k)^n}\right\}$ is Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is completes space, the sequence $\left\{\frac{f\left((2k)^n x\right)}{(2k)^n}\right\}$ coverges. So one can define the mapping $H\colon \mathbf{X} \to \mathbf{Y}$ by

$$H\big(x\big) := \lim_{n\to\infty}\frac{f\Big((2k)^n x\Big)}{(2k)^n} \tag{3.23}$$

for all $x \in \mathbf{X}$.

It follows from (3.17), (3.23) and lemma 2.6 that

$$\left\|H\left(\sum_{j=1}^{k}x_j+\frac{1}{k}\sum_{j=1}^{k}x_{k+j}\right)-\sum_{j=1}^{k}H\Big(x_j\Big)-\sum_{j=1}^{k}H\left(\frac{x_{k+j}}{k}\right), z_1, z_2, ..., z_{l-1}\right\|_\beta$$
$$= \lim_{n\to\infty}\left|\Big(2k\Big)^{-n\beta}\right|\left\|f\left[\Big(2k\Big)^n\left(\sum_{j=1}^{n}x_j+\frac{1}{k}\sum_{j=1}^{n}x_{k+j}\right)\right]-\sum_{j=1}^{k}f\left(\Big(2k\Big)^n x_j\right)\right.$$
$$\left.-\sum_{j=1}^{k}f\left(\Big(2k\Big)^n\frac{x_{k+j}}{k}\right), z_1, z_2, ..., z_{l-1}\right\|_\beta$$
$$\le \lim_{n\to\infty}\left\|\Big(2k\Big)^{-n\beta}\right|\varphi\Big(\Big(2k\Big)^n x, \Big(2k\Big)^n x, ..., \Big(2k\Big)^n x,$$
$$\Big(2k\Big)^n kx, \Big(2k\Big)^n kx, ..., \Big(2k\Big)^n kx\Big)\psi\big(z_1, z_2, ..., z_{2k-1}\big)$$

and so for all x_j, $x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$.we get

$$\left\|H\left(\sum_{j=1}^{k}x_j+\frac{1}{k}\sum_{j=1}^{k}x_{k+j}\right)-\sum_{j=1}^{k}H\Big(x_j\Big)-\sum_{j=1}^{k}H\left(\frac{x_{k+j}}{k}\right), z_1, z_2, ..., z_{l-1}\right\| = 0$$

for all x_j, $x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. By lemma 2.6, we get

$$H\left(\sum_{j=1}^{k}x_j+\frac{1}{k}\sum_{j=1}^{k}x_{k+j}\right)-\sum_{j=1}^{k}H\Big(x_j\Big)-\sum_{j=1}^{k}H\left(\frac{x_{k+j}}{k}\right) = 0$$

for all x_j, $x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. So mapping H is additive.

Replace x by 2kx in (3.20) and dividing both sides by $\left|(2k)^\beta\right|$, we get

$$\left\|\frac{f\Big((2k)^2 x\Big)}{(2k)^2}-\frac{f\big(2kx\big)}{2k}, z_1, z_2, ..., z_{l-1}\right\|_\beta$$
$$\le \left\|\Big(2k\Big)^{-2\beta}\right|\varphi\Big(\Big(2k\Big)^n x, \Big(2k\Big)^n x, ..., \Big(2k\Big)^n x,$$
$$\Big(2k\Big)^n kx, \Big(2k\Big)^n kx, ..., \Big(2k\Big)^n kx\Big)\psi\big(z_1, z_2, ..., z_{l-1}\big) \tag{3.24}$$

for all $x \in \mathbf{X}$, $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Considering (3.20), we get

$$\begin{aligned}&\left\|f\big(x\big)-\frac{f\big((2k)^2x\big)}{(2k)^2},z_1,z_2,...,z_{l-1}\right\|_\beta\\&\leq max\Bigg\{\Big|\Big|(2k)^{-\beta}\Big|\varphi\big(x,x,...,x,kx,kx,...,kx\big)\Big|(2k)^{-2\beta}\Big|\varphi\big((2k)x,(2k)x,...,(2k)x,\\&(2k)kx,(2k)kx,...,(2k)kx\big)\Bigg\}\psi\big(z_1,z_2,...,z_{l-1}\big)\end{aligned} \tag{3.25}$$

for all $x \in \mathbf{X}$, $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. By induction on n, we get

$$\begin{aligned}&\left\|f\big(x\big)-\frac{f\big((2k)^nx\big)}{(2k)^n},z_1,z_2,...,z_{l-1}\right\|_\beta\\&\leq max\Bigg\{\frac{\varphi\big((2k)^{h-1}x,(2k)^{h-1}x,...,(2k)^{h-1}x,(2k)^{h-1}kx,(2k)^{h-1}kx,...,(2k)^{h-1}kx\big)}{\Big|(2k)^{h\beta}\Big|},\\&1\leq h\leq n\Bigg\}\psi\big(z_1,z_2,...,z_{l-1}\big)\end{aligned} \tag{3.26}$$

replacing x by 2kx in (3.26)and dividing both sides by $\Big|(2k)^\beta\Big|$, we get

$$\begin{aligned}&\left\|\frac{f\big((2k)x\big)}{(2k)}-\frac{f\big((2k)^{n+1}x\big)}{(2k)^{n+1}},z_1,z_2,...,z_{l-1}\right\|_\beta\\&\leq max\Bigg\{\frac{\varphi\big((2k)^hx,(2k)^hx,...,(2k)^hx,(2k)^hkx,(2k)^hkx,...,(2k)^hkx\big)}{\Big|(2k)^{(h+1)\beta}\Big|},\\&1\leq h\leq n\Bigg\}\psi\big(z_1,z_2,...,z_{l-1}\big)\end{aligned} \tag{3.27}$$

for all $x \in \mathbf{X}$, $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$ and $n \in \mathbb{N}$, which together with (3.20) implies .

$$\left\|f\big((2k)x\big)-\frac{f\big((2k)^{n+1}x\big)}{(2k)^{n+1}},z_1,z_2,...,z_{l-1}\right\|_\beta$$

$$\leq max\left\{\frac{\varphi\big(x,x,...,x,kx,kx,...,kx\big)}{\left|(2k)^\beta\right|},\right.$$

$$\frac{\varphi\big((2k)^hx,(2k)^hx,...,(2k)^hx,(2k)^hkx,(2k)^hkx,...,(2k)^hkx\big)}{\left|(2k)^{(h+1)\beta}\right|}$$

$$\left.,1\leq h\leq n\right\}\psi\big(z_1,z_2,...,z_{l-1}\big)$$

$$=max\left\{\frac{\varphi\big((2k)^hx,(2k)^hx,...,(2k)^hx,(2k)^hkx,(2k)^hkx,...,(2k)^hkx\big)}{\left|(2k)^{(h+1)\beta}\right|}\right.$$

$$\left.,1\leq h\leq n\right\}\psi\big(z_1,z_2,...,z_{l-1}\big)$$

$$=max\left\{\frac{\varphi\big((2k)^hx,(2k)^hx,...,(2k)^hx,(2k)^hkx,(2k)^hkx,...,(2k)^hkx\big)}{\left|(2k)^{h\beta}\right|}\right.$$

$$\left.,1\leq h\leq n+1\right\}\psi\big(z_1,z_2,...,z_{l-1}\big) \tag{3.28}$$

for all $x\in\mathbf{X}$, $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$ and $n\in\mathbb{N}$. This is completes the proof of (3.26) Taking the limit as $n\to\infty$ in (3.26). Now we need to prove the uniqueness of H. Let H' be another additive mapping satisfying (3.18). Sence

$$\lim_{h\to\infty}\left|\frac{1}{(2k)^{h\beta}}\right|\widetilde{\varphi}((2k)^hx)=\lim_{h\to\infty}\lim_{n\to\infty}max\left\{\left|\frac{1}{(2k)^{i\beta}}\right|\varphi\big((2k)^{i+h-1}x_1,(2k)^{i+h-1}x_2,...,(2k)^{h+i-1}x_k,\right.$$

$$\left.(2k)^{h+i-1}kx_{k+1},(2k)^{h+i-1}kx_{k+2},...,(2k)^{h+i-1}kx_{2k}\big),1\leq i\leq n\right\}$$

$$=\lim_{h\to\infty}\lim_{n\to\infty}max\left\{\left|\frac{1}{(2k)^{i\beta}}\right|\varphi\big((2k)^{i-1}x_1,(2k)^{i-1}x_2,...,(2k)^{i-1}x_k,\right.$$

$$\left.(2k)^{i-1}kx_{k+1},(2k)^{i-1}kx_{k+2},...,(2k)^{i-1}kx_{2k}\big),1+h\leq i\leq n+h\right\} \tag{3.29}$$

for all $x\in\mathbf{X}$, $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$, it follows from then H is a unique additive mapping satisfying (3.19) that.

$$\begin{aligned}
&\left\|H\big(x\big)-H'\big(x\big),z_1,z_2,...,z_{l-1}\right\|_\beta\\
&\quad=\lim_{h\to\infty}\left|\frac{1}{(2k)^{h\beta}}\right|\left\|H\big((2k)^hx\big)-H'\big((2k)^hx\big),z_1,z_2,...,z_{l-1}\right\|_\beta\\
&\quad\le\lim_{h\to\infty}\left|\frac{1}{(2k)^{h\beta}}\right|max\Bigg\{\left\|H\big((2k)^hx\big)-f\big((2k)^hx\big),z_1,z_2,...,z_{l-1}\right\|_\beta,\\
&\quad\left\|f\big((2k)^hx\big)-H'\big((2k)^hx\big),z_1,z_2,...,z_{l-1}\right\|_\beta\\
&\quad\le\lim_{h\to\infty}\left|\frac{1}{(2k)^{h\beta}}\right|\widetilde{\varphi}((2k)^{-h\beta}x)\psi\big(z_1,z_2,...,z_{l-1}\big)=0 \qquad (3.30)
\end{aligned}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Considering lemma 2.6 we prove that H is unique □

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4. Stability of the Jensen type functional equation in non-Archimdean (l,β)-normed space

In section, we assumer that $\big|2\big|\neq 1$. Under this condition we prove the Hyers-Ulam-Rassias type stability of the Jensen type additive functional equations in non-Archimdean (l,β)-normed space when $\mathbf{X}$ is a vector and $\mathbf{Y}$ is a complete non-Archimdean (l,β)-normed space with norm $\|\cdot\|_\beta$.
Under this setting, we can show that the mapping satisfying (1.2) is Jensen additive. These results are give in the following.

Theorem 4.1.

Suppose That $\mathbf{X}$ be a vector space and that $\mathbf{Y}$ is a complete non-Archimedean $(l,\beta)-$normed space, where $l\geq 2$, $0<\beta\leq 1$. Let

$$\varphi:\mathbf{X^{2k}}\to[0,\infty)$$

be a function such that

$$\lim_{n\to\infty}\left|2^{n\beta}\right|\varphi\left(\frac{x_1}{2^n},\frac{x_2}{2^n},...,\frac{x_k}{2^n},\frac{x_{k+1}}{2^n},\frac{x_{k+2}}{2^n},...,\frac{x_{2k}}{2^n}\right)=0 \qquad (4.1)$$

for all $x_j,x_{k+j}\in\mathbf{X}$ for all $j=1\to k$, and

suppose that a mapping

$$\psi\colon\mathbf{Y^{l-1}}\to[0,\infty)$$

be a function. The limit

$$\lim_{n\to\infty}max\left\{\left|2^{i\beta}\right|\varphi\big(\frac{x_1}{2^i},\frac{x_2}{2^i},...,\frac{x_k}{2^i},0,0,...,0\big),1\leq i\leq n-1\right\} \qquad (4.2)$$

exists for $x\in\mathbf{X}$, which is denoted by $\widetilde{\varphi}(x)$. Suppose that a mapping

$$f:\mathbf{X}\to\mathbf{Y}$$

and $f\left(0\right)=0$ satisfying the inequality

$$\left\|2kf\left(\frac{1}{2k}\sum_{j=1}^{k}x_j+\frac{1}{2k}\sum_{j=1}^{k}\frac{x_{k+j}}{k}\right)-\sum_{j=1}^{k}f\left(x_j\right)-\sum_{j=1}^{k}f\left(\frac{x_{k+j}}{k}\right),z_1,z_2,...,z_{l-1}\right\|_{\beta}$$
$$\leq\varphi\left(x_1,x_2,...,x_k,x_{k+1},x_{k+2},...,x_{2k}\right)\cdot\psi\left(z_1,z_2,...,z_{l-1}\right) \tag{4.3}$$

for all $x_j, x_{k+j}\in\mathbf{X}$ for all $j=1\rightarrow k$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Then there exists a unique additive mapping

$$H:\mathbf{X}\rightarrow\mathbf{Y}$$

satisfying

$$\left\|f\left(x\right)-H\left(x\right),z_1,z_2,...,z_{l-1}\right\|_{\beta}\leq\widetilde{\varphi}(x)\varphi\left(z_1,z_2,...,z_{2k-1}\right) \tag{4.4}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Moreover, if

$$\lim_{h\to\infty}\lim_{n\to\infty}max\left\{\left|2^{i\beta}\right|\varphi\big(\frac{x_1}{2^i},\frac{x_2}{2^i},...,\frac{x_k}{2^i},0,0,...,0\big),h\leq i\leq n+h-1\right\}=0 \tag{4.5}$$

for all $x\in\mathbf{X}$, then H is a unique additive mapping satisfying (4.4).

Proof. Put $x_j=x, x_{k+j}=0$ for all $j=1\rightarrow k$ in (4.3)we get

$$\left\|2f\left(\frac{x}{2}\right)-f\left(x\right),z_1,z_2,...,z_{l-1}\right\|_{\beta}$$
$$\leq\left|k^{-\beta}\right|\varphi\left(x,x,...,x,0,0,...,0\right)\psi\left(z_1,z_2,...,z_{l-1}\right) \tag{4.6}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Replacing x by $\frac{x}{2^n}$ in (4.6)

and multiplying both sides by $\left|2^{n\beta}\right|$, we get

$$\left\|2^{n+1}f\left(\frac{x}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right),z_1,z_2,...,z_{l-1}\right\|_{\beta}$$
$$\leq\left|k^{-\beta}\right|\left|2^{n\beta}\right|\varphi\big(\frac{x_1}{2^n},\frac{x_2}{2^n},...,\frac{x_k}{2^n},0,0,...,0\big)\psi\left(z_1,z_2,...,z_{l-1}\right) \tag{4.7}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Taking the limit as $i\rightarrow\infty$ and considering (4.1)

$$\lim_{n\to\infty}\left\|2^{n+1}f\left(\frac{x}{2^{n+1}}\right)-2^nf\left(\frac{x}{2^n}\right),z_1,z_2,...,z_{l-1}\right\|_{\beta}=0 \tag{4.8}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$

. It follows from (3.22) that the sequence $\left\{2^nf\left(\frac{x}{2^n}\right)\right\}$ is Cauchy sequence for all $x\in\mathbf{X}$. Since $\mathbf{Y}$ is completes space, the sequence $\left\{2^nf\left(\frac{x}{2^n}\right)\right\}$ coverges. So one can define the mapping H: $\mathbf{X}\rightarrow\mathbf{Y}$ by

$$H\left(x\right):=\lim_{n\to\infty}2^nf\left(\frac{x}{2^n}\right) \tag{4.9}$$

for all $x\in\mathbf{X}$.

By inductionon n, we have

$$\left\|2^n f\left(\frac{x}{2^n}\right) - f(x), z_1, z_2, ..., z_{l-1}\right\|_\beta \leq max\left\{\left|2^{h\beta}\right|\varphi\left(\frac{x_1}{2^h}, \frac{x_2}{2^h}, ..., \frac{x_k}{2^h}, 0, 0, ..., 0\right), 1 \leq h \leq n-1\right\}\psi\left(z_1, z_2, ..., z_{l-1}\right) \tag{4.10}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$ and $n \in \mathbb{N}$. Replacing x by $\frac{x}{2^n}$ in (4.10)

and multiplying both sides by $\left|2^\beta\right|$, we get

$$\left\|2^{n+1} f\left(\frac{x}{2^{n+1}}\right) - 2f\left(\frac{x}{2}\right), z_1, z_2, ..., z_{l-1}\right\|_\beta \leq max\left\{\left|2^{(h+1)\beta}\right|\varphi\left(\frac{x_1}{2^{h+1}}, \frac{x_2}{2^{h+1}}, ..., \frac{x_k}{2^{h+1}}, 0, 0, ..., 0\right), 1 \leq h \leq n-1\right\}\psi\left(z_1, z_2, ..., z_{l-1}\right) \tag{4.11}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$ and $n \in \mathbb{N}$. Considering the above inequality and (4.6) we have

$$\begin{aligned} &\left\|2^{n+1} f\left(\frac{x}{2^{n+1}}\right) - 2f\left(\frac{x}{2}\right), z_1, z_2, ..., z_{l-1}\right\|_\beta \\ &\leq max\left\{\varphi\left(x_1, x_2, ..., x_k, 0, 0, ..., 0\right),\right. \\ &\left.\left|2^{(h+1)\beta}\right|\varphi\left(\frac{x_1}{2^{h+1}}, \frac{x_2}{2^{h+1}}, ..., \frac{x_k}{2^{h+1}}, 0, 0, ..., 0\right), 1 \leq h \leq n-1\right\}\psi\left(z_1, z_2, ..., z_{l-1}\right) \\ &= max\left\{\left|2^{h\beta}\right|\varphi\left(\frac{x_1}{2^h}, \frac{x_2}{2^h}, ..., \frac{x_k}{2^h}, 0, 0, ..., 0\right), 1 \leq h \leq n-1\right\} \end{aligned} \tag{4.12}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$ and $n \in \mathbb{N}$. This completes the proof of (4.6) Taking the limit as $n \to \infty$ in (4.10), we obtain (4.4)

Next, we prve that H is additive. Considering (4.1), (4.3) and (4.9)

$$\begin{aligned} &\left\|2kH\left(\frac{1}{2k}\sum_{j=1}^{k} x_j + \frac{1}{2k}\sum_{j=1}^{k}\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} H\left(x_j\right) - \sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right), z_1, z_2, ..., z_{l-1}\right\|_\beta \\ &= \lim_{n\to\infty}\left|2^{n\beta}\right|\left\|2kf\left[\left(\frac{1}{k}\sum_{j=1}^{n}\frac{x_j}{2^{n+1}} + \frac{1}{k^2}\sum_{j=1}^{n}\frac{x_{k+j}}{2^{n+1}}\right)\right] - \sum_{j=1}^{k} f\left(\frac{x_j}{2^n}\right) - \sum_{j=1}^{k} f\left(\frac{x_{k+j}}{2^n k}\right), z_1, z_2, ...\right. \\ &\left., z_{l-1}\right\|_\beta \\ &\leq \lim_{n\to\infty}\left|2^{n\beta}\right|\varphi\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}, ..., \frac{x_k}{2^n}, 0, 0, ..., 0\right)\psi\left(z_1, z_2, ..., z_{l-1}\right) \end{aligned}$$

and so for all $x_j,\, x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k.$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$.we get

$$\left\|2kH\left(\frac{1}{2k}\sum_{j=1}^{k} x_j + \frac{1}{2k^2}\sum_{j=1}^{k} x_{k+j}\right) - \sum_{j=1}^{k} H\Big(x_j\Big) - \sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right), z_1, z_2, ..., z_{l-1}\right\| = 0$$

for all $x_j,\, x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k.$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. By lemma 2.6, we get

$$H\Big(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k} x_{k+j}\Big) - \sum_{j=1}^{k} H\Big(x_j\Big) - \sum_{j=1}^{k} H\big(\frac{x_{k+j}}{k}\big) = 0$$

for all $x_j,\, x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. So mapping H is additive.

. Now we need to prove the uniqueness of H. Let H' be another additive mapping satisfying (4.4). Sence

$$\begin{aligned}\lim_{h\to\infty}\left|\frac{1}{(2k)^{h\beta}}\right|\widetilde{\varphi}((2k)^h x) &= \lim_{h\to\infty}\lim_{n\to\infty} max\Bigg\{\left|\frac{1}{(2k)^{i\beta}}\right|\varphi\Big((2k)^{i+h-1}x_1, (2k)^{i+h-1}x_2, ..., (2k)^{h+i-1}x_k,\\ &(2k)^{h+i-1}kx_{k+1}, (2k)^{h+i-1}kx_{k+2}, ..., (2k)^{h+i-1}kx_{2k}\Big), 1 \le i \le n\Bigg\}\\ &= \lim_{h\to\infty}\lim_{n\to\infty} max\Bigg\{\left|\frac{1}{(2k)^{i\beta}}\right|\varphi\Big((2k)^{i-1}x_1, (2k)^{i-1}x_2, ..., (2k)^{i-1}x_k,\\ &(2k)^{i-1}kx_{k+1}, (2k)^{i-1}kx_{k+2}, ..., (2k)^{i-1}kx_{2k}\Big), 1+h \le i \le n+h\Bigg\}\end{aligned} \tag{4.13}$$

for all $x \in \mathbf{X},\ z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$, it follows from then H is a unique additive mapping satisfying (3.19) that.

$$\begin{aligned}&\left\|H\big(x\big) - H'\big(x\big), z_1, z_2, ..., z_{l-1}\right\|_\beta\\ &= \lim_{h\to\infty}\left|\frac{1}{(2k)^{h\beta}}\right|\left\|H\big((2k)^h x\big) - H'\big((2k)^h x\big), z_1, z_2, ..., z_{2k-1}\right\|_\beta\\ &\le \lim_{h\to\infty}\left|\frac{1}{(2k)^{h\beta}}\right| max\Bigg\{\left\|H\big((2k)^h x\big) - f\big((2k)^h x\big), z_1, z_2, ..., z_{l-1}\right\|_\beta,\\ &\left\|f\big((2k)^h x\big) - H'\big((2k)^h x\big), z_1, z_2, ..., z_{l-1}\right\|_\beta\\ &\le \lim_{h\to\infty}\left|\frac{1}{(2k)^{h\beta}}\right|\widetilde{\varphi}((2k)^{-h\beta}x)\psi\big(z_1, z_2, ..., z_{l-1}\big) = 0\end{aligned} \tag{4.14}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Considering lemma 2.6 we prove that H is unique □

.

5. Stability of the pexiderized Cauchy Functional equation

is devoted to prove the Hyers-Ulam-Rassias type stability of the pexiderized Cauchy type functional equations in non-Archimdean (l,β)-normed space when $\mathbf{X}$ is a vector and $\mathbf{Y}$ is a complete non-Archimdean (l,β)-normed space with norm $\|\cdot\|_\beta$.

Theorem 5.1.

Suppose That $\mathbf{X}$ be a vector space and that $\mathbf{Y}$ is a complete non-Archimedean $(l,\beta)-$normed space, where $l \geq 2$, $0 < \beta \leq 1$. Let

$$\varphi : \mathbf{X^{2k}} \to [0,\infty)$$

be a function such that

$$\begin{aligned}
&\Gamma\big(x_1, x_2, ..., x_k, x_{k+1}, x_{k+2}, ..., x_{2k}\big)\\
&= \sum_{i=1}^{\infty} k^{i\beta}\Big(\varphi\big(\frac{x_1}{2k^{i-1}}, \frac{x_2}{2k^{i-1}}, ..., \frac{x_k}{2k^{i-1}}, 0, 0, ..., 0\big)\\
&+ \varphi\big(0, 0, ..., 0, \frac{x_{k+1}}{2k^{i-1}}, \frac{x_{k+2}}{2k^{i-1}}, ..., \frac{x_{2k}}{2k^{i-1}}\big)\\
&+ \varphi\big(\frac{x_1}{2k^{i-1}}, \frac{x_2}{2k^{i-1}}, ..., \frac{x_k}{2k^{i-1}}, \frac{x_{k+1}}{2k^{i-1}}, \frac{x_{k+2}}{2k^{i-1}}, ..., \frac{x_{2k}}{2k^{i-1}}\big)\Big)\\
&< \infty
\end{aligned} \tag{5.1}$$

and

$$\lim_{n\to\infty} \left|k^{n\beta}\right| \varphi\left(\frac{x_1}{2k^n}, \frac{x_2}{2k^n}, ..., \frac{x_k}{2k^n}, \frac{x_{k+1}}{2k^n}, \frac{x_{k+2}}{2k^n}, ..., \frac{x_{2k}}{2k^n}\right) = 0 \tag{5.2}$$

for all $x_j, x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$, and

suppose that a mapping

$$\psi\colon \mathbf{Y^{l-1}} \to [0,\infty)$$

be a function.

If mapping

$$f, g, p : \mathbf{X} \to \mathbf{Y}$$

satisfying the inequality

$$\begin{aligned}
&\left\|f\left(\sum_{j=1}^{k} x_j + \sum_{j=1}^{k}\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} g\big(x_j\big) - \sum_{j=1}^{k} p\left(\frac{x_{k+j}}{k}\right), z_1, z_2, ..., z_{l-1}\right\|_\beta\\
&\leq \varphi\big(x_1, x_2, ..., x_k, x_{k+1}, x_{k+2}, ..., x_{2k}\big)\cdot\psi\big(z_1, z_2, ..., z_{l-1}\big)
\end{aligned} \tag{5.3}$$

for all $x_j, x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$, then there exists a unique additive mapping

$$H : \mathbf{X} \to \mathbf{Y}$$

satisfying

$$\begin{aligned}
&\left\|f\big(x\big) - H\big(x\big), z_1, z_2, ..., z_{l-1}\right\|_\beta\\
&\leq \Gamma\big(x\big)\psi\big(z_1, z_2, ..., z_{l-1}\big) + \left\|p\big(0\big), z_1, z_2, ..., z_{l-1}\right\|_\beta + \left\|g\big(0\big), z_1, z_2, ..., z_{l-1}\right\|_\beta
\end{aligned} \tag{5.4}$$

$$\begin{aligned}&\Big\|g\big(x\big)-H\big(x\big),z_1,z_2,...,z_{l-1}\Big\|_\beta\\&\le\Gamma\big(x\big)\psi\big(z_1,z_2,...,z_{l-1}\big)+\Big\|g\big(0\big),z_1,z_2,...,z_{l-1}\Big\|_\beta+2\Big\|p\big(0\big),z_1,z_2,...,z_{l-1}\Big\|_\beta\\&+\varphi\big((2k)^{i-1}x_1,(2k)^{i-1}x_2,...,(2k)^{i-1}x_k,0,0,...,0\big)\psi\big(z_1,z_2,...,z_{l-1}\big)\end{aligned}\tag{5.5}$$

$$\begin{aligned}&\Big\|p\big(x\big)-H\big(x\big),z_1,z_2,...,z_{l-1}\Big\|_\beta\\&\le\Gamma\big(x\big)\psi\big(z_1,z_2,...,z_{l-1}\big)+\Big\|p\big(0\big),z_1,z_2,...,z_{l-1}\Big\|_\beta+2\Big\|g\big(0\big),z_1,z_2,...,z_{l-1}\Big\|_\beta\\&+\varphi\big(0,0,...,0,(2k)^{i-1}x_{k+1},(2k)^{i-1}x_{k+2},...,(2k)^{i-1}x_{2k}\big)\psi\big(z_1,z_2,...,z_{l-1}\big)\end{aligned}\tag{5.6}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$.

Proof. Put $x_j=\frac{x}{2k},x_{k+j}=\frac{x}{2}$ for all $j=1\to k$ in (5.3)we get

$$\begin{aligned}&\Big\|f\big(x\big)-kg\big(\frac{x}{2k}\big)-kp\big(\frac{x}{2k}\big),z_1,z_2,...,z_{l-1}\Big\|_\beta\\&\le\varphi\big(\frac{x}{2k},\frac{x}{2k},...,\frac{x}{2k},\frac{x}{2},\frac{x}{2},...,\frac{x}{2}\big)\psi\big(z_1,z_2,...,z_{l-1}\big)\end{aligned}\tag{5.7}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$.
Put $x_j=\frac{x}{2k}$ $x_{k+j}=0$ for all $j=1\to k$ in (5.3)

we get

$$\begin{aligned}&\Big\|f\big(\frac{x}{2}\big)-kg\big(\frac{x}{2k}\big)-kp\big(0\big),z_1,z_2,...,z_{l-1}\Big\|_\beta\\&\le\varphi\big(\frac{x}{2k},\frac{x}{2k},...,\frac{x}{2k},0,0,...,0\big)\psi\big(z_1,z_2,...,z_{l-1}\big)\end{aligned}\tag{5.8}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. It then follows from (5.8)

$$\begin{aligned}&\Big\|f\big(\frac{x}{2}\big)-kg\big(\frac{x}{2k}\big),z_1,z_2,...,z_{l-1}\Big\|_\beta\\&\le\varphi\big(\frac{x}{2k},\frac{x}{2k},...,\frac{x}{2k},0,0,...,0\big)\psi\big(z_1,z_2,...,z_{l-1}\big)+\Big\|kp\big(0\big),z_1,z_2,...,z_{l-1}\Big\|_\beta\end{aligned}\tag{5.9}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$.
Put $x_j=0$, $x_{k+j}=\frac{x}{2}$ for all $j=1\to k$ in (5.3)

we get

$$\begin{aligned}&\Big\|f\big(\frac{x}{2}\big)-kg\big(0\big)-kp\big(\frac{x}{2k}\big),z_1,z_2,...,z_{l-1}\Big\|_\beta\\&\le\varphi\big(0,0,...,0,\frac{x}{2},\frac{x}{2},...,\frac{x}{2}\big)\psi\big(z_1,z_2,...,z_{l-1}\big)\end{aligned}\tag{5.10}$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Thus, we obtain

$$
\begin{aligned}
&\left\|f\big(\frac{x}{2}\big)-kp\big(\frac{x}{2k}\big),z_1,z_2,...,z_{l-1}\right\|_\beta \\
&\qquad\leq\varphi\big(0,0,...,0,\frac{x}{2},\frac{x}{2},...,\frac{x}{2}\big)\psi\big(z_1,z_2,...,z_{l-1}\big)+\left\|kg\big(0\big),z_1,z_2,...,z_{l-1}\right\|_\beta
\end{aligned}
\tag{5.11}
$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$.

$$
\begin{aligned}
&\eta\big(\frac{x}{2},z_1,z_2,...,z_k,z_{k+1},z_{k+2},...,z_{l-1}\big)\\
&\quad=\left\|kg\big(0\big),z_1,z_2,...,z_{2k-1}\right\|_\beta+\left\|kp\big(0\big),z_1,z_2,...,z_{l-1}\right\|_\beta\\
&\quad+\varphi\big(\frac{x}{2k},\frac{x}{2k},...,\frac{x}{2k},\frac{x}{2},\frac{x}{2},...,\frac{x}{2}\big)\psi\big(z_1,z_2,...,z_{l-1}\big)+\\
&\quad+\varphi\big(\frac{x}{2k},\frac{x}{2k},...,\frac{x}{2k},0,0,...,0\big)\psi\big(z_1,z_2,...,z_{l-1}\big)+\\
&\quad+\varphi\big(0,0,...,0,\frac{x}{2},\frac{x}{2},...,\frac{x}{2}\big)\psi\big(z_1,z_2,...,z_{l-1}\big)
\end{aligned}
\tag{5.12}
$$

Using (5.7), (5.9) and (5.11), we have

$$
\begin{aligned}
&\left\|f\big(x\big)-2f\big(\frac{x}{2}\big),z_1,z_2,...,z_{l-1}\right\|_\beta\\
&\leq\left\|f\big(x\big)-kg\big(\frac{x}{2k}\big)-kp\big(\frac{x}{2k}\big),z_1,z_2,...,z_{l-1}\right\|_\beta+\left\|kg\big(\frac{x}{2k}\big)-f\big(\frac{x}{2}\big),z_1,z_2,...,z_{2k-1}\right\|_\beta\\
&+\left\|kp\big(\frac{x}{2k}\big)-f\big(\frac{x}{2}\big),z_1,z_2,...,z_{l-1}\right\|_\beta\\
&\leq\left\|kg\big(0\big),z_1,z_2,...,z_{l-1}\right\|_\beta+\left\|kp\big(0\big),z_1,z_2,...,z_{l-1}\right\|_\beta\\
&+\varphi\big(\frac{x}{2k},\frac{x}{2k},...,\frac{x}{2k},0,0,...,0\big)\psi\big(z_1,z_2,...,z_{l-1}\big)+\varphi\big(0,0,...,0,\frac{x}{2},\frac{x}{2},...,\frac{x}{2}\big)\psi\big(z_1,z_2,...,z_{l-1}\big)\\
&+\varphi\big(\frac{x}{2k},\frac{x}{2k},...,\frac{x}{2k},\frac{x}{2},\frac{x}{2},...,\frac{x}{2}\big)\psi\big(z_1,z_2,...,z_{l-1}\big)\\
&=\eta\big(\frac{x}{2},z_1,z_2,...,z_{l-1}\big)
\end{aligned}
\tag{5.13}
$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$.
So

$$
\left\|f\big(\frac{x}{2}\big)-\frac{1}{2}f\big(x\big),z_1,z_2,...,z_{l-1}\right\|_\beta\leq2^{-\beta}\eta\big(\frac{x}{2},z_1,z_2,...,z_{l-1}\big)
\tag{5.14}
$$

for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Replacing x by $\frac{x}{2}$ in (5.14), we get

$$
\left\|f\big(\frac{x}{2^2}\big)-\frac{1}{2}f\big(\frac{x}{2}\big),z_1,z_2,...,z_{l-1}\right\|_\beta\leq\eta\big(\frac{x}{2^2},z_1,z_2,...,z_{l-1}\big)
\tag{5.15}
$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. It then follows from (5.14) and (5.15)

$$\left\|f\left(\frac{x}{2^2}\right)-\frac{1}{2^2}f\left(x\right), z_1, z_2, ..., z_{l-1}\right\|_\beta \leq \left\|f\left(\frac{x}{2^2}\right)-\frac{1}{2}f\left(\frac{x}{2}\right), z_1, z_2, ..., z_{l-1}\right\|_\beta$$
$$+2^{-\beta}\left\|f\left(\frac{x}{2}\right)-\frac{1}{2}f\left(x\right), z_1, z_2, ..., z_{l-1}\right\|_\beta \leq \eta\left(\frac{x}{2^2}, z_1, z_2, ..., z_{l-1}\right)+2^{-\beta}\eta\left(\frac{x}{2}, z_1, z_2, ..., z_{2k-1}\right) \tag{5.16}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Applying an induction argument on n, we will prove that

$$\left\|f\left(\frac{x}{2^n}\right)-\frac{1}{2^n}f\left(x\right), z_1, z_2, ..., z_{l-1}\right\|_\beta \leq \sum_{i=1}^{n}\frac{1}{2^{(i-1)\beta}}\varphi\left(\frac{x}{2^{n-1}}, z_1, z_2, ..., z_{l-1}\right) \tag{5.17}$$

for all $x \in \mathbf{X}$, $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$ and $n \in \mathbb{N}$. In view of (5.14) is true for n=1. Assume that (5.14) is true for $n > 1$. Substituting $\frac{x}{2}$ for x in (5.14), we obtain

$$\left\|f\left(\frac{x}{2^{n+1}}\right)-\frac{1}{2^n}f\left(\frac{x}{2}\right), z_1, z_2, ..., z_{l-1}\right\|_\beta \leq \sum_{i=1}^{n}\frac{1}{2^{(i-1)\beta}}\eta\left(\frac{x}{2^{n-1}}, z_1, z_2, ..., z_{l-1}\right) \tag{5.18}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Hence, it follows from (5.17) that

$$\begin{aligned}&\left\|f\left(\frac{x}{2^{n+1}}\right)-\frac{1}{2^{n+1}}f\left(x\right), z_1, z_2, ..., z_{l-1}\right\|_\beta\\ &\leq \left\|f\left(\frac{x}{2^{n+1}}\right)-\frac{1}{2^n}f\left(x\right), z_1, z_2, ..., z_{l-1}\right\|_\beta\\ &+2^{-n\beta}\left\|f\left(\frac{x}{2}\right)-\frac{1}{2}f\left(x\right), z_1, z_2, ..., z_{l-1}\right\|_\beta\\ &\leq \sum_{i=1}^{n}\frac{1}{2^{(i-1)\beta}}\eta\left(\frac{1}{2^{n+1-i}}x, z_1, z_2, ..., z_{l-1}\right)+2^{-n\beta}\eta\left(\frac{x}{2}, , z_1, z_2, ..., z_{l-1}\right)\\ &=\sum_{i=1}^{n}\frac{1}{2^{(i-1)\beta}}\eta\left(\frac{1}{2^{n+1-i}}x, z_1, z_2, ..., z_{l-1}\right)\end{aligned} \tag{5.19}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$, which proves inequality (5.17) by (5.17), we have

$$\left\|2^n f\left(\frac{x}{2^n}\right)-f\left(x\right), z_1, z_2, ..., z_{l-1}\right\|_\beta \leq \sum_{i=1}^{n}\frac{2^{n\beta}}{2^{(i-1)\beta}}\eta\left(\frac{1}{2^{n-1}}x, z_1, z_2, ..., z_{l-1}\right) \tag{5.20}$$

for all $x \in \mathbf{X}$, $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$ and $n \in \mathbb{N}$. Moreover, if $n, q \in \mathbb{N}$ with $n < q$, then it follows from (5.14) that

$$
\begin{aligned}
&\left\|2^q f\big(\frac{x}{2^q}\big)-2^n f\big(\frac{x}{2^n}\big), z_1, z_2, \ldots, z_{l-1}\right\|_\beta \\
&\leq \sum_{i=n}^{q-1}\left\|2^i f\big(\frac{x}{2^i}\big)-2^{(i+1)} f\big(\frac{x}{2^{i+1}}\big), z_1, z_2, \ldots, z_{l-1}\right\|_\beta \\
&\leq \sum_{i=n}^{q-1} 2^{(i+1)\beta}\left\|2^{-1} f\big(\frac{x}{2^i}\big)-f\big(\frac{x}{2^{i+1}}\big), z_1, z_2, \ldots, z_{l-1}\right\|_\beta \\
&=\sum_{i=n}^{q-1} 2^{(i+1)\beta} \eta\big(\frac{1}{2^i} x, z_1, z_2, \ldots, z_{l-1}\big) \\
&=\sum_{i=1}^{\infty} 2^{(i+1)\beta}\Big[\varphi\left(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \ldots, \frac{x_k}{2^i}, 0,0, \ldots, 0\right) \psi\big(z_1, z_2, \ldots, z_{l-1}\big) \\
&+\varphi\big(0,0, \ldots, 0, \frac{x_{k+1}}{2^i}, \frac{x_{k+2}}{2^i}, \ldots, \frac{x_{2k}}{2^i}\big) \psi\big(z_1, z_2, \ldots, z_{l-1}\big) \\
&+\varphi\left(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \ldots, \frac{x_k}{2^i}, \frac{x_{k+1}}{2^i}, \frac{x_{k+2}}{2^i}, \ldots, \frac{x_{2k}}{2^i}\right) \psi\big(z_1, z_2, \ldots, z_{l-1}\big) \\
&+\left\|p\big(0\big), z_1, z_2, \ldots, z_{l-1}\right\|_\beta+\left\|g\big(0\big), z_1, z_2, \ldots, z_{l-1}\right\|_\beta\Big] \\
&\leq \sum_{i=1}^{\infty} 2^{-(i+1)\beta}\Big[\varphi\left(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \ldots, \frac{x_k}{2^i}, 0,0, \ldots, 0\right)+\varphi\left(0,0, \ldots, 0, \frac{x_{k+1}}{2^i}, \frac{x_{k+2}}{2^i}, \ldots, \frac{x_{2k}}{2^i}\right) \\
&+\varphi\left(\frac{x_1}{2^i}, \frac{x_2}{2^i}, \ldots, \frac{x_k}{2^i}, \frac{x_{k+1}}{2^i}, \frac{x_{k+2}}{2^i}, \ldots, \frac{x_{2k}}{2^i}\right)\Big] \psi\big(z_1, z_2, \ldots, z_{l-1}\big) \\
&+2^{-n}\left\|p\big(0\big), z_1, z_2, \ldots, z_{l-1}\right\|_\beta+\left\|g\big(0\big), z_1, z_2, \ldots, z_{l-1}\right\|_\beta
\end{aligned} \tag{5.21}
$$

for all $x \in \mathbf{X}$ $z_1, z_2, \ldots, z_{l-1} \in \mathbf{Y}$. Hence, it follows from

Taking the limit as $n, q \rightarrow \infty$ and considering (5.1)

$$
\lim_{n, q \rightarrow \infty}\left\|2^q f\left(\frac{1}{2^q} x\right)-2^n f\left(\frac{1}{2^q} x\right), z_1, z_2, \ldots, z_{l-1}\right\|_\beta=0 \tag{5.22}
$$

for all $x \in \mathbf{X}$ and $z_1, z_2, \ldots, z_{l-1} \in \mathbf{Y}$. Accoding to Difinition 2.4, we know that

that the sequence $\left\{2^q f\big(\frac{1}{2^q} x\big)\right\}$ is Cauchy sequence for all $x \in \mathbf{X}$. Since $\mathbf{Y}$ is completes $\big(n, \beta\big)$ space, the sequence $\left\{2^q f\big(\frac{1}{2^q} x\big)\right\}$ coverges.

So one can define the mapping H: $\mathbf{X} \rightarrow \mathbf{Y}$ by

$$
H\big(x\big):=\lim_{n \rightarrow \infty} 2^n f\left(\frac{1}{2^n} x\right) \tag{5.23}
$$

for all $x \in \mathbf{X}$. in (5.3)

$$
\begin{aligned}
&\left\|f\left(\sum_{j=1}^k x_j+\frac{1}{k} \sum_{j=1}^k \frac{x_{k+j}}{k}\right)-\sum_{j=1}^k g\Big(x_j\Big)-\sum_{j=1}^k p\Big(\frac{x_{k+j}}{k}\Big), z_1, z_2, \ldots, z_{l-1}\right\|_\beta \\
&\leq \lim_{n \rightarrow \infty}\left|2^{n \beta}\right| \varphi\Big(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \ldots, \frac{x_k}{2^n}, \frac{x_{k+1}}{2^n}, \frac{x_{k+2}}{2^n}, \ldots, \frac{x_{2k}}{2^n}\big) \psi\big(z_1, z_2, \ldots, z_{l-1}\big)
\end{aligned}
$$

and so for all $x_j,\ x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$.It follows from (5.9)

$$\left\|k^n f\left(\frac{x}{2k^n}\right) - k^n g\left(\frac{x}{2k^n}\right), z_1, z_2, ..., z_{l-1}\right\|_\beta \leq k^{n\beta}\Big[\varphi\left(\frac{x}{k}, \frac{x}{k}, ..., \frac{x}{k}, 0, 0, ..., 0\right)\psi\left(z_1, z_2, ..., z_{l-1}\right) + \left\|kp(0), z_1, z_2, ..., z_{l-1}\right\|_\beta\Big] \tag{5.24}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Considering (5.1)

$$\begin{aligned}
&k^{n\beta}\varphi\left(\frac{x_1}{2k^n}, \frac{x_2}{2k^n}, ..., \frac{x_k}{2k^n}, 0, 0, ..., 0\right)\psi\left(z_1, z_2, ..., z_{l-1}\right)\\
&\leq k^\beta \sum_{i=1}^{\infty} k^{(i+1)\beta}\Big[\varphi\left(\frac{x_1}{2k^i}, \frac{x_2}{2k^i}, ..., \frac{x_k}{2k^i}, 0, 0, ..., 0\right)\psi\left(z_1, z_2, ..., z_{l-1}\right)\\
&+ \varphi\left(0, 0, ..., 0, \frac{x_{k+1}}{2k^i}, \frac{x_{k+2}}{2k^i}, ..., \frac{x_{2k}}{2k^i}\right)\psi\left(z_1, z_2, ..., z_{l-1}\right)\\
&+ \varphi\left(\frac{x_1}{2k^i}, \frac{x_2}{2k^i}, ..., \frac{x_k}{2k^i}, \frac{x_{k+1}}{2k^i}, \frac{x_{k+2}}{2k^i}, ..., \frac{x_{2k}}{2k^i}\right)\psi\left(z_1, z_2, ..., z_{l-1}\right)
\end{aligned} \tag{5.25}$$

$\to 0$ as $n \to \infty$

It follows from (5.25) that

$$H(x) := \lim_{n\to\infty} k^n f\left(\frac{1}{2k^n}x\right) = \lim_{n\to\infty} k^n g\left(\frac{1}{2k^n}x\right) \tag{5.26}$$

for all $x \in \mathbf{X}$. Also, by (5.11)

$$\left\|k^n f\left(\frac{x}{2k^n}\right) - k^n p\left(\frac{x}{2k^n}\right), z_1, z_2, ..., z_{l-1}\right\|_\beta \leq k^{n\beta}\Big[\varphi\left(0, 0, ..., 0, \frac{x}{2k^n}, \frac{x}{2k^n}, ..., \frac{x}{2k^n}\right)\psi\left(z_1, z_2, ..., z_{l-1}\right) + \left\|kg(0), z_1, z_2, ..., z_{l-1}\right\|_\beta\Big] \tag{5.27}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Similarly, it follows from (5.27) that

$$H(x) := \lim_{n\to\infty} k^n f\left(\frac{1}{2k^n}x\right) = \lim_{n\to\infty} k^n p\left(\frac{1}{2k^n}x\right) \tag{5.28}$$

for all $x \in \mathbf{X}$ and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$. Thus, by (5.2), (5.25), (5.28) and lemma 2.7, we get

$$\begin{aligned}
&\left\|H\left(\sum_{j=1}^{k} x_j + \frac{1}{k}\sum_{j=1}^{k}\frac{x_{k+j}}{k}\right) - \sum_{j=1}^{k} H\left(x_j\right) - \sum_{j=1}^{k} H\left(\frac{x_{k+j}}{k}\right), z_1, z_2, ..., z_{l-1}\right\|_\beta\\
&= \lim_{n\to\infty}\left|k^{n\beta}\right|\left\|f\left[\left(\sum_{j=1}^{n}\frac{1}{2k^n}x_j + \frac{1}{k}\sum_{j=1}^{n}\frac{1}{2k^n}x_{k+j}\right)\right] - \sum_{j=1}^{k} f\left(\frac{1}{2k^n}x_j\right) - \sum_{j=1}^{k} f\left(\frac{1}{2k^n}\frac{x_{k+j}}{k}\right), z_1, z_2, ...\right.\\
&\left., z_{l-1}\right\|_\beta\\
&\leq \lim_{n\to\infty}\left|k^{n\beta}\right|\varphi\left(\frac{x_1}{2k^n}, \frac{x_2}{2k^n}, ..., \frac{x_k}{2k^n}, \frac{x_{k+1}}{2k^n}, \frac{x_{k+2}}{2k^n}, ..., \frac{x_{2k}}{2k^n}\right)\psi\left(z_1, z_2, ..., z_{l-1}\right)\\
&= 0
\end{aligned}$$

and so for all $x_j,\ x_{k+j} \in \mathbf{X}$ for all $j = 1 \to k$. and $z_1, z_2, ..., z_{l-1} \in \mathbf{Y}$.

we get

$$\left\|H\left(\sum_{j=1}^{k}x_j+\frac{1}{k}\sum_{j=1}^{k}x_{k+j}\right)-\sum_{j=1}^{k}H\Big(x_j\Big)-\sum_{j=1}^{k}H\left(\frac{x_{k+j}}{k}\right),z_1,z_2,...,z_{l-1}\right\|=0$$

for all $x_j,\ x_{k+j}\in\mathbf{X}$ for all $j=1\to k.$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. By lemma 2.6, we get

$$H\Big(\sum_{j=1}^{k}x_j+\frac{1}{k}\sum_{j=1}^{k}x_{k+j}\Big)-\sum_{j=1}^{k}H\Big(x_j\Big)-\sum_{j=1}^{k}H\big(\frac{x_{k+j}}{k}\big)=0$$

for all $x_j,\ x_{k+j}\in\mathbf{X}$ for all $j=1\to k$. So mapping H is additive. Taking the limit as $n\to\infty$ in (5.17)

$$\begin{aligned}
&\Big\|H\big(x\big)-f\big(x\big),z_1,z_2,...,z_l\Big\|_\beta\\
&\leq\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2k^{(i-1-n)\beta}}\varphi\big(\frac{x}{2k^{n-1}},z_1,z_2,...,z_{l-1}\big)\\
&=\lim_{n\to\infty}\big(1-2k^{n\beta}\big)\Big(\Big\|kg\big(0\big),z_1,z_2,...,z_{l-1}\Big\|_\beta+\Big\|kp\big(0\big),z_1,z_2,...,z_{l-1}\Big\|_\beta\Big)\\
&+\lim_{n\to\infty}\sum_{i=1}^{n}k^{(i-n-1)\beta}\Big(\varphi\big(\frac{x_1}{2k^{i-1}},\frac{x_2}{2k^{i-1}},...,\frac{x_k}{2k^{i-1}},0,0,...,0\big)\psi\big(z_1,z_2,...,z_{l-1}\big)\\
&+\varphi\big(0,0,...,0,\frac{x_{k+1}}{2k^{i-1}},\frac{x_{k+2}}{2k^{i-1}},...,\frac{x_{2k}}{2k^{i-1}}\big)\psi\big(z_1,z_2,...,z_{l-1}\big)\\
&+\varphi\big(\frac{x_1}{2k^{i-1}},\frac{x_2}{2k^{i-1}},...,\frac{x_k}{2k^{i-1}},\frac{x_{k+1}}{2k^{i-1}},\frac{x_{k+2}}{2k^{i-1}},...,\frac{x_{2k}}{2k^{i-1}}\big)\psi\big(z_1,z_2,...,z_{l-1}\big)\Big)\\
&=\Big\|kg\big(0\big),z_1,z_2,...,z_{l-1}\Big\|_\beta+\Big\|kp\big(0\big),z_1,z_2,...,z_{l-1}\Big\|_\beta+\Gamma\big(x\big)\psi\big(z_1,z_2,...,z_{l-1}\big)
\end{aligned}\tag{5.29}$$

for all $x\in\mathbf{X}$, $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$, Which prover (5.4) Prover the uniqueness of H. Assume That

$$H':\mathbf{X}\to\mathbf{Y}$$

is another additive mapping which satisfying (5.4)

$$\begin{aligned}
&\Big\|H\big(x\big)-H'\big(x\big),z_1,z_2,...,z_{l-1}\Big\|_\beta\\
&\leq k^{n\beta}\left\|H\left(\frac{x}{2k^n}\right)-f\left(\frac{x}{2k^n}\right),z_1,z_2,...,z_{l-1}\right\|_\beta+k^{n\beta}\left\|f\left(\frac{x}{2k^n}\right)-H'\left(\frac{x}{2k^n}\right),z_1,z_2,...,z_{l-1}\right\|_\beta\\
&k^{n\beta+1}\Big(\Big\|kg\big(0\big),z_1,z_2,...,z_{l-1}\Big\|_\beta+\Big\|kp\big(0\big),z_1,z_2,...,z_{l-1}\Big\|_\beta+\Gamma\big(\frac{x}{2k^n}\big)\psi\big(z_1,z_2,...,z_{l-1}\big)\Big)\\
&=k^{n\beta-1}\Big(\Big\|kg\big(0\big),z_1,z_2,...,z_{l-1}\Big\|_\beta+\Big\|kp\big(0\big),z_1,z_2,...,z_{l-1}\Big\|_\beta\\
&+\varphi\big(\frac{x_1}{2k^{i-1}},\frac{x_2}{2k^{i-1}},...,\frac{x_k}{2k^{i-1}},0,0,...,0\big)\psi\big(z_1,z_2,...,z_{2k-1}\big)\\
&+\varphi\big(0,0,...,0,\frac{x_{k+1}}{2k^{i-1}},\frac{x_{k+2}}{2k^{i-1}},...,\frac{x_{2k}}{2k^{i-1}}\big)\psi\big(z_1,z_2,...,z_{l-1}\big)\\
&+\varphi\big(\frac{x_1}{2k^{i-1}},\frac{x_2}{2k^{i-1}},...,\frac{x_k}{2k^{i-1}},\frac{x_{k+1}}{2k^{i-1}},\frac{x_{k+2}}{2k^{i-1}},...,\frac{x_{2k}}{2k^{i-1}}\big)\psi\big(z_1,z_2,...,z_{l-1}\big)\Big)
\end{aligned}\tag{5.30}$$

$n\to0$ as $n\to\infty$ for all $x\in\mathbf{X}$ and $z_1,z_2,...,z_{l-1}\in\mathbf{Y}$. Which together with lemma 2.6 implies that $H\big(x\big)=H'\big(x\big)$ for all $x\in\mathbf{X}$. □

6. Conclusion

In this paper, I have shown that the solutions of the (1.1) and (1.2) are additive mappings. The Hyers-Ulam-Rassia stability for these given from theorems. These are the main results of the paper , which are the generalization of the results [10, 11, 25] .

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