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Original Article

f_q-Derivations of BN_1-Algebras

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Abstract *- In this paper, the notion of left-right (respectively right-left) derivation and left derivation of BN-algebra are discussed and some related properties are investigated. Also, the notion of f_q-derivation on BN_1-algebra is studied and some of its properties are investigated.*

Keywords *- (l, r)-derivation, (r, l)-derivation, left derivation, f_q- derivation, BN-algebra, BN_1-algebra.*

I. INTRODUCTION

An algebraic structure called *B*-algebra was introduced by Neggers and Kim [15]. Besides *B*-algebra, there was a new algebraic structure instituted by Kim and Kim [10] called *BN*-algebra. In the same article Kim and Kim also introduced *BN*-algebra with condition (*D*), which is a *BN*-algebra with added particular property. Walendziak [16] defined a new *BN*-algebra called BN_1-algebra by adding an axiom to *BN*-algebra.

Algebraic structure such as *B*-algebra has been discussed by researchers. One of the interesting topics is derivation. The notion of derivation from analytic theory was introduced by Posner in 1957 in prime ring discussion. Jun and Xin [8] applied the notion of derivation in ring and near-ring theory to *BCI*-algebras. Some of the results are defining (*l,r*) and (*r,l*)-derivations in *BCI*-algebra *X*. Moreover, they also defined a notion of *regular* of *BCI* -algebra, where a self-map *d* of a *BCI*-algebra is said to be *regular* if $d(0) = 0$. The notion of derivation and *regular* of *BCI*-algebra resulting in some interesting properties.

Abujabal and Al-Shehri [1] introduced a notion of left derivation of *BCI*-algebra and investigated its properties. They also defined a notion of *regular* and *p*-semisimple *BCI*-algebra which admits left derivation. In [3], Al-Shehrie applied the notion of derivation on *BCI*-algebra [8] to *B*-algebra and gave some of the related properties. Lee [12] instituted a new kind of derivation of *BCI*-algebra, which is f_q-derivation and discussed its properties as well. Furthermore, Abujabal and Al-Shehri [2] also gave some derivation results of *BCI*-algebra. More discussion on algebra derivation can be found in [5], [9] and [11].

From the properties of *BN*-algebra, *BN*-algebra with condition (*D*), and BN_1-algebra authors interested to discuss the notion that they have. Therefore, derivation and left derivation of *BN*-algebra and their properties are defined in this article. Lastly, derivation of f_q-derivation in BN_1-algebra and their properties are investigated.

II. PRELIMINARIES

In this section, some necessary definitions needed to construct the main result are given starting with the notion of *B*-algebra and derivation in *B*-algebra, *BCI*-algebra and f_q-derivation in *BCI*-algebra. Later, the notion of *BN*-algebra and its properties and BN_1-algebra and its properties discussed in [1], [3], [10], [12], [15], and [17].

Definition 2.1 [15] A *B*-algebra is a non-empty set *X* with a constant 0 and a binary operation $*$ satisfying the following axioms:

(B1) $x * x = 0$,
(B2) $x * 0 = x$,
(B3) $(x * y) * z = x * (z * (0 * y))$,
for all $x, y, z \in X$.

Example 2.2 Let $A = \{0, 1, 2\}$ be a set with Cayley table as follows:

Table 1: Cayley table for (A; $*$, 0)

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

It can be proven that (A; $*$, 0) is a B-algebra.

Definition 2.3 [12] A *BCI*-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms*:*
(*BCI1*) $\big((x * y) * (x * z)\big) * (z * y) = 0,$
(*BCI2*) $(x * (x * y)) * y = 0,$
(*BCI3*) $x * x = 0,$
(*BCI4*) $x * y = 0$ dan $y * x = 0$ implies $x = y$.
for all $x, y, z \in X$.

For (X; $*$, 0) be an algebra, we denote $x \wedge y = y * (y * x)$ for all $x, y \in X$. Definitions of derivation and regular of *BCI*-algebra are equivalent to B-algebra and as given below.

Definition 2.4 [3] Let (X; $*$, 0) be a B-algebra and d is a self-map of X. A self-map d is a (l, r)-derivation of X if it satisfies $d(x * y) = (d(x) * y) \wedge (x * d(y))$ for all $x, y \in X$. If d satisfies $d(x * y) = (x * d(y)) \wedge (d(x) * y)$, then d is a (r, l)-derivation of X. Moreover, if d is both a (l, r)-derivation and a (r, l)-derivation, we say that d is a derivation of X.

Definition 2.5 [3] Let (X; $*$, 0) be a B-algebra. A self-map d of X is said to be regular if $d(0) = 0$.

The following is the notion of left derivation in *BCI*-algebra.

Definition 2.6 [1] Let (X; $*$, 0) be a *BCI*-algebra. A self-map d is left derivation of X if it satisfies

$$d(x * y) = (x * d(y)) \wedge (y * d(x)) \textit{ for all } x, y \in X.$$

Now, let X be a *BCI* -algebra and f be an endomorphism of X. d_q^f is a self-map of X by $d_q^f(x) = f(x) * q$, for all $q, x \in X$.

Definition 2.7 [10] A *BN*-algebra is a non-empty set X with a constant 0 and a binary operation $*$ satisfying axioms (B1), (B2), and the following axiom
(BN) $(x * y) * z = (0 * z) * (y * x)$,
for all $x, y, z \in X$.

Example 2.8 Let $X = \{0, 1, 2\}$ be a set with Cayley table as follows:

Table 2: Cayley table for (X; $*$, 0)

$*$	0	1	2
0	0	1	2
1	1	0	1
2	2	1	0

It can be proven that (X; $*$, 0) is a *BN* -algebra.

Theorem 2.9 [10] If $(X; *, 0)$ is a *BN*-algebra, then

(i) $0 * (0 * x) = x$,
(ii) $y * x = (0 * x) * (0 * y)$,
(iii) $(0 * x) * y = (0 * y) * x$,
(iv) if $x * y = 0$ then $y * x = 0$,
(v) if $0 * x = 0 * y$ then $x = y$,
(vi) $(x * z) * (y * z) = (z * y) * (z * x)$,
for any $x, y, z \in X$.

Proof. Theorem 2.9 has been proved in [10].

Definition 2.10. [10] Let $(X; *, 0)$ be a *BN*-algebra. $(X; *, 0)$ is said to be a *BN*-algebra with condition (D) if it satisfies $(x * y) * z = x * (z * y)$ for all $x, y, z \in X$.

Example 2.11 Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set with the following Cayley table:

Table 3: Cayley table for $(X; *, 0)$

$*$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

It can be proven that $(X; *, 0)$ is a *BN*-algebra with condition (D).

Theorem 2.12 [10] If $(X; *, 0)$ is a *BN*-algebra with the condition (D), then
(i) $0 * x = x$,
(ii) $x * y = y * x$,
for all $x, y \in X$.

Proof. Theorem 2.12 has been proved in [10].

The following is definition of BN_1-algebra and its properties necessary to construct the notion of f_q-derivation on BN_1-algebra which has been discussed in [17].

Definition 2.13 A *BN*-algebra is said to be BN_1-algebra if it satisfies $(x * y) * y = x$ for all $x, y \in X$.

Proposition 2.14 *If* $(X; *, 0)$ is a BN_1-algebra, then for all $x, y \in X$:

(i) $0 * x = x$,
(ii) $x = (x * y) * (0 * y)$,
(iii) $x * y = y * x$,
(iv) $x = y * (y * x)$.

Proof. Proposition 2.14 has been proved in [17].

Definition 2.15 [14] *Loop* is an algebra $(L, *)$ with an unique solution and satisfies $x * 0 = 0 * x = x$ for all $x \in L$.

Theorem 2.16 [17] Every BN_1-algebra is a *loop*.

Proof. Theorem 2.16 has been proved in [17].

III. DERIVATIONS ON *BN*-ALGEBRAS

In this section, notion of derivation on *BN*-algebra and *BN*-algebra with condition (*D*) are defined by employing definition of derivation on *B* -algebra.

Let $(X; *, 0)$ be a *BN* -algebra defined as $x \wedge y = y * (y * x)$ for all $x, y \in X$.

Definition 3.1 Let $(X; *, 0)$ be a *BN*-algebra. A self-map d is called (l, r)-derivation of $\underline{X}$ if it satisfies

$$d(x * y) = (d(x) * y) \wedge (x * d(y)) \text{ for all } x, y \in X.$$

d is said to be (r, l)-derivation of X if it satisfies

$$d(x * y) = (x * d(y)) \wedge (d(x) * y)$$

If d is both a (l, r)-derivation and a (r, l)-derivation of X, then d is called derivation of X.

Definition 3.2 Let $(X; *, 0)$ be a *BN*-algebra. A self-map d is called left derivation of X if it satisfies

$$d(x * y) = (x * d(y)) \wedge (y * d(x)) \text{ for all } x, y \in X.$$

Example 3.3 Let $X = \{0, a, b, c\}$ be a set with the following Cayley table*:*

Table 4: *Cayley* table for $(X; *, 0)$

$*$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

It can be proven that $(X; *, 0)$ is a *BN* -algebra. A map $d : X \to X$ is defined by

$$d(x) = \begin{cases} b & if\ x = 0, \\ c & if\ x = a, \\ 0 & if\ x = b, \\ a & if\ x = c. \end{cases}$$

Then it can be proven that d is a derivation in X and d is a left derivation in X.

From Example 3.3 it can be seen that d satisfies all the notions of derivation in X. The following example shows that there is a mapping d which is (l, r)-derivation, but does not satisfy (r, l)-derivation and left derivation in X.

Example 3.4 Let $(Z; -, 0)$ be a set of integer numbers Z with subtraction operation. It can be shown that $(Z; -, 0)$ is a *BN*-algebra.

Assume that $d : Z \to Z$ where $d(x) = x - 5$ for all $x \in Z$. Then

$$\begin{aligned} (d(x) - y) \wedge (x - d(y)) &= (x - 5 - y) \wedge (x - (y - 5)) \\ &= (x - y + 5) - ((x - y + 5) - (x - y - 5)) \\ &= (x - y) - 5 \\ &= d(x - y), \end{aligned}$$

for all $x, y \in Z$. Hence, it has been shown that d is (l, r)-derivation in X. Furthermore,

$$\begin{aligned}(x - d(y)) \wedge (d(x) - y) &= (x - (y - 5)) \wedge ((x - 5) - y) \\ &= (x - y - 5) - ((x - y - 5) - (x - y + 5)) \\ &= (x - y) + 5 \\ &\neq d(x - y),\end{aligned}$$

The above statement shows that d is not a (r, l)-derivation in Z. Then,

$$\begin{aligned}(x - d(y)) \wedge (y - d(x)) &= (x - (y - 5)) \wedge (y - (x - 5)) \\ &= (y - x + 5) - ((y - x + 5) - (x - y + 5)) \\ &= (x - y) + 5 \\ &\neq d(x - y),\end{aligned}$$

The above statement shows that d is not a left derivation in Z.

In the following proposition, the properties of left derivation in BN -algebra is given.

Proposition 3.5 Let $(X; *, 0)$ be BN-algebra. If d is a left derivation in X, then $d(0) = x * d(x)$ for all $x \in X$.

Proof. Let $(X; *, 0)$ be a BN -algebra. Since d is a left derivation in X, then by axioms $(B1)$ and $(B2)$ yields

$$\begin{aligned}d(0) &= d(x * x) \\ &= [x * d(x)] \wedge [x * d(x)] \\ &= [x * d(x)] * [(x * d(x)) * (x * d(x))] \\ &= [x * d(x)] * 0 \\ &= x * d(x).\end{aligned}$$

Hence, it has been shown that $d(0) = x * d(x)$ for all $x \in X$.

The following is notion of regular in BN-algebra and proposition applied if d is regular.

Definition 3.6 Let $(X; *, 0)$ be a BN-algebra and d be a self-map of X. d is regular if it satisfies $d(0) = 0$.

Proposition 3.7 Let $(X; *, 0)$ be BN-algebra and d be a regular of X.

(i). If d is a (l, r)-derivation of X, then $d(x) = d(x) \wedge x$ for all $x \in X$,

(ii). If d is a (r, l)-derivation of X, then $d(x) = x \wedge d(x)$ for all $x \in X$.

Proof.

(i) Since d is a (l, r)-derivation of X, then by axiom $(B2)$ we have

$$\begin{aligned}d(x) &= d(x * 0) \\ &= [d(x) * 0] \wedge [x * d(0)] \\ &= d(x) \wedge [x * 0] \\ &= d(x) \wedge x.\end{aligned}$$

Therefore, $d(x) = d(x) \wedge x$ for all $x \in X$.

(ii) Since d is a (r, l)-derivation in X, then by axiom $(B2)$ obtained

$$\begin{aligned}d(x) &= d(x * 0) \\ &= [x * d(0)] \wedge [d(x) * 0] \\ &= [x * 0] \wedge d(x) \\ &= x \wedge d(x).\end{aligned}$$

Hence, $d(x) = x \wedge d(x)$ for all $x \in X$.

Let $(X; *, 0)$ is a BN -algebra with condition (D) and d be a (r, l)-derivation in X. Then, by using $d(x * y) = [x * d(y)] \wedge [d(x) * y]$ and Theorem 2.12 (ii) we have

$$d(x * y) = [d(y) * x] \wedge [d(x) * y]. \qquad (1)$$

If d is a left derivation in X, then

$$d(x * y) = [x * d(y)] \wedge [y * d(y)]. \qquad (2)$$

From Theorem 2.12 (ii) we have

$$d(x * y) = [d(y) * x] \wedge [d(x) * y]. \qquad (3)$$

From equation(1) and (3) it can be concluded that in BN-algebra with condition (D), notion of (r, l)-derivation is similar to the notion of left derivation.

Proposition 3.8 Let $(X; *, 0)$ be a BN -algebra with condition *(D)* and d be a self-map of X.

(i) If d is a (l, r)-derivation of X, then $d(x * y) = d(x) * y$ for all $x, y \in X$,

(ii) If d is a (r, l)-derivation of X, then $d(x * y) = d(y) * x$ for all $x, y \in X$,

(iii) If d is a (l, r)-derivation of X, then d is also a (r, l)-derivation and it is called a derivation of X.

Proof.

(i) Since d is a (l, r)-derivation of X and by Theorem 2.12(ii) we have

$$\begin{aligned} d(x * y) &= [d(x) * y] \wedge [x * d(y)] \\ &= [d(x) * y] \wedge [d(y) * x] \\ &= [d(y) * x] * [(d(y) * x) * (d(x) * y)] \\ &= [(d(y) * x) * (d(x) * y)] * [d(y) * x)] \\ &= [(d(x) * y) * (d(y) * x)] * [d(y) * x)] \\ &= [d(x) * y] * [(d(y) * x)) * (d(y) * x)] \\ &= [d(x) * y] * 0 \\ &= d(x) * y. \end{aligned}$$

Hence, $d(x * y) = d(x) * y$ for all $x, y \in X$.

(ii) Since d is a (r, l)-derivation of X and by Theorem 2.12(ii) we get

$$\begin{aligned} d(x * y) &= [x * d(y)] \wedge [d(x) * y] \\ &= [d(y) * x] \wedge [d(x) * y] \\ &= [d(x) * y] * [(d(x) * y) * (d(y) * x)] \\ &= [(d(x) * y) * (d(y) * x)] * [d(x) * y)] \\ &= [(d(y) * x) * (d(x) * y)] * [d(x) * y)] \\ &= [d(y) * x] * [(d(x) * y)) * (d(x) * y)] \end{aligned}$$

$$= [d(y) * x] * 0$$
$$= d(y) * x.$$

Thus, $d(x * y) = d(y) * x$ for all $x, y \in X$.

(iii) Let d is a (l, r)-derivation of X, then by Theorem 2.12(i) we obtain $d(x * y) = d(x) * y$. From Theorem 2.12 (ii) obtained $d(x * y) = d(y * x) = d(y) * x$ for all $x, y \in X$, such that from Proposition 3.8 (ii) it can be concluded that d is also a (r, l)-derivation of X. It is enough to show that d is a derivation of X.

The converse of Proposition 3.8 needed to be true in general.

Proposition 3.9 Let $(X; *, 0)$ is a BN-algebra with condition (D) and d be a derivation of X. Then

(i) $d(0) = d(x) * x$ for all $x \in X$,

(ii) $d(x) * d(y) = x * y$ for all $x, y \in X$,

(iii) d is a one-one function,

(iv) If d is a regular, then it is an identity map.

Proof.

(i) From axiom $(B1)$ and Proposition 3.8(i) obtained $d(0) = d(x * x) = d(x) * x$ for all $x \in X$.

(ii) From Proposition 3.9(i), $d(0) = d(x) * x$ and $d(0) = d(y) * y$, and from axiom $(B1)$ we get

$$d(0) = d(0)$$
$$d(x) * x = d(y) * y$$
$$[d(x) * x] * x = [d(y) * y] * x$$
$$d(x) * (x * x) = d(y) * (x * y)$$
$$d(x) * 0 = d(y) * (x * y)$$
$$d(x) = d(y) * (x * y)$$
$$d(x) * d(y) = [d(y) * (x * y)] * d(y)$$
$$d(x) * d(y) = [(x * y) * d(y)] * d(y)$$
$$= (x * y) * [d(y) * d(y)]$$
$$= (x * y) * 0$$
$$= x * y.$$

Hence, $d(x) * d(y) = x * y$ for all $x, y \in X$.

(iii) Let $x, y \in X$ such that $d(x) = d(y)$. From Proposition 3.9(i) we have $d(0) = d(x) * x$ and $d(0) = d(y) * y$, such that

$$d(0) = d(0)$$
$$d(x) * x = d(y) * y$$
$$d(x) * x = d(x) * y$$
$$x = y.$$

Thus, d is a one-one function.

(iv) From Proposition 3.9(i) we have $d(0) = d(x) * x$ for all $x \in X$. Since d is a regular, then

$$d(0) = 0$$

$$d(x) * x = 0$$
$$d(x) * x = x * x$$
$$d(x) = x,$$

for all $x \in X$. Let $y \in X$, since d is a regular and from Proposition 3.9(i), then $d(y) * y = d(0) = 0$, thus

$$d(y) * y = y * y$$
$$d(y) = y,$$

for all $y \in X$. Hence, d is an identity map.

Let $(X; *, 0)$ be a BN-algebra and d_1, d_2 be a self-maps of X. Given that $d_1 \circ d_2 : X \to X$ as $d_1 \circ d_2 = d_1\left(d_2(x)\right)$ for all $x \in X$.

Theorem 3.10 Let $(X; *, 0)$ is a BN-algebra with (D) condition, d_1 and d_2 be derivations of X, then
(i) $d_1 \circ d_2$ is also a derivation of X,
(ii) $d_1 \circ d_2 = d_2 \circ d_1$.

Proof.

(i) From Proposition 3.8(i)

$$\begin{aligned} d_1 \circ d_2(x * y) &= d_1(d_2(x * y)) \\ &= d_1(d_2(x) * y) \\ &= d_1(d_2(x)) * y \\ &= d_1 \circ d_2(x) * y. \end{aligned}$$

Thus, $d_1 \circ d_2$ is also a derivation of X.

(ii) From Theorem 2.12 (ii) and Proposition 3.8(i), for all $x, y \in X$ we get

$$\begin{aligned} d_1 \circ d_2(x * y) &= d_1(d_2(x * y)) \\ &= d_1(d_2(x) * y) \\ &= d_1(y * d_2(x)) \\ &= d_1(y) * d_2(x) \\ &= d_2(x) * d_1(y), \end{aligned}$$

and

$$\begin{aligned} d_2 \circ d_1(x * y) &= d_2(d_1(x * y)) \\ &= d_2(d_1(y * x)) \\ &= d_2(d_1(y) * x) \\ &= d_2(x * d_1(y)) \\ &= d_2(x) * d_1(y). \end{aligned}$$

This shows that $d_1 \circ d_2 = d_2 \circ d_1$.

IV. f_q-DERIVATIONS ON BN_1-ALGEBRAS

In this section, notion of f_q -derivation of BN_1-algebra and its properties are given. It is clear that f_q-derivation of BN_1-algebra generally does not apply in BN-algebra since in BN_1-algebra X, $x \wedge y = x$ applies for all $x, y \in X$.

Definition 4.1 Let $(X; *, 0)$ be a BN_1-algebra and let f be a self-map of X. f is an endomorphism of X if it satisfies $f(x *$

$y) = f(x) * f(y)$ for all $x, y \in X$.

Let $(X; *, 0)$ is a BN_1-algebra. $d_q^f : X \longrightarrow X$ is defined by $d_q^f(x) = q * f(x)$ for all $q, x \in X$ and f is an endomorphism of X. For all $x, y \in X$, it is denoted that $x \wedge y = y * (y * x)$. Since $x \wedge y = x$ in BN_1-algebra, then notion of f_q-derivation in [12] is redefined to get the following definition.

Definition 4.2. Let $(X; * ,0)$ be a BN_1-algebra. A self-map d_q^f is called inside f_q-derivation of X if it satisfies $d_q^f(x * y) = d_q^f(x) * f(y)$ for all $x, y \in X$. If d_q^f satisfies $d_q^f(x * y) = d_q^f(y) * f(x)$ for all $x, y \in X$, then it is called an outside f_q-derivation of X. If d_q^f is both an inside and an outside f_q-derivation of X, then d_q^f is a f_q-derivation of X.

Example 4.3 Let $X = \{0, 1, 2, 3\}$ and binary operation $*$ is given by the Cayley table below:

Table 5: Cayley table for $(X; *, 0)$

$*$	0	1	2	3
0	0	0	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

It can be proven that X is BN_1-algebra. Endomorphism $f : X \rightarrow X$ is defined as follows

$$f(x) = \begin{cases} 0 & if\ x = 0, \\ 1 & if\ x = 3, \\ 2 & if\ x = 2, \\ 3 & if\ x = 1. \end{cases}$$

Then d_q^f will be investigated if it is an inside and an outside f_q-derivation in X using the following table:

Table 6: $d_q^f(x * y) = d_q^f(x) * f(y) = d_q^f(y) * f(x)$

x	0	0	0	0	1	1	1	1	2	2	2	2	3	3	3	3
y	0	1	2	3	0	1	2	3	0	1	2	3	0	1	2	3
$x * y$	0	1	2	3	1	0	3	2	2	3	0	1	3	2	1	0
$d_0^f(x * y)$	0	3	2	1	3	0	1	2	2	1	0	3	1	2	3	0
$d_0^f(x)$	0	0	0	0	3	3	3	3	2	2	2	2	1	1	1	1
$f(y)$	0	3	2	1	0	3	2	1	0	3	2	1	0	3	2	1
$d_0^f(x) * f(y)$	0	3	2	1	3	0	1	2	2	1	0	3	1	2	3	0
$d_0^f(y)$	0	3	2	1	0	3	2	1	0	3	2	1	0	3	2	1
$f(x)$	0	0	0	0	3	3	3	3	2	2	2	2	1	1	1	1
$d_0^f(y) * f(x)$	0	3	2	1	3	0	1	2	2	1	0	3	1	2	3	0

Table 6 shows that d_0^f is f_q-derivation in X. Likewise, it can be shown that d_q^f is a f_q-derivation in X.

Every BN_1-algebra has at least one f_q-derivation. The following proposition shows the existence of f_q-derivation in BN_1-algebra.

Proposition 4.4 Let $(X; *, 0)$ be a BN_1-algebra and f be an endomorphism of X. Then, there exist at least one f_q-derivation in X, which is d_0^f.

Proof. Let $q = 0$, by Proposition 2.14 (i) for all $x, y \in X$ gives

$$\begin{aligned} d_0^f(x * y) &= 0 * f(x * y) \\ &= f(x) * f(y) \\ &= [0 * f(x)] * f(y) \\ d_0^f(x * y) &= d_0^f(x) * f(y). \end{aligned}$$

Thus, d_0^f is an inside f_q-derivation of X. On the other hand

$$\begin{aligned} d_0^f(x * y) &= 0 * f(x * y) \\ &= f(y * x) \\ &= f(y) * f(x) \\ &= [0 * f(y)] * f(y) \\ d_0^f(x * y) &= d_0^f(y) * f(x). \end{aligned}$$

Thus, d_0^f is an outside f_q-derivation of X. Therefore, it has been proven that there exist at least one f_q -derivation in X, which is d_0^f.

Proposition 4.5 Let $(X; *, 0)$ is a BN_1-algebra and f is an identity endomorphism of X. Then

(i) d_0^f is an identity function,
(ii) d_0^f is a f_q-derivation in X.

Proof. Let X be a BN_1-algebra.

(i) Since f is an identity endomorphism of X, then $f(x) = x$ for all $x \in X$. If $q = 0$, then $d_0^f(x) = 0 * f(x) = f(x) = x$. Hence, d_0^f is an identity function.

(ii) From (i) it is obtained that $d_0^f(x * y) = x * y$, $d_0^f(x) * f(y) = x * y$, and $d_0^f(y) * f(x) = y * x = x * y$. Hence, d_0^f is a f_q-derivation of X.

The following is definition of regular in BN_1-algebra.

Definition 4.6 Let $(X; *, 0)$ is a BN_1-algebra and d_q^f is a self-map of X. d_q^f is regular in X if $d_q^f(0) = 0$.

For all $x, y \in X$, if $x = y$ then an inside f_q-derivation is similar to an outside f_q-derivation. Hence, from this statement is followed by the proposition below.

Proposition 4.7 Let $(X; *, 0)$ is a BN_1-algebra. If d_q^f is an inside or an outside f_q-derivation in X, then

(i) $d_q^f(0) = d_q^f(x) * f(x)$ for all $x \in X$,

(ii) If d_q^f is a regular, then $d_q^f(x) = f(x)$ for all $x \in X$.

Proof.

(i) Let d_q^f is an inside or an outside f_q-derivation of X, then by (B1) axiom we have

$$d_q^f(0) = d_q^f(x * x) = d_q^f(x) * f(x)$$

for all $x \in X$.

(ii) Let d_q^f is a regular, then $d_q^f(0) = 0$. If d_q^f is an inside f_q-derivation of X by Proposition 2.14 (i) obtained

$$d_q^f(x) = d_q^f(0 * x) = d_q^f(0) * f(x) = 0 * f(x) = f(x),$$

for all $x \in X$. If d_q^f is an outside f_q -derivation of X, then by (B2) axiom we get

$$d_q^f(x) = d_q^f(x * 0) = d_q^f(0) * f(x) = 0 * f(x) = f(x),$$

for all $x \in X$.

Theorem 4.8 If $(X; *, 0)$ be a BN_1-algebra with condition (D), then

(i) d_q^f is an inside f_q-derivation of X,
(ii) If d_q^f is an outside f_q-derivation of X, then d_q^f is a f_q-derivation of X.

Proof.

(i) Since X be a BN_1-algebra with condition (D), then by Proposition 2.14 (iii) for all $x, y \in X$ obtained

$$\begin{aligned} d_q^f(x * y) &= q * f(x * y) \\ &= q * [f(x) * f(y)] \\ &= q * [f(y) * f(x)] \\ &= [q * f(x)] * f(y) \\ d_q^f(x * y) &= d_q^f(x) * f(y). \end{aligned}$$

Therefore, d_q^f is an inside f_q-derivation of X.

(ii) Let X be a BN_1-algebra with condition (D), then by (i) it is known that d_q^f is an inside f_q-derivation of X. Since d_q^f is an outside fq -derivation of X, then it is proven that d_q^f is a f_q-derivation in X.

Definition 4.9 Let $(X; *, 0)$ be a BN_1-algebra and d_q^f and D_q^f are two self-maps of X. $d_q^f \circ D_q^f : X \to X$ is defined by

$$(d_q^f \circ D_q^f)(x) = d_q^f\left(D_q^f(x)\right) \text{ for all } x \in X.$$

Proposition 4.10 Let $(X; *, 0)$ be a BN_1-algebra and f is an identity endomorphism of X.

(i) If d_q^f and D_q^f are two inside f_q-derivations of X, then $d_q^f \circ D_q^f$ is an inside f_q-derivation in X.
(ii) If d_q^f and D_q^f are two outside f_q-derivations of X, then $d_q^f \circ D_q^f$ is an outside f_q-derivation in X.
(iii) If d_q^f and D_q^f are two f_q-derivations of X, then $d_q^f \circ D_q^f$ is a f_q-derivation in X.

Proof. Let X be a BN_1-algebra. Since f is an identity endomorphism of X, then $f(x) = x$ for all $x \in X$.

(i). Let d_q^f and D_q^f are two inside f_q-derivations in X, then for all $x, y \in X$ we obtain

$$(d_q^f \circ D_q^f)\ (x * y) = d_q^f(D_q^f\ (x * y))$$
$$= d_q^f(D_q^f\ (x) * f(y))$$
$$= d_q^f\left(D_q^f\ (x)\right) * f(f(y))$$
$$(d_q^f \circ D_q^f)\ (x * y) = (d_q^f \circ D_q^f)(x) * f(y).$$

Therefore, it is proven that $d_q^f \circ D_q^f$ ia an inside f_q-derivations in X.

(ii). Let d_q^f and D_q^f are two outside f_q-derivations of X, then for all $x, y \in X$ we get

$$(d_q^f \circ D_q^f)\ (x * y) = d_q^f(D_q^f\ (x * y))$$
$$= d_q^f(D_q^f\ (y) * f(x))$$
$$= d_q^f\left(D_q^f\ (y)\right) * f(f(x))$$
$$(d_q^f \circ D_q^f)\ (x * y) = (d_q^f \circ D_q^f)(y) * f(x).$$

Hence, it is enough to show that $(d_q^f \circ D_q^f)(x * y)$is an outside f_q-derivations in X.

(iii). From (i) and (ii) it can be proven that $d_q^f \circ D_q^f$ is a f_q-derivations in X.

Proposition 4.11 Let $(X; *, 0)$ be a BN_1-algebra, f is an identity endomorphism of X, and d_q^f and D_q^f are f_q- derivations of X. If $d_q^f \circ f = f \circ d_q^f$ and $D_q^f \circ f = f \circ D_q^f$, then $d_q^f \circ D_q^f = D_q^f \circ d_q^f$.

Proof. Since d_q^f and D_q^f are f_q-derivations of X, then we say that d_q^f is an inside f_q-derivation and D_q^f is an outside f_q-derivation of X. For all $x, y \in X$ obtained

$$(d_q^f \circ D_q^f)\ (x * y) = d_q^f(D_q^f\ (x * y))$$
$$= d_q^f(D_q^f\ (y) * f(x))$$
$$= d_q^f(f(x)) * f(D_q^f(y))$$
$$\left(d_q^f \circ D_q^f\right)(x * y) = \left(d_q^f \circ f\right)(x) * (f \circ D_q^f)(y).$$

On the other side we have

$$(D_q^f \circ d_q^f)\ (x * y) = D_q^f(d_q^f\ (x * y))$$
$$= D_q^f(d_q^f\ (x) * f(y))$$
$$= D_q^f(f(y)) * f(d_q^f(x))$$
$$= f\left(d_q^f(x)\right) * D_q^f(f(y))$$
$$= \left(f \circ d_q^f\right)(x) * (D_q^f \circ f)(y)$$

$$(D_q^f \circ d_q^f)(x * y) = (d_q^f \circ f)(x) * (f \circ D_q^f)(y).$$

Thus, it can be seen that $(d_q^f \circ D_q^f)(x * y) = (D_q^f \circ d_q^f)(x * y)$. Changing $y = 0$ yields $(d_q^f \circ D_q^f)(x) = (D_q^f \circ d_q^f)(x)$. Therefore, $d_q^f \circ D_q^f = D_q^f \circ d_q^f$ is true.

From Theorem 2.16 it is confirmed that BN_1-algebra X defined in Example 4.3 is a *loop*. Therefore, it can be concluded that *loop* can be applied to the notion of f_q-derivation. However, the following example shows that the notion of f_q-derivation does not apply to *loop* that is not a BN_1-algebra.

Example 4.12 Let $X = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $*$ on X is defined by the following Cayley table:

Table 7: Cayley table for (X; $*$, 0)

$*$	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	4	2
2	2	4	0	1	3
3	3	2	4	0	1
4	4	3	1	2	0

It can be proven that X is a *loop* but not a BN_1-algebra. Given an endomorphism $f : X \to X$ where $f(x) = x$. If $q = 0$ it can be proven that d_0^f is an *inside* f_q-derivation of X. However, since d_0^f $(1 * 2) = 3$ and d_0^f $(2) * f(1) = 4$, then d_0^f $(1 * 2) \neq d_0^f$ $(2) * f(1)$. Hence, d_0^f is an *outside* f_q-derivation of X.

From Example 4.12 it can be seen that f is an identity endo-morphism of X, but d_0^f is not a f_q-derivation in X. This statement contradicts with Proposition 4.5 (ii). Therefore, generally the notion of f_q-derivation in BN_1-algebra does not apply on a *loop*.

V. CONCLUSIONS

Since every BN_1-algebra is a BN -algebra it is clear that the notion of derivation of BN -algebra can also be applied to BN_1-algebra. On the other hand, the notion of f_q-derivation defined in BN_1-algebra generally does not hold neither in BN -algebra nor in *loop*.

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