International Journal of Mathematics Trends and Technology
ISSN: 2231 – 5373/doi:10.14445/22315373/IJMTT-V67I10P503

Volume 67 Issue 10, 34-40, October, 2021

Some Properties of Analytic and P-Valent Functions Involving Certain Convolution Operators

Yakubu Gambo[1], Kunle Oladeji Babalola[2], Adamu Umar Mustapha[3], Daniel Eneojo Emmanuel[4]

[1,3,4]*Department of Mathematics and Computer Science Federal University of Kashere, Gombe, Nigeria.*

[2]*Professor Department of Mathematics, University of Ilorin, Nigeria*

Abstract - *Let* $A(p)$ *be denote the class of functions that are analytic in the unit disk E which have the form;* $f(z)=z^p+a_{p+1}z^{p+1}+a_{p+2}z^{p+2}+a_{p+3}z^{p+3}+\dots$ *the class of functions satisfying the geometric condition* $\mathrm{Re}\left\{(M_n^{\sigma,p}f(z))/z^p\right\}>\beta$ *was defined where,* $M_n^{\sigma,p}: A(p)\to A(p)$ *is an operator define using convolution * The main concern of this work is to obtain some basic properties of the class with geometric condition above. These properties include; Inclusion, Growth, and Covering theorem.*

Keywords - *Convolution operators, analytic and p-valent functions.*

I. INTRODUCTION

Let $A(p)$ denotes the class of functions

$$f(z)=z^p+a_{p+1}z^{p+1}+a_{p+2}z^{p+2}+a_{p+3}z^{p+3}+\dots \quad \text{and} \quad p\in \mathrm{N}=\{1,2,3,\dots\} \tag{1.1}$$

which are analytic in *E*.

Let *P* be the class of all functions of the form $p(z)=1+c_1z+c_2z^2+\dots$ (1.2)

which are analytic in *E* such that for $z\in E$, $\mathrm{Re}\,p(z)>0$ and $p(0)=1$.

For $0\le\beta<1$, let $P(\beta)$ denote the subclass of *P* consisting of analytic function of the form

$$p_\beta(z)=\beta+(1-\beta)p(z), \qquad p(z)\in P$$

Let $g(z)=z^p+a_{p+1}z^{p+1}+a_{p+2}z^{p+2}+a_{p+3}z^{p+3}+\dots\in A(p)$ Then, the convolution of f and *g*, written as $(f*g)(z)$ or $(g*f)(z)$ is defined as $(f*g)=z^p+\sum_{k=1}^{\infty}a_{p+k}b_{p+k}z^{p+k}$

The Gauss hypergeometric function is defined for $|z|<1,\ z\in E$ by the power series as

$${}_2F_1(a,b;c;z)=\sum_{k=1}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!} \tag{1.3}$$

Where $(\lambda)_k$ is known as the Pochhammer symbol defined in term of Gamma by

$$(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)} = \begin{cases} 1 & \text{if } k=0 \\ \lambda(\lambda+1)(\lambda+2)\ldots(\lambda+k-1), & \text{if } k>0 \end{cases}$$

Where $\lambda \neq (0,-1,-2,-3,\ldots)$

Let σ be any fixed real number and p be a natural number, such that $(\sigma+p-n)>0,$ for $n \in \mathrm{N}_0$ such that $n>-p$. Define by

$$f_{\sigma+p,n}(z) = z^p\,{}_2F_1(\sigma+p-n,1;1;z), \quad (\sigma+p-n)>0 \tag{1.4}$$

Let $f^{(-1)}_{\sigma+p,n}(z)$ be defined such that

$$\left(f_{\sigma+p,n} * f^{(-1)}_{\sigma+p,n}\right)(z) = z^p\,{}_2F_1(1,1;1;z) \tag{1.5}$$

For n=0, we write $f_{\sigma+p}$ instead of $f_{\sigma+p,0}$ and $f^{(-1)}_{\sigma+p}$ instead of $f^{(-1)}_{\sigma+p,0}$

Let $f \in A(1)$, define the operator $D^\sigma : A(1) \to A(1)$ by $D^\sigma f(z) = (f_{\sigma+1} * f)(z)$

The operator D^σ is called Rusheweyh derivative [8], Analogous to D^σ Noor [4] defined the integral operator $I_\sigma : A(1) \to A(1)$ by $I_\sigma f(z) = \left(f^{(-1)}_{\sigma+1} * f\right)(z)$

Let $f \in A(1)$ Babalola [2] respectively defined the differential and integral operators as
$L^\sigma_n : A(1) \to A(1)$ by $L^\sigma_n f(z) = \left(f_{\sigma+1} * f^{(-1)}_{\sigma+1,n} * f\right)(z)$ and
$l^\sigma_n : A(1) \to A(1)$ by $l^\sigma_n f(z) = \left(f^{(-1)}_{\sigma+1} * f_{\sigma+1,n} * f\right)(z)$
The differential and the integral operators above D^σ and I_σ respectively defined various classes of functions in different literature (see [1, 3, 5, 6, 8]).

Now, we simply defined the following operators

Definition 1: Let $f \in A(p)$ we define the operator $M^{\sigma,p}_n : A(p) \to A(p)$ by

$M^{\sigma,p}_n f(z) = \left(f_{\sigma+p} * f^{(-1)}_{\sigma+p,n} * f\right)(z)$ and

Definition 2: Let $f \in A(p)$ we define the operator $m^{\sigma,p}_n : A(p) \to A(p)$ by

$$m^{\sigma,p}_n f(z) = \left(f^{(-1)}_{\sigma+p} * f_{\sigma+p,n} * f\right)(z)$$

Note that; $M^{\sigma,p}_0 f(z) = M^{0,p}_0 f(z) = f(z)$ Similarly, $m^{\sigma,p}_0 f(z) = m^{0,p}_0 f(z) = f(z)$

Remark 1: Let $f \in A(p)$. Then $M_n^{\sigma,p}\left(m_n^{\sigma,p} f(z)\right) = m_n^{\sigma,p}\left(M_n^{\sigma,p} f(z)\right) = f(z)$

Definition 3: Let $f \in A(p)$ and σ be any fixed real number, $n \in N_0$ and p is a natural number, then a function $f \in A(p)$ is said to be in the class $\lambda_n^{\sigma,p}(\beta)$ if and only if

$$\operatorname{Re}\left\{\frac{M_n^{\sigma,p} f(z)}{z^p}\right\} > \beta, \qquad 0 \le \beta < 1. \tag{1.6}$$

Remark 2: From the Remark 1 and the geometric condition **1.6** the function in the class $\lambda_n^{\sigma,p}(\beta)$ can be represented in terms of function in $p(\beta)$ as $f(z) = m_n^{\sigma,p}\left[z^p p_\beta(z)\right]$. The class $\lambda_n^{\sigma,p}(\beta)$ will be investigated in section 3. However, we require some preliminary discussions and results, which we present in the next section.

II. ITERATED INTEGRAL TRANSFORM OF THE CLASS P

Definition 4: Let $h \in P$ and σ be any fixed real number such that $(\sigma + p - n) > 0$ for each $n \ge 1$ we define *p*-sigma *nth* iterated integral transform of $h(z)$, $z \in E$ as

$$h_{\sigma+p,n}(z) = \frac{\sigma + p - n}{z^{\sigma+p-n}} \int_0^z t^{\sigma+p-(n+1)} h_{\sigma+p,n-1}(t)dt, \qquad \mathrm{n} \ge 1 \tag{2.1}$$

With $h_{\sigma+p,0}(z) = p(z)$

Since, $h_{\sigma+p,0}(z)$ belong to *P*, the transformation $h_{\sigma+p,n}(z)$ is analytic, satisfying $h_{\sigma+p,0}(0) = 1$ and $h_{\sigma+p,0}(z) \ne 0$ We denote the family of iterations above by $P_n^{\sigma,p}$ With p(z) given by (1.1) it is easily verified that $h_{\sigma+p,n}(z) = 1 + \sum_{k=1}^{\infty} C_{n,k}^{\sigma,p} z^k$ (2.2)

Where $C_{n,k}^{\sigma,p} = \dfrac{(\sigma + p - 1)(\sigma + p - 2)(\sigma + p - 3)\ldots(\sigma + p - n)}{(\sigma + p + k - 1)(\sigma + p + k - 2)\ldots(\sigma + p + k - n)} c_k$

The multiplier of c_k above can be written in factorial form as

$$\frac{(\sigma + p - 1)(\sigma + p - 2)(\sigma + p - 3)\ldots(\sigma + p - n)}{(\sigma + p + k - 1)(\sigma + p + k - 2)\ldots(\sigma + p + k - n)} = \frac{(\sigma + p - 1)!}{(\sigma + p - n - 1)!} \cdot \frac{(\sigma + p + k - n - 1)!}{(\sigma + p + k - 1)!}$$

From the fact that $\Gamma(\lambda+1)=\lambda!$ and $(\lambda)_k=\dfrac{\Gamma(\lambda+k)}{\Gamma(k)}$ the above expression can also be represented as $(\sigma+p-n)_n/(\sigma+p+k-n)_n$ where $(\sigma+p-n)_n$ and $(\sigma+p+k-n)_n$ are known as Pochhammer symbols. Therefore we have

$$C_{n,k}^{\sigma,p}=\frac{(\sigma+p-n)_n}{(\sigma+p+k-n)_n} \qquad (2.3)$$

with $$\frac{(\sigma+p-n)_0}{(\sigma+p+k-n)_0}=1$$

By setting $h_{\sigma+p,0}(z)=L_0(z)=(1+z)/(1-z)$ we get the p-sigma *nth* integral iterations of Möbius function as

$$L_{\sigma+p,n}(z)=\frac{\sigma+p-n}{z^{\sigma+p-n}}\int_0^z t^{\sigma+p-(n+1)}L_{\sigma+p,n-1}(t)dt, \qquad \mathrm{n}\geq 1 \qquad (2.4)$$

$$=1+2\sum_{k=1}^{\infty}\frac{(\sigma+p-n)_n}{(\sigma+p+k-n)_n}z^k$$

Remark 3: From (2.3) above and well known inequality (caratheodory Lemma), we have the following inequality

$$\left|C_{n,k}^{\sigma,p}\right|\leq 2\frac{(\sigma+p-n)_n}{(\sigma+p+k-n)_n}$$

With equality if and only if $h_{\sigma+p,n}(z)=L_{\sigma+p,n}(z)$

Remark 4: Let $h_{\sigma+p,n}(z)=\chi_n^{(\sigma)}\left(p(z)\right)$ then for any $p\in P$ and $n,m\in N_o=N\cup\{0\}$ we have

$\chi_m^{(v)}\left(\chi_n^{(\sigma)}\left(p(z)\right)\right)=\chi_n^{(\sigma)}\left(\chi_m^{(v)}\left(p(z)\right)\right)$ where $v>0$ is real. This can be seen easily using (2.2) and (2.3) above. For $v=\sigma$ gives $\chi_m^{(\sigma)}\left(\chi_n^{(\sigma)}\left(p(z)\right)\right)=\chi_n^{(\sigma)}\left(\chi_m^{(\sigma)}\left(p(z)\right)\right)=\chi_{m+n}^{(\sigma)}\left(p(z)\right)$

We use this remark to study functions in the family $P_n^{\sigma,p}$.

Remark 5: We note that for $p=1$, $P_n^{\sigma,1}=P_n^{\sigma}$ a class of function established by Babalola [2]. The following results characterizing the family $P_n^{\sigma,p}$ can be obtained *mutatis mutandis* as in Section 2 of [1], thus we omit the proofs.

Theorem 2.1: Let $\gamma \neq 1$ be a non-negative real number then for any fixed σ a real, $p \in \mathrm{N}$ and each $n \geq 1$.

$$\operatorname{Re} h_{\sigma+p,n-1}(z) > \gamma \Rightarrow \operatorname{Re} h_{\sigma+p,n}(z) > \gamma, \qquad 0 \leq \gamma < 1$$

and

$$\operatorname{Re} h_{\sigma+p,n-1}(z) < \gamma \Rightarrow \operatorname{Re} h_{\sigma+p,n}(z) < \gamma, \qquad \gamma > 1$$

Corollary 2.1: $P_n^{\sigma,p} \subset P, \quad \mathrm{n} \geq 1$

Theorem 2.2: $P_{n+1}^{\sigma,p} \subset P_n^{\sigma,p}, \quad \mathrm{n} \geq 1$

Theorem 2.3: Let $h_{\sigma+p,n}(z) \in P_n^{\sigma,p}$ Then,

(a) $$\left|\mathrm{h}_{\sigma+\mathrm{p,n}}(z)\right| \leq 1 + 2\sum_{k=1}^{\infty} \frac{(\sigma+p-n)_n}{(\sigma+p+k-n)_n} r^k, \qquad |\mathrm{z}| = r$$

(b) $$\left|\mathrm{h}_{\sigma+\mathrm{p,n}}(z)\right| \geq 1 + 2\sum_{k=1}^{\infty} \frac{(\sigma+p-n)_n}{(\sigma+p+k-n)_n} (-r)^k, \qquad |\mathrm{z}| = r$$

The equality is attained in the upper bound for $h_{\sigma+p,n}(z) = L_{\sigma+p,n}(z)$ while for the lower bound the equality is realized if $h_{\sigma+p,n}(z) = L_{\sigma+p,n}(-z)$.

III. CHARACTERIZATIONS OF THE CLASS $\lambda_n^{\sigma,p}(\beta)$

we present the main results of this work in this section. These include; Inclusion, Growth, Covering, and Closure under certain integral transformation. A basic relationship between the classes $P_n^{\sigma,p}$ and $\lambda_n^{\sigma,p}(\beta)$ was given by the following lemma.

Lemma 3.1: Let $f \in A(p)$, σ be any fixed real number and $0 \leq \beta < 1$ where $n \in \mathrm{N}$. Then, the following are equivalent.

(i) $f \in \lambda_n^{\sigma,p}(\beta)$

(ii) $\left(M_n^{\sigma,p} f(z)/z^p - \beta\right)/(1-\beta) \in P$

(iii) $\left(f(z)/z^p - \beta\right)/(1-\beta) \in P_n^{\sigma,p}$

Proof

That (i) $\Leftrightarrow$ (ii) is clear from the definition (3). Now, (ii) is true $\Leftrightarrow$ there exist $p \in P$ such that

$$M_n^{\sigma,p} f(z) = z^p[\beta - (1-\beta)p(z)]$$

$$= z^p + (1-\beta)\sum_{k=1}^{\infty} c_k z^{p+k} \tag{3.1}$$

Now, by applying the operator $m_n^{\sigma,p}$ on (3.1) we get

$$f(z)=z^{p}+(1-\beta)\sum_{k=1}^{\infty}C_{n,k}^{\sigma,p}z^{k+p} \tag{3.2}$$

We get from (3.2) above that

$$\left(f(z)/z^{p}-\beta\right)/(1-\beta)=1+\sum_{k=1}^{\infty}C_{n,k}^{\sigma,p}z^{k} \tag{3.3}$$

Thus, the right hand side of (3.3) is a function in $P_{n}^{\sigma,p}$ This completes the proof.

MAIN RESULTS

Theorem 3.1: For any fixed σ satisfying $(\sigma + p - n) > 0$. The following inclusion holds $\lambda_{n+1}^{\sigma,p}(\beta)\subset\lambda_{n}^{\sigma,p}(\beta)$

Proof

Let $f\in\lambda_{n+1}^{\sigma,p}(\beta)$ then by Lemma 3.1 we have $\left(f(z)/z^{p}-\beta\right)/(1-\beta)\in P_{n+1}^{\sigma,p}$ By theorem 2.2 we have $\left(f(z)/z^{p}-\beta\right)/(1-\beta)\in P_{n}^{\sigma,p}$ by Lemma 3.1 again $f\in\lambda_{n}^{\sigma,p}(\beta)$. This completes the proof.

Theorem 3.2

Let $f\in\lambda_{n}^{\sigma,p}(\beta)$, then

$$r^{p}+2(1-\beta)\sum_{k=1}^{\infty}\frac{(\sigma+p-n)_{n}}{(\sigma+p+k-n)_{n}}(-1)^{k}r^{p+k}\le\left|f(z)\right|\le r^{p}+2(1-\beta)\sum_{k=1}^{\infty}\frac{(\sigma+p-n)_{n}}{(\sigma+p+k-n)_{n}}r^{p+k}$$

$|z|=r,\quad 0\le r<1.$ And the inequalities are sharp

Proof

The result follows by setting $h_{\sigma+p}(z)=\left(f(z)/z^{p}-\beta\right)/(1-\beta)$ in Theorem 2.3.

Equality is attained in upper bound for the function $f(z)=|z|^{p}+2(1-\beta)\sum_{k=1}^{\infty}\frac{(\sigma+p-n)_{n}}{(\sigma+p+k-n)_{n}}|z|^{p+k}$

While equality in the lower bound is realized for the function given by

$$f(z)=|z|^{p}+2(1-\beta)\sum_{k=1}^{\infty}\frac{(\sigma+p-n)_{n}}{(\sigma+p+k-n)_{n}}(-1)^{k}|z|^{p+k}$$

Hence, the proved.

Theorem 3.3

Each function $f(z)$ in the class $\lambda_n^{\sigma,p}(\beta)$ maps the unit disk onto a domain which covers the disk

$$|w| < r^p + 2(1-\beta)\sum_{k=1}^{\infty}\frac{(\sigma+p-n)_n}{(\sigma+p+k-n)_n}(-1)^k r^{p+k}$$

Proof

From theorem 3.2 above we have the inequality below

$$|f(z)| \geq r^p + 2(1-\beta)\sum_{k=1}^{\infty}\frac{(\sigma+p-n)_n}{(\sigma+p+k-n)_n}(-1)^k r^{p+k}$$

This implies that the range of every function $f(z)$ in the class $\lambda_n^{\sigma,p}(\beta)$ covers the disk

$$|w| < r^p + 2(1-\beta)\sum_{k=1}^{\infty}\frac{(\sigma+p-n)_n}{(\sigma+p+k-n)_n}(-1)^k r^{p+k}$$

$$|w| < 1 + 2(1-\beta)\sum_{k=1}^{\infty}\frac{(\sigma+p-n)_n}{(\sigma+p+k-n)_n}(-1)^k$$

$$= \inf_{\mathrm{r}\to 1}\left\{ r^p + 2(1-\beta)\sum_{k=1}^{\infty}\frac{(\sigma+p-n)_n}{(\sigma+p+k-n)_n}(-1)^k r^{p+k} \right\}$$ Hence, proved.

REFERENCES

[1] Babalola, K. O., & Opoola, T. O. (2006). Iterated integral transforms of Caratheodory functions and their applications to analytic and univalent functions. Tamkang Journal of Mathematics, 37(4), 355-366.

[2] Babalola, K. O., (2008). New subclasses of analytic and univalent functions involving certain convolution operators. Mathematica Tome, 50(73), 3-12.

[3] Goel, R. M., & Sohi, N. S. (1980). Subclasses of univalent functions. Tamkang. J.. Math. 77-81.

[4] Noor, K.I. (1998). On an integral operator, J. Nat. Geometry, 13 (2), 127200.

[5] Noor, K. I. (2005). Generalized integral operator and multivalent functions. J. Inequal. Pure Appl. Math, 6, 1-7.

[6] Noor, K. I. (2004). Some classes of p-valent analytic functions defined by certain integral operator. Applied Mathematics and Computation, 157(3), 835-840.

[7] Pommerenke, Ch., Univalent functions, Springer-Verlag, New York Inc., 1983.

[8] Ruscheweyh, S. (1975). New criteria for univalent functions, Proc. Amer. Math. Soc., 49(1), 109115.