International Journal of Mathematics Trends and Technology Volume 67 Issue 10, 59-66, October, 2021
ISSN: 2231 – 5373/doi:10.14445/22315373/IJMTT-V67I10P505

Average Operator of Double-Framed Fuzzy Multicossets

[1,2]G.Rethnarexlin, [3]G.Subbiah and [4]V.Nagarajan

[1]Research scholar, Reg.No: 18123152092017,Research Department of Mathematics,S.T.Hindu College, Nagercoil-629 002,Tamil Nadu, India.

[2]Assistant Professor in Mathematics, Sarah Tucker College, Tirunelveli-627 007, Tamil Nadu, India.

[3]Associate Professor in Mathematics, Sri K.G.S. Arts College, Srivaikuntam-628 619, Tamil Nadu, India.

[4]Assistant Professor in Mathematics, S.T.Hindu College, Nagercoil-629 002,Tamil Nadu, India.

Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627 012, Tamil Nadu, India.

***Abstract** - The concept of the double-framed fuzzy multi group offers a useful technique for real life transportation problems. This special phenomenon is used to model the structure of the controlling an intersection of two-way sheets. In this paper, we study the concept of double-framed fuzzy multi group structures, double-framed fuzzy multi cossets and double-framed multi normal subgroup structures. Finally, we define double-framed fuzzy multi homomorphism between any two double-framed fuzzy multi groups and establish some important properties of this phenomenon.*

Keywords - *Fuzzy set, multi set. multi group, double-framedfuzzy multi set, normal subgroup, cosset, homomorphism, image.*

AMS Subject classification (2020) : 03E72 , 08A05 , 20N25

I. INTRODUCTION

\The theory of collections is a necessary mathematical tool.It gives mathematical models for the class of problems that explains with exactness, precision and uncertainty. Characteristically, non crisp set theory is extensional.More often than not, the real life problems inherently involve uncertainties, imprecision and not clear. In particular, such classes of problems arise in economics, engineering, environmental sciences, medical sciences, and social sciences etc.Zadeh [12] defined fuzzy set theory in his pioneering paper in 1965. In order to solve various types of uncertainties and complex MAGDM problems, the theory of fuzzy sets is proposed by Zadeh [12]. Later on, Atanassov [1] introduced the intuitionistic fuzzy set (IFS) theory to extend the concept of fuzzy set.In an attempt to model uncertainty, the notion of fuzzy sets was proposed by Zadeh [12] as a method for representing imprecision in real-life situations. One can say, a fuzzy set (or a fuzzy subset of a set) is the fuzzification of crisp set to capture uncertainty in a collection. The concept of fuzzy set has grown stupendously over the years giving birth to fuzzy group which is the application of fuzzy sets to the elementary theory of groups and groupoids as noted in [5]. Several works has been done on fuzzy groups since inception; some could be found in [3]. Motivated by Zadeh [12], the idea of fuzzy multisets was introduced in [3] as the generalization of fuzzy sets or the fuzzification of multi sets in [5]. Recently, Shinoj et.al [8] followed the foot steps of Rosenfeld [7] and introduced a non-classical group called fuzzy multigroup. In particular, the idea of fuzzy multigroups generalized fuzzy groups.R.Nagarajan studied fermatean fuzzy multi groups over multi homomorphisms in [6].In this paper, we study the concept of double-framed fuzzy multi group structures, double-framed fuzzy multi cossets and double-framed multi normal subgroup structures. Finally, we define double-framed fuzzy multi homomorphism between any two double-framed fuzzy multi groups and establish some important properties of this phenomenon.

II. PRELIMINARIES

***2.1Definition*:**Let X be a set. A multi set M is characterized by a count function C_M: X $\rightarrow$N , when N = N∪{0}. For each x∈ X, $C_M(x)$ is the characteristic value of x in M. The set of all multi sets of X is denoted by $M_S(X)$.

***2.2 Definition*:** Let X be a group. A multi set M is called a multi group of X if it satisfies the following conditions;

(i) $C_M(xy) \geq \min \{ C_M(x), C_M(y)\}$

(ii) $C_M(x^{-1}) \geq C_M(x)$, for all x ∈X. We denote the set of all multi groups of X by $M_G(X)$.

2.3 Definition: If X is a collection of objects, then a fuzzy set A in X is a set of ordered pairs; A = $\{(x , \mu_A(x) / x \in X, \mu_A : X \rightarrow [0,1]\}$,where$\mu_A$ is called the membership function of A and is defined from X into [0, 1].

For convenience ($C_{MF}(x)$, $C_{NF}(x)$) is called double-framed fuzzy multi number (DFMN) denoted by ($C_{MF}(x)$, $C_{NF}(x)$). A double-framed fuzzy multi sets (DFMSS) F is denoted by

F = $\{ < x , (\mu_A{}^1(x) , \mu_A{}^2(x), \ldots\ldots\ldots\ldots,\mu_A{}^p(x)), (v_A{}^1(x), v_A{}^2(x),\ldots\ldots\ldots\ldots,v_A{}^p(x)) > / x \in X\}$.

2.4 Remark : we arrange the membership sequence in decreasing order but the corresponding non- membership sequence may not be in decreasing or increasing order.

2.5 Definition: Let X be a group. A DFMS 'F' is called a double-framed fuzzy multi group of X if it satisfies the conditions

(i) $C_{MF}(xy) \geq \min \{ C_{MF}(x), C_{MF}(y)\}$and $C_{NF}(xy) \leq \max \{ C_{NF}(x), C_{NF}(y)\}$

(ii) $C_{MF}(x^{-1}) \geq C_{MF}(x)$ and $C_{NF}(x^{-1}) \leq C_{NF}(x)$, for all x ∈ X.

We denote the set of all double-framed fuzzy multi groups of X by DFMG(X).

2.6 Example : Let F_1 = { < 0.7 , 0.3 / x >, < 1 , 0.7 / y > , < 0.2 , 0.7 / z >} and F_2 = { < 0.8 , 0.5 / x > , < 0.7 , 0.9 / y > , < 0.1 , 0.4 / z > } for X = { x,y,z}. Then clearly F_1 and F_2 are double-framed fuzzy multi groups of X.

2.7 Example : Assume that X = {a ,b, c} is a set. Then for

$C_{MF}(a)$ = { 1,0.5. 0.7}, $C_{MF}(b)$ = { 0.3. 0.7}, $C_{MF}(c)$ = { 0},

$C_{NF}(a)$ = { 0.2,0.5. 0.7}, $C_{NF}(b)$ = { 1,0.7}, $C_{NF}(c)$ = { 0.3},

A DFMS of X written as F ={< 1, 0.5, 0.7 / a > , < 0.3, 0.7 / b >, < 0.2, 0.5, 0.7 / a >, < 0.3 /c >}.

2.8 Definition : Let C be a non-empty crisp set and $\mathcal{F}_1 = (\mu_1(m), \upsilon_1(m))$ and $\mathcal{F}_2 = (\mu_2(m), \upsilon_2(m))$ be DFS's on C. Then

(i) $\mathcal{F}_1 = \mathcal{F}_2$ if and only if $\mu_1(m) \leq \mu_2(m)$ and $\upsilon_1(m) \geq \upsilon_2(m)$.

(ii) $\mathcal{F}_1 = \mathcal{F}_2$ if and only if $\mathcal{F}1 \subset \mathcal{F}2$ and $\mathcal{F}2 \subset \mathcal{F}1$.

(iii) $\mathcal{F}^c{}_1 = (\upsilon_1(m), \mu_1(m))$.

(iv) $\mathcal{F}1 \cap \mathcal{F}2 = (\min \{ \mu_1(m) ,\mu_2(m)\}, \max \{\mu_1(m) ,\mu_2(m)\})$.

(v) $\mathcal{F}1 \cup \mathcal{F}2 = (\max \{ \mu_1(m) ,\mu_2(m)\}, \min \{\mu_1(m) ,\mu_2(m)\})$.

(vi) $[\ \ ]\mathcal{F}1 = (\mu_1(m) ,1 - \mu_2(m)\}$, $<>\mathcal{F}1 = (1 - \upsilon_2(m), \upsilon_1(m))$.

***2.9 Definition*:** Let F_1, F_2∈ DFMS(X). Then F_1 is called a double-framed fuzzy submulti set of F_2 written as $F_1 \subseteq F_2$ if $C_{MF1}(x) \leq C_{MF2}(x)$, $C_{NF1}(x) \leq C_{NF2}(x)$, for all x ∈ X.

***2.10 Definition*:** Let α and β be positive real numbers lie in the closed unit interval such that $0 \leq \alpha + \beta \leq 1$.Then (α , β)-cut set of DFMS 'F' of the universe 'U' is a crisp set consisting of all these elements of U for which $C_{MF}(x) \geq \alpha$ and $C_{NF}(x) \leq \beta$ for all $x \in U$.

***2.11 Remark*:** A DFMS 'F' of a group is DFMG if each of its (α , β)-cut set is a subgroup of G.

***2.12 Definition*:** A DFMS 'F' is said to be double-framed fuzzy multi normal subgroup (DFMNG) if it meets the following conditions;

(i) $C_{MF}(xy) = C_{MF}(yx)$ and (ii) $C_{NF}(xy) = C_{NF}(yx)$, for all $x,y \in G$.

2.13 Definition: Let F_1 and F_2 be two DFMS's of the universe U.Then the average operator F_1 \$ F_2 is defined as $F_1\$F_2 = \{ < s_1 , \sqrt{C_{MF1}(x)\, C_{MF2}(x)} , \sqrt{C_{NF1}(x)\, C_{NF2}(x)} > / x \in U\}$.

III. PROPERTIES OF DOUBLE-FRAMED FUZZY MULTI GROUP

In this section, we studydouble-framed fuzzy multi subgroup. Moreover, numerous useful results and algebraic properties are introduced.

Definition 3.1: Suppose F is a DFMS of a universe S and $\Lambda \in [\, 0 , 1]$. Then DFMS

$F^{\Lambda} = (C_{MF}^{\Lambda} , C_{NF}^{\Lambda})$ is called Λ – DFMS of universe S with respect to DFMS 'F'; where

$C_{MF}^{\Lambda}(x) = ℘ (C_{MF}(x) , \Lambda)$, $C_{NF}^{\Lambda}(x) = \Omega (C_{NF}(x) , 1-\Lambda)$ and ℘ , Ω denote the average operators defined in 2.13.

Proposition 3.2: Let F_1 and F_2 be two Λ – DFMS's of the universe U. Then so is $F_1^{\Lambda} \cap F_2^{\Lambda}$.

Proof: Consider

$C_{M(F1 \cap F2)}^{\Lambda} (x) = ℘ \{ C_{M(F1 \cap F2)}(x) , \Lambda \} = ℘ \{ \min \{ C_{MF1}(x) , C_{MF2}(x)\} , \Lambda \}$

$= ℘ \{ \min \{ C_{MF1}(x) , \Lambda\} , \min \{C_{MF2}(x) , \Lambda \}\}$

$= \min \{ C_{MF1}^{\Lambda} (x) , C_{MF2}^{\Lambda} (x) \} = C_{M(F1^{\Lambda} \cap F2^{\Lambda})}$for all $x \in U$.

Similarly, it can be proved that

$C_{N(F1 \cap F2)}^{\Lambda} (x) = C_{N(F1^{\Lambda} \cap F2^{\Lambda})}(x)$, for all $x \in U$.Hence $(F_1 \cap F_2)^{\Lambda} = F_1^{\Lambda} \cap F_2^{\Lambda}$.

Remark 3.3: The union of any two DFMS's is also DFMS.

Definition 3.4: A DFMS of a group G is DFMG 'F^{Λ}' satisfying the following conditions(i) $C_{MF}^{\Lambda}(xy) \geq \min \{ C_{MF}^{\Lambda}(x), C_{MF}^{\Lambda}(y)\}$and $C_{NF}^{\Lambda}(xy) \leq \max \{ C_{NF}^{\Lambda}(x), C_{NF}^{\Lambda}(y)\}$ (ii) $C_{MF}^{\Lambda}(x^{-1}) \geq C_{MF}^{\Lambda}(x)$ and $C_{NF}^{\Lambda}(x^{-1}) \leq C_{NF}^{\Lambda}(x)$, for all $x, y \in G$.

Remark3.5: Let 'e' be an identity element of G. Then $F^{\Lambda}(x) \leq F^{\Lambda}(e)$ for all $x \in G$. Also, $F^{\Lambda}(xy^{-1}) = F^{\Lambda}(e)$ implying that $F^{\Lambda}(x) = F^{\Lambda}(y)$, for all $y \in G$.

Proposition 3.6: Every double-framed fuzzy multi group of G is an double-framedfuzzy multi normal subgroup of G.

Proof: Let $x,y \in G$.Then by using the fact that F is double-framed fuzzy multi group. we have

$C_{MF}^{\Lambda} (xy) = ℘\{ C_{MF}(xy) , \Lambda \}$

$\geq ℘\{ \min \{C_{MF}(x) , C_{MF}(y) , \Lambda \}$

$= ℘\{ \min \{ C_{MF}(x) , \Lambda\} , \min \{C_{MF}(y) , \Lambda \}$

$= \min \{ C_{MF}^{\Lambda}(x) , C_{MF}^{\Lambda}(y) \}$

Similarly, it can be proved that $C_{NF}^{\Lambda} (xy) \leq \max \{ C_{NF}^{\Lambda}(x) , C_{NF}^{\Lambda}(y) \}$.

Moreover,$C_{MF}^{\Lambda} (x^{-1}) = ℘ \{C_{MF} (x^{-1}) , \Lambda \} = ℘ \{C_{MF} (x) , \Lambda \} = C_{MF}^{\Lambda} (x)$.

Similarly, $C_{NF}^{\Lambda}(x^{-1}) = \wp\{C_{NF}(x^{-1}), \Lambda\} = \wp\{C_{NF}(x), \Lambda\} = C_{NF}^{\Lambda}(x)$.

Consequently, F^{Λ} is a double-framedfuzzy multi normal subgroup of G.

***Remark 3.7*:** A double-framedfuzzy multi group need not be a double-framed fuzzy multi normal subgroup, that is the converse of proposition 3.6 does not hold.

The above fact can be explained in the following example.

Example 3.8: Let G = {e,a,b,c } be the Kelin's -4 group. we define DFMS F of G as

F = { < e,0.4 ,05 > , < a, 0.3 ,0.5> , < b, 0.2, 0.8 > , < c ,0.1,0.7 > }. Note that, F is not DFMG of G.Let $\Lambda = 0.5$. ThenF^{Λ}= {< e,0.4 ,0.5 > , < a, 0.3 ,0.5> , < b, 0.2, 0.8 > , < c ,0.1,0.7 > }.It is clear that (0.4 , 0.5)-cut set of 0.5 . A DFMS is $\Lambda^{0.5}$ = {a} and $\Lambda^{0.6}$= {a} , whereas (0.5 , 0.8)-cut set of 0.5. DFMS is given by $\Lambda^{0.5}$ = {e , a} and $\Lambda^{0.6}$ = {e,a,b,c}.

Note that, each of the above cut set of double-framedfuzzy multi set is a subgroup of G. Hence it is adouble-framedfuzzy multi group.

The following results present the condition under which a given double-framedfuzzy multi group is a double-framedfuzzy multi group.

Proposition 3.9: Let F be any DFMS of a group G such that $C_{MF}(x^{-1}) = C_{MF}(x)$ and $C_{NF}(x^{-1}) = C_{NF}(x)$ for all $x \in G$.Moreover , $\Lambda < \min\{a, 1-b\}$,

where $a = \min\{C_{MF}(x) / x \in G\}$

and $b = \max\{C_{NF}(x) / x \in G\}$. Then F is adouble-framedfuzzy multi group of G.

Proof: In view of given conditions, we have $a > \Lambda$ and $b < 1- \Lambda$.

It follows that $C_{MF}(x) > \Lambda$ and $C_{NF}(x) < 1- \Lambda$, for all $x \in G$.

Therefore , $C_{MF}^{\Lambda}(xy) \geq \min\{C_{MF}^{\Lambda}(x), C_{MF}^{\Lambda}(y)\}$ and

$C_{NF}^{\Lambda}(xy) \leq \max\{C_{NF}^{\Lambda}(x), C_{NF}^{\Lambda}(y)\}$ for all $x,y \in G$.

Moreover, for any $x \in G$, we obtain

$C_{MF}(x^{-1}) = C_{MF}(x)$ and

$C_{NF}(x^{-1}) = C_{NF}(x)$. This shows that

$C_{MF}^{\Lambda}(x^{-1}) = C_{MF}^{\Lambda}(x)andC_{NF}^{\Lambda}(x^{-1}) = C_{NF}^{\Lambda}(x)$.

The subsequent result indicates that the intersection of any two double-framedfuzzy multi groups is a double-framedfuzzy multi group of G.

***Proposition 3.10*:** The intersection of twodouble-framedfuzzy multi groups of a group G is also adouble-framedfuzzy multi group.

Proof: Suppose F_1^{Λ} and F_2^{Λ} are double-framedfuzzy multi groups of a group G.Then

$C_{M(F1 \cap F2)}^{\Lambda}(xy) = \wp\{C_{M(F1 \cap F2)}(xy), \Lambda\}$

$= \wp\{\min\{C_{MF1}(xy), C_{MF2}(xy)\}, \Lambda\}$

$= \wp\{\min\{C_{MF1}(xy), \Lambda\}, \min\{C_{MF2}(xy), \Lambda\}\}$

$= \min\{C_{MF1}^{\Lambda}(xy), C_{MF2}^{\Lambda}(xy)\}$

$= C_{M(F1^{\Lambda} \cap F2^{\Lambda})}(xy)$, for all $x \in U$.

$\geq \min\{\min\{C_{MF1}^{\Lambda}(x), C_{MF2}^{\Lambda}(x)\}, \min\{C_{MF1}^{\Lambda}(y), C_{MF2}^{\Lambda}(y)\}\}$

$= \min\{C_{M(F1^{\Lambda} \cap F2^{\Lambda})}(x), C_{M(F1^{\Lambda} \cap F2^{\Lambda})}(y)\}$.

Similarly, it can be proved that for all $x,y \in U$,

$C_{N(F1 \cap F2)^{\Lambda}}(xy) \leq \max \{ C_{N(F1^{\Lambda} \cap F2^{\Lambda})}(x) , C_{N(F1^{\Lambda} \cap F2^{\Lambda})}(y)\}$.Also

$C_{M(F1 \cap F2)^{\Lambda}}(x^{-1}) = \wp \{C_{M(F1 \cap F2)}(x^{-1}), \Lambda \}$

$= \wp\{ \min\{C_{MF1}(x^{-1}), \Lambda\}, \min \{C_{MF2}(x^{-1}), \Lambda\}$

$= \min \{C_{MF1}(x^{-1}),C_{MF2}(x^{-1})\}$

$= C_{M(F1^{\Lambda} \cap F2^{\Lambda})}(x)$.

Similarly,$C_{N(F1 \cap F2)^{\Lambda}}(x^{-1}) = C_{N(F1^{\Lambda} \cap F2^{\Lambda})}(x)$.

***Corollary 3.11*:** The intersection of any number of double-framedfuzzy multi groups of a group G is also adouble-framedfuzzy multi group of G.

***Remark 3.12*:** The Union of any number of double-framedfuzzy multi groups of a group G may not be a double-framedfuzzy multi group of G.

***Definition 3.13*:** Let F^{Λ} be adouble-framedfuzzy multi group ofG and $x \in G$.Adouble-framedfuzzy multi right cosset of 'F' in G, denoted by $F^{\Lambda}x$ is defined as

$F^{\Lambda} x(g) = (C_{MF}^{\Lambda} x(g) , C_{NF}^{\Lambda}x(g))$,

where $C_{MF}^{\Lambda}x (g) = \wp (C_{MF}(gx^{-1}) , \Lambda)$, $C_{NF}^{\Lambda}x(g) = \Omega (C_{NF}(gx^{-1}) , 1-\Lambda)$ for all $g \in G$.

Similarly, we can define a double-framedfuzzy multi left cosset of F in G.

Definition 3.14: Adouble-framedfuzzy multi group F^{Λ} of G is called double-framedfuzzy multi normal subgroup of G if $xF^{\Lambda} = F^{\Lambda}x$, for all $x \in G$.

The following results shows that every double-framed fuzzy multi normal subgroup of G is also double-framedfuzzy multi normal subgroup of G.

***Proposition 3.15*:** If F is a double-framed fuzzy multi normal subgroup of a group G, then F^{Λ} is also adouble-framedfuzzy multi normal subgroup of G.

Proof: Let F be a double-framedfuzzy multi normal subgroup of G. Then for all $x,g \in G$.

$C_{MF}(gx^{-1}) = C_{MF}(x^{-1}g)$ and

$C_{NF}(gx^{-1}) = C_{NF}(x^{-1}g)$.

Implying that$C_{MF}^{\Lambda}(x^{-1}g) = \wp (C_{MF}(x^{-1}g) ,\Lambda)$

$= \wp (C_{MF}(gx^{-1}) , \Lambda)$

$= C_{MF}^{\Lambda}(gx^{-1})$.

Similarly, we can prove that $C_{NF}(gx^{-1}) = C_{NF}(x^{-1}g)$. Consequently, $xF^{\Lambda} = F^{\Lambda}x$.

The converse of the given result does not hold generally. This fact can be viewed in the successive example.

***Example 3.16*:** Let $D_3 = \langle a,b : a^3 = b^2 = e, ba = a^2b \rangle$ be dihedral group of order 6.Define double-framed fuzzy multi normal subgroup of G 'F' of D_3 by

$$C_{MF}(x) = \begin{cases} 0.63 \text{ if } x \in \langle 6 \rangle \\ 0.40 \text{ otherwise and} \end{cases}$$

$$C_{NF}(x) = \begin{cases} 0.70 \text{ if } x \in \langle 6 \rangle \\ 0.83 \text{ otherwise} \end{cases}$$

Then $F = \{ \langle e,0.63 ,0.70 \rangle , \langle a, 0.40 ,0.83\rangle , \langle a^2, 0.40, 0.86 \rangle , \langle b ,0.63,0.70\rangle, \langle ab, 0.40. 0.63\rangle, \langle a^2b, 0.63, 0.63\rangle\}$.

Since $C_{MF}(xy) = 0.40 \neq 0.63 = C_{MF}(yx)$, therefore F is not a double-framed fuzzy multi normal subgroup of G.

Next, let $\Lambda = 0.2$, then $\Lambda^{0.2} = \{ <e,0.4 ,0.4 > , < a, 0.4 ,0.4> , < a^2, 0.4, 0.4 > , < b ,0.4,0.4> , <ab, 0.4. 0.4>, <a^2b, 0.4, 0.4>\}$.

One can see that $\Lambda^{0.2}$ is a double-framedfuzzy multi normal subgroup of G.

In the following result, a condition for double-framedfuzzy multi subgroup of G to be double-framedfuzzy multi normal subgroup of G is established.

***Proposition 3.17*:** Let F^{Λ} be a double-framedfuzzy multi subgroup of a group G such that

$\Lambda < \min \{ a , 1\text{-}b\}$, where a = = ₰ ($C_{MF}(x) / x \in G\}$ and b = Ω ($C_{NF}(x) / x \in G\}$.

Then F^{Λ} is a double-framedfuzzy multi subgroup of a group G.

Proof: Since $a > \Lambda$ and $b < 1\text{-} \Lambda$, thereforemin $\{ C_{MF}(x) / x \in G\} > \Lambda$ and

$\max \{ C_{NF}(x) / x \in G\} < 1\text{-}\Lambda$.Thus $C_{MF}(x) > \Lambda$ for all $x \in G$. Also, $C_{NF}(x) < 1\text{-} \Lambda$, so

$C_{MF}{}^{\Lambda}(xg)$ = ₰ $(C_{MF}(gx^{-1}) , \Lambda) = \Theta$ and

$C_{NF}{}^{\Lambda}(xg) = \Omega (C_{MF}(gx^{-1}) , 1\text{-}\Lambda) = \acute{\emptyset}$, for all $g \in G$.

Similarly, $C_{MxF}{}^{\Lambda} (g)$ = ₰ $(C_{MF}(x^{-1}g) , \Lambda) = \Theta$ and $C_{N\,xF}{}^{\Lambda} (g) = \Omega (C_{NF}(x^{-1}g) , 1\text{-}\Lambda) = \acute{\emptyset}$.

This concludes the proof.

IV. DOUBLE-FRAMEDFUZZY MULTI HOMOMORPHISMS

In this section, we define double-framedfuzzy multi homomorphism between any two double-framed fuzzy multi groups and establish some important properties of this phenomenon.

Defnition 4.1: Let $F_1{}^{\Lambda}$and $F_2{}^{\Lambda}$ be two double-framedfuzzy multi groups of the groups G_1 and G_2 respectively and $\varsigma : G_1 \rightarrow G_2$ be a group homomorphism from F^{Λ} to $F_2{}^{\Lambda}$ if $\varsigma (F_1{}^{\Lambda}) = F_2{}^{\Lambda}$.

The following results indicates that adouble-framedfuzzy homomorphic image of the double-framedfuzzy multi group is a double-framedfuzzy multi group.

Theorem 4.2: Let F^{Λ} be adouble-framedfuzzy multi group of the groupG and $\varsigma : G_1 \rightarrow G_2$ be a surjective homomorphism. Then $\varsigma (F^{\Lambda})$ is a double-framedfuzzy multi group of G.

Proof: In view of the given condition, for any two elements $p, q \in G_2$, there exist $x ,y \in G$ such that $\varsigma (x) = p$ and $\varsigma (y) = q$.

Consider $\varsigma (F)^{\Lambda} (pq) = (C_{M\varsigma (F)}{}^{\Lambda} (pq) , C_{M\varsigma (F)}{}^{\Lambda} (pq))$. Which implies that

$C_{M\varsigma (F^{\Lambda})} (pq) = C_{M\,\varsigma (F^{\Lambda})}(pq)$ = ₰ ($C_{M\varsigma (F)} (\varsigma (x) \varsigma (y)) , \Lambda)$

= ₰ ($C_{M\varsigma (F)} (\varsigma (xy)) , \Lambda)$

$\geq$ ₰ ($C_{MF}(xy) , \Lambda)$

$= C_{MF}{}^{\Lambda} (xy)$

$\geq \min \{ C_{MF}{}^{\Lambda}(x) , C_{MF}{}^{\Lambda}(y) \}$

$= \min \{ C_{MF}{}^{\Lambda}(p) , C_{MF}{}^{\Lambda}(p) \}$.

Thus $C_{M\,\varsigma (F^{\Lambda})} (pq) \geq \min \{ C_{M\varsigma(F)}{}^{\Lambda}(p) , C_{M\varsigma(F)}{}^{\Lambda}(p) \}$.

Similarly, it can be proved that

$C_{N\,\varsigma (F^{\Lambda})} (pq) \leq \max \{ C_{N\varsigma(F)}{}^{\Lambda}(p) , C_{N\varsigma(F)}{}^{\Lambda}(p) \}$.Also,

$C_{M\,\varsigma (F^{\Lambda})} (p^{-1}) = \max \{ C_{MF}{}^{\Lambda} (x^{-1}) : \varsigma(x^{-1}) = p^{-1} \}$

$= \max \{ C_{MF}{}^{\Lambda} (x) : \varsigma(x) = p \}= C_{M\,\varsigma (F^{\Lambda})} (p)$.

Similarly , $C_{N\varsigma(F^{\Lambda})}(p^{-1}) = C_{N\varsigma(F^{\Lambda})}(p)$.

Theorem 4.3: Let F^{Λ} be adouble-framedfuzzy multi normal subgroup of the group G_1 and $\varsigma : G_1 \rightarrow G_2$ be a bijective homomorphism. Then $\varsigma(F^{\Lambda})$ is adouble-framedfuzzy multi normal subgroup of G_2.

Proof: In view of the given condition , for any two elements $p, q \in G_2$, there exists a unique pair of elements $x, y \in G$ such that $\varsigma(x) = p$ and $\varsigma(y) = q$.

Consider $\varsigma(F^{\Lambda})(pq) = (C_{M\varsigma(F^{\Lambda}))}(pq) , C_{M\varsigma(F^{\Lambda}))}(pq))$.Then

$C_{M\varsigma(F)}{}^{\Lambda}(pq)=$ ʃϑ $(C_{M\varsigma(F)}(\varsigma(x)\varsigma(y)) , \Lambda)$

$=$ ʃϑ $(C_{M\varsigma(F)}(\varsigma(xy)) , \Lambda)$

$\geq$ ʃϑ $(C_{MF}(xy) , \Lambda)$

which follows that$=$ ʃϑ $(C_{MF}{}^{\Lambda}(yx))$

(since $C_{MF}{}^{\Lambda}(xy) = C_{MF}{}^{\Lambda}(yx)=$ ʃϑ$\{ C_{M\varsigma(F)}(_{\varsigma(yx)}) , \Lambda \}$

$=$ ʃϑ$\{ C_{M\varsigma(F)}(qp), \Lambda \}$

$= C_{M\varsigma(F)}{}^{\Lambda}(qp)$.

Similarly , $C_{M\varsigma(F)}{}^{\Lambda}(pq) = C_{M\varsigma(F)}{}^{\Lambda}(qp)$.

The following theorem shows that every double-framedfuzzy multi inverse homomorphic image of double-framedfuzzy multi group is always double-framedfuzzy multi group.

***Theorem 4.4*:** Let $F_2{}^{\Lambda}$ be adouble-framedfuzzy multi group of the group G and $\varsigma : G_1 \rightarrow G_2$ be a group homomorphism. Then $\varsigma^{-1}(F_2{}^{\Lambda})$ is adouble-framedfuzzy multi group of G_1.

Proof: Suppose $F_2{}^{\Lambda}$ be adouble-framedfuzzy multi group of the group G_2 , then there exists a unique pair of elements $x,y \in G_1$ such that

$\varsigma^{-1}(F_2{}^{\Lambda})(xy) = (C_{M\varsigma-1(F2^{\Lambda})}(xy) , C_{N\varsigma-1(F2^{\Lambda})}(xy))$.Also

$C_{M\varsigma-1(F2^{\Lambda})}(xy) = C_{MF2}{}^{\Lambda}(\varsigma(xy)) = C_{MF2}{}^{\Lambda}(\varsigma(x)\varsigma(y))$

$\geq \min \{ C_{MF2}{}^{\Lambda}(\varsigma(x)), C_{MF2}{}^{\Lambda}(\varsigma(y)) \}$

$= \min \{ C_{M\varsigma-1(F2^{\Lambda})}(x) , C_{M\varsigma-1(F2^{\Lambda})}(y)\}$

Thus $C_{M\varsigma^{-1}(F2^{\Lambda})}(xy) \geq \min \{ C_{M\varsigma^{-1}(F2^{\Lambda})}(x) , C_{M\varsigma^{-1}(F2^{\Lambda})}(y)\}$.

Similarly, we will show that

$C_{N\varsigma^{-1}(F2^{\Lambda})}(xy) \leq \max \{ C_{N\varsigma^{-1}(F2^{\Lambda})}(x) , C_{N\varsigma^{-1}(F2^{\Lambda})}(y)\}$.Also

$C_{M\varsigma^{-1}(F2^{\Lambda})}(p^{-1}) = C_{MF2}{}^{\Lambda}(\varsigma(p^{-1}))$

$= C_{M(F2)}{}^{\Lambda}(\varsigma(p)^{-1})$

$= C_{M(F2)}{}^{\Lambda}(\varsigma(p)) = C_{M\varsigma^{-1}(F2^{\Lambda})}(p)$.

Similarly, $C_{N(F2)}{}^{\Lambda}(\varsigma(p)^{-1}) = C_{N(F2)}{}^{\Lambda}(\varsigma(p)) = C_{N\varsigma^{-1}(F2^{\Lambda})}(p)$.

consequently $\varsigma^{-1}(F_2{}^{\Lambda})$ is a double-framedfuzzy multi group of $G_{1.}$

The following theorem, we show that every double-framedfuzzy multi homomorphic inverse image of double-framedfuzzy multi normal subgroup of G is a double-framedfuzzy multi normal subgroup ofG.

***Theorem 4.5*:** Let $F_2{}^{\Lambda}$ be adouble-framedfuzzy multi normal subgroup of the group G_2 and $\varsigma : G_1 \rightarrow G_2$ be a group homomorphism. Then $\varsigma^{-1}(F_2{}^{\Lambda})$ is adouble-framedfuzzy multi normal subgroup of G_1.

Proof: Suppose F_2^{Λ} is adouble-framedfuzzy multi normal subgroup of the group G_2 , then there exists a unique pair of elements x,y $\in G_1$ such that

$\varsigma^{-1} (F_2^{\Lambda}) (xy) = (C_{M \varsigma^{-1} (F2^{\Lambda})} (xy) , C_{N \varsigma^{-1} (F2^{\Lambda})} (xy))$.Also

$C_{M \varsigma^{-1} (F2^{\Lambda})} (xy) = C_{MF2^{\Lambda}} (\varsigma(xy))$

$= C_{MF2^{\Lambda}} (\varsigma(x) \varsigma (y))$

$= C_{MF2^{\Lambda}} (\varsigma(y) \varsigma (x))$

$= C_{M \varsigma^{-1} (F2^{\Lambda})} (yx)$

Similarly, we can prove that$C_{N \varsigma^{-1} (F2^{\Lambda})} (xy)= C_{N \varsigma^{-1} (F2^{\Lambda})} (yx)$.

Thus $\varsigma^{-1}(F_2^{\Lambda})$ is adouble-framedfuzzy multi normal subgroup of G_1.

***Conclusion*:** In this article, double-framedfuzzy multi set generalizes the idea of classical fuzzy multi set intending to assess the fuzziness level of uncertainty situation. We have presented cossets, normal subgroups of double-framedfuzzy multi set and its applications. Also we demonstrated the effectiveness of the image and inverse image of double-framedfuzzy multi normal subgroup followed by fuzzy multi homomorphism.

REFERENCES

[1] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems., 20(1) (1986) 87-96.
[2] A. Baby,T.K.Shinojand J.J.Sunil, On abelian fuzzy multigroups and order of fuzzy multigroups, J. New Theory, 5(2) (2015) 80–93.
[3] S. Miyamoto, Basic operations of fuzzy multisets, J.JapanSoci. Fuzzy Theory Systems, 8(4) (1996) 639–645.
[4] J.M.Mordeson,K.R.Bhutaniand A.Rosenfeld, Fuzzy group theory, SpringerVerlag Berlin Heidelberg, (2005).
[5] Sk.Nazmul, P.Majumdar and S.K.Samanta, On multisets and multigroups, Ann. Fuzzy Math. Inform., 6(3) (2013)643–656.
[6] R.Nagarajan, Fermatean fuzzy multigroup over multi homomorphisms, International journal of Computer Science Engineering andTechnology, 11 (2021), 18-22.
[7] A.Rosenfeld, Fuzzy subgroups, J.Mathl. Analy. Appl., 35(1971)512-517.
[8] T.K.Shinoj, A.Baby and J.J.Sunil, On some algebraic structures of fuzzy multisets, Ann. Fuzzy Math. Inform., 9(1) (2015) 77-90.
[9] Yager.R.R., Pythagorean uncertainty subsets, In Proceedings Joint IFSA World Congress and NAFIPS Annual Meeting, Edmonton, Canada, (2013) 57–61.
[10] Yager.R.R., Pythagorean membership grades in multi criteria decision making, IEEE Transactions on Fuzzy Systems, 22 (2014) 958–965.
[11] Yager.R.Rand Abbasov.A.M., Pythagorean membership grades, complex numbers, and decision making, International Journal of Intelligent Systems, 28 (2013) 436–452.
[12] L.A.Zadeh, Fuzzy sets, Information and Control, 8(1965) 338 – 353.