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On the Proof of an Inequality Related to the Mathieu Series

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Abstract - *In this paper, we first show that there is a mistake in the proof of an inequality related to the Mathieu series which given by J. Ernest Wilkins Jr., but his proof method is still valuable and can be modified. Follow this method we have improved the corresponding proof process in the last section.*

Keywords - *Mathieu series, Alzer-Brenner-Ruehr's inequality, the trigamma function, Riemann Zeta function*.

I. INTRODUCTION

Horst Alzer and Joel Brenner Proposed the following conjecture as an open question in [1], prove and disprove that

$$\left(\sum_{n=1}^{\infty}\frac{n}{(n^2+z)^2}\right)^2 \geq \sum_{n=1}^{\infty}\frac{n}{(n^2+z)^3}\ (z>0). \tag{1.1}$$

Soon after, J. Ernest Wilkins, Jr. gave a solution of this conjecture in [2]. Then based on this result, Horst Alzer, Joel Brenner and O.G. Ruehr gave a graceful inequality for the Mathieu series as follows.

$$\frac{1}{x^2+1/(2\varsigma(3))} < \sum_{n=1}^{\infty}\frac{2n}{(n^2+x^2)^2} < \frac{1}{x^2+1/6}. \tag{1.2}$$

The inequality (1.2) plays an important role in the study of the Mathieu series. There is a huge literature about the Mathieu series (interested readers could find useful information in paper [4-10], and the references therein.). However, there is a mistake in the J. Ernest Wilkins, Jr.'s solution of inequality (1.1). we will first show in the next section that the inequality (1.1) can only be proved for $0 \leq z \leq 0.08648913729$ and $z > 1.19323408312446$ according to the method given in [2]. But J. Ernest Wilkins, Jr.'s proof method is still valuable and can be modified. Follow this method we will give a complete proof of the inequality (1.1) in the third section.

II. THE MISTAKE OF WILKINS'S PROOF AND THE RESULT OF HIS METHOD

We recall some necessary marks and expressions used in [2]. Let $F(z)=\{F_1(z)\}^2-F_2(z)$, in which

$$F_1(z)=\sum_{n=1}^{\infty}\frac{n}{(n^2+z)^2},\quad F_2(z)=\sum_{n=1}^{\infty}\frac{n}{(n^2+z)^3}. \tag{2.1}$$

Obviously, the inequality (1.1) is equivalent to $F(z) \geq 0$ $(z > 0)$.

For the trigamma function

$$\psi'(u)=\sum_{n=0}^{\infty}\frac{1}{(n^2+u)^2}\ (u>0),$$

it holds

$$F_1(z)=\frac{i}{4y}\{\psi'(iy)-\psi'(-iy)\},\ (y=z^{1/2}). \tag{2.2}$$

By using the following formula (see[2,(4)])

$$\psi'(u)=\frac{1}{u}+\frac{1}{2u^2}+\sum_{s=1}^{m-1}\frac{B_{2s}}{u^{2s+1}}+(2m+1)\int_0^{\infty}\{B_{2m}-B_{2m}(v)\}(x+u)^{-(2m+2)}dx, \tag{2.3}$$

in which B_{2m} is the Bernoulli number of order $2m$, $B_{2m}(v)$ is the Bernoulli polynomial of degree $2m$, and $v = x - [x]$. Specially, when $m = 3$, we obtain that

$$\psi'(iy)-\psi'(-iy)=\frac{2}{iy}-\frac{1}{3iy^3}-\frac{1}{15iy^5}+7\int_0^\infty\{B_6-B_6(v)\}\{(x+iy)^{-8}-(x-iy)^{-8}\}dx\,. \tag{2.4}$$

We write

$$I_{2m}=\frac{(2m+1)i}{4y}\int_0^\infty\{B_{2m}-B_{2m}(v)\}\{(x+iy)^{-(2m+2)}-(x-iy)^{-(2m+2)}\}dx\,. \tag{2.5}$$

Then the expression (5) in [2] may write as follows

$$F_1(z)=\frac{1}{2z}-\frac{1}{12z^2}-\frac{1}{60z^3}+I_6\,. \tag{2.6}$$

Using the follow equation

$$\begin{aligned}(x+iy)^{-8}-(x-iy)^{-8}&=(x^2+y^2)^{-8}\{(x-iy)^{-8}-(x+iy)^{-8}\}\\&=-16x(iy)(x^2+y^2)^{-8}(x^6-7x^4y^2+7x^2y^4-y^6)\,,\end{aligned}$$

we get

$$I_6=28\int_0^\infty\{B_6-B_6(v)\}(x^2+z)^{-8}(x^6-7x^4z+7x^2z^2-z^3)xdx\,. \tag{2.7}$$

However, the above expression is written in the original proof as follows (see[2, (7)]).

$$I_6=14\int_0^\infty\{B_6-B_6(v)\}(x^2+z)^{-8}(2x^6-7x^4z+7x^2z^2-2z^3)xdx\,, \tag{2.8}$$

this is obviously a mistake. Thus, each step next the equation (7) in [2] must be recalculated. Following the steps of [2], we only will prove that (1.1) is true when $z\geq 1.19323408312446$ for the part of sufficiently large z. Now we show it as follows.

By [2], we have

$$F_2(z)=-\frac{1}{2}F_1'(z)=\frac{1}{4z^2}-\frac{1}{12z^3}-\frac{1}{40z^4}-\frac{1}{2}I_6'\,. \tag{2.9}$$

in which

$$-\frac{1}{2}I_6'=14\int_0^\infty\{B_6-B_6(v)\}(x^2+z^2)^{-9}(15x^6-63x^4z+45x^2z^2-5z^3)xdx\,. \tag{2.10}$$

Now make the substitution, $x=(sz)^{1/2}$, then (2.7) and (2.10) can be transformed into the following formula successively.

$$I_6=\frac{14}{z^4}\int_0^\infty\{B_6-B_6(v)\}(s+1)^{-8}(s^3-7s^2+7s-1)ds\,. \tag{2.11}$$

$$-\frac{1}{2}I_6'=\frac{7}{z^5}\int_0^\infty\{B_6-B_6(v)\}(s+1)^{-9}(15s^3-63s^2+45s-5)ds\,. \tag{2.12}$$

Combining (2.6), (2.9), (2.11) and (2.12), we obtain that

$$F(z)=F_1^2(z)-F_2(z)=\frac{11}{720z^4}+\frac{1}{360z^5}+\frac{1}{3600z^6}+I_6^2+J\,, \tag{2.13}$$

in which

$$\begin{aligned}J&=\left(\frac{1}{z}-\frac{1}{6z^2}-\frac{1}{30z^3}\right)I_6+\frac{1}{2}I_6'\\&=\left(\frac{1}{z}I_6+\frac{1}{2}I_6'\right)-\left(\frac{1}{6z^2}+\frac{1}{30z^3}\right)I_6\\&=\frac{7}{z^5}J_1-\left(\frac{7}{3z^6}+\frac{7}{15z^7}\right)J_2\,,\end{aligned} \tag{2.14}$$

where

$$J_1=\int_0^\infty\{B_6-B_6(v)\}(s+1)^{-9}f_1(s)ds\,,\ \text{and}\ \ f_1(s)=2s^4-27s^3+63s^2-33s+3\,,$$

$$J_2=\int_0^\infty\{B_6-B_6(v)\}(s+1)^{-8}f_2(s)ds\,,\ \text{and}\ f_2(s)=s^3-7s^2+7s-1\,.$$

Furthermore, $f_1(s)=2(s-s_{11})(s-s_{12})(s-s_{13})(s-s_{14})$, and $s_{11}\approx 0.114870309645755$, $s_{12}\approx 0.579199667765806$, $s_{13}\approx 2.10730378832133$, $s_{14}\approx 10.69862623426710$, $f_2(s)=(s-s_{21})(s-s_{22})(s-s_{23})$, and $s_{21}=3-2\sqrt{2}$, $s_{22}=1$, $s_{23}=3+2\sqrt{2}$. Because $f_1(s)<0$ when $s_{11}<s<s_{12}$ and when $s_{13}<s<s_{14}$, and $f_2(s)>0$ when $s_{21}<s<s_{22}$ and when $s>s_{23}$.

Using the following inequality

$$0\le B_6-B_6(v)\le\frac{3}{64}\,,\ \text{for}\ 0<v<1,$$

we get

$$J_1>\frac{3}{64}\left(\int_{s_{11}}^{s_{12}}+\int_{s_{13}}^{s_{14}}\right)(s+1)^{-9}f_1(s)ds\approx -0.00271014346498955\,, \tag{2.15}$$

$$J_2<\frac{3}{64}\left(\int_{s_{21}}^{s_{22}}+\int_{s_{23}}^{\infty}\right)(s+1)^{-8}f_2(s)ds\approx 0.000994001116462756\,. \tag{2.16}$$

It then follows from (2.13)~(2.16) that

$$F(z)>\frac{11}{720}z^{-4}-0.016193226477149z^{-5}-0.002041558160635z^{-6}-0.000463867187673z^{-7}\,.$$

We conclude that $F(z)>0$ when $z>1.19323408312446$. From this it is clear that the inequality (1.1) has not been fully proved by [2].

III. PROOF OF INEQUALITY (1.1)

Case 1. $F(z)>0$, when $z>0.389273748122060$.

Let m=2 in (2.3), we have

$$\psi'(u)=\frac{1}{u}+\frac{1}{u^2}+\frac{1}{6u^3}+5\int_0^\infty\{B_4-B_4(v)\}(x+u)^{-6}dx\,. \tag{3.1}$$

We deduce from (3.1) that

$$\psi'(iy)-\psi'(-iy)=\frac{2}{iy}-\frac{1}{3iy^3}+5\int_0^\infty\{B_4-B_4(v)\}\{(x+iy)^{-6}-(x-iy)^{-6}\}dx\,, \tag{3.2}$$

furthermore,

$$F_1(z)=\frac{1}{2z}-\frac{1}{12z^2}+I_4\,, \tag{3.3}$$

in which

$$I_4=5\int_0^\infty\{B_4-B_4(v)\}(x+z)^{-6}(3x^4-10x^2z+3z^2)xdx\,. \tag{3.4}$$

Therefore

$$F_1^2(z)=\frac{1}{4z^2}-\frac{1}{12z^3}+\frac{1}{144z^4}+I_4^2+\left(\frac{1}{z}-\frac{1}{6z^2}\right)I_4\,, \tag{3.5}$$

$$F_2(z)=-\frac{1}{2}F_1'(z)=\frac{1}{4z^2}-\frac{1}{12z^3}-\frac{1}{2}I_4'\,. \tag{3.6}$$

Then

$$F(z)=F_1^2(z)-F_2(z)=\frac{1}{144z^2}+\left(\frac{1}{z}-\frac{1}{6z^2}\right)I_4+\frac{1}{2}I_4'+I_4^2\,, \tag{3.7}$$

where

$$\frac{1}{2}I_4'=-10\int_0^\infty\{B_4-B_4(v)\}(x^2+z)^{-7}(7x^4-14x^2z+3z^2)xdx\,. \tag{3.8}$$

Now make the substitution, $x=(sz)^{1/2}$, then we have

$$I_4 = \frac{5}{2z^3}\int_0^\infty \{B_4 - B_4(v)\}(s+1)^{-6}(3s^2 - 10s + 3)ds := \frac{5}{2z^3}\beta_1, \tag{3.9}$$

and

$$\frac{1}{2}I_4' = -\frac{5}{z^4}\int_0^\infty \{B_4 - B_4(v)\}(s+1)^{-7}(7s^2 - 14s + 3)ds := -\frac{5}{z^4}\beta_2 \,. \tag{3.10}$$

It then follows from (3.7), (3.9), and (3.10) that

$$F(z) = \frac{1}{z^4}\left(\frac{1}{144} + \frac{5}{2}(\beta_1 - 2\beta_2)\right) - \frac{5}{12z^5}\beta_1 + I_4^2. \tag{3.11}$$

We note that

$$-\frac{1}{16} \le B_4 - B_4(v) = -v^2(v-1)^2 \le 0, \text{ for } 0 < v < 1, \tag{3.12}$$

then we get

$$\begin{aligned}\beta_1 - 2\beta_2 &= 3\int_0^\infty \{B_4 - B_4(v)\}(s+1)^{-7}(s^3 - 7s^2 + 7s - 1)ds \\ &> -\frac{3}{16}\left(\int_{3-2\sqrt{2}}^{1} + \int_{3+2\sqrt{2}}^{\infty}\right)(s+1)^{-7}(s^3 - 7s^2 + 7s - 1)ds \\ &\approx -0.000436344948745758,\end{aligned} \tag{3.13}$$

then

$$\frac{1}{144} + \frac{5}{2}(\beta_1 - 2\beta_2) > 0.00585358207258004, \tag{3.14}$$

and

$$\begin{aligned}-\frac{5}{12}\beta_1 &= -\frac{5}{12}\int_0^\infty \{B_4 - B_4(v)\}(s+1)^{-6}(3s^2 - 10s + 3)ds \\ &> \frac{5}{12}\cdot\frac{1}{16}\int_{1/3}^{3}(s+1)^{-6}(3s^2 - 10s + 3)ds \\ &\approx -0.00227864583333333.\end{aligned} \tag{3.15}$$

It then follows from (3.11), (3.14) and (3,15) that

$$F(z) > 0.00585358207258004z^{-4} - 0.00227864583333333z^{-5}. \tag{3.16}$$

By (3.16) ,we conclude that $F(z) > 0$ when $z > 0.389273748122060$.

Case 2. $F(z) > 0$, when $0 \le z \le 0.420977962526003$.

For $x \ge 0$, $\alpha < 0$, using Taylor's expansion formula, we have

$$(1+x)^\alpha = 1 + \sum_{k=1}^{n}\binom{\alpha}{k}x^k + \binom{\alpha}{n+1}x^{n+1}(1+\theta x)^{\alpha-n-1},\ 0 < \theta < 1. \tag{3.17}$$

We get the following inequality

$$\sum_{k=0}^{2n-1}\binom{\alpha}{k}x^k \le (1+x)^\alpha \le \sum_{k=0}^{2n-2}\binom{\alpha}{k}x^k,\ n \ge 1. \tag{3.18}$$

By (3.18), if $z \ge z_1 > 0$, we have

$$\begin{aligned}F_1(z) &= \sum_{n=1}^\infty \frac{n}{(n^2+z)^2} = \sum_{n=1}^\infty \frac{1}{n^3}\left(1 + \frac{z}{n^2}\right)^{-2} \\ &> \sum_{n=1}^\infty \frac{1}{n^3}\sum_{k=0}^{2m-1}\binom{-2}{k}\left(\frac{z}{n^2}\right)^k\end{aligned}$$

$$=\sum_{k=1}^{2m}(-z)^{k-1}k\cdot\varsigma(2k+1)\,, \tag{3.19}$$

and

$$\begin{aligned}F_2(z)&=\sum_{n=1}^{\infty}\frac{n}{(n^2+z)^3}<\sum_{n=1}^{\infty}\frac{1}{n^5}\left(1+\frac{z_1}{n^2}\right)^{-3}\\&<\sum_{n=1}^{\infty}\frac{1}{n^5}\sum_{k=0}^{2m-2}\binom{-3}{k}\left(\frac{z_1}{n^2}\right)^k\\&=\sum_{k=1}^{2m-1}(-z_1)^{k-1}\frac{k(k+1)}{2}\cdot\varsigma(2k+3)\,,\end{aligned} \tag{3.20}$$

thus $F(z) > 0$, only if z satisfies the following inequality

$$\sum_{k=1}^{2m}(-z)^{k-1}k\cdot\varsigma(2k+1)\geq\left\{\sum_{k=1}^{2m-1}(-z_1)^{k-1}\frac{k(k+1)}{2}\cdot\varsigma(2k+3)\right\}^{1/2}. \tag{3.21}$$

Now let $m = 4$, then we use the following recursive solution method for (3.21).
First, let $z_1 = 0$ in (3.21), the solution set of (3.21) is $0 \le z \le a_1 = 0.101954321777668$.
Second, let $z_1 = a_1$ in (3.21), the solution set of (3.21) is $0 \le z \le a_2 = 0.198730783364018$.
Third, let $z_1 = a_2$ in (3.21), the solution set of (3.21) is $0 \le z \le a_3 = 0.290024566178641$.
Fourth, let $z_1 = a_3$ in (3.21), the solution set of (3.21) is $0 \le z \le a_4 = 0.370665047677767$.
Last, let $z_1 = a_4$ in (3.21), the solution set of (3.21) is $0 \le z \le a_5 = 0.420977962526003$.

This finishes the proof.

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