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On an integro differential equation with parameter

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abstract: In this paper, we study the existence of at least one and exact one solution x for an initial value problem of an implicit differential equation with parameter in the two classes $x \in C^1[0,T]$ and $x \in AC[0,T]$. The maximal and minimal solution will be proved. The continuous dependence of the unique solution will be studied.

keywords: Implicit differential equation, existence of solutions, continuous dependence, maximal and minimal solution.

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1 Introduction

It is well known that the nonlinear initial value problems create an important branch of nonlinear analysis and have numerous applications in describing of miscellaneous real world problems. For papers studying such kind of equations (see [1]-[7], [9]-[11] and [13] and references therein).

In this paper, we are concerned with the initial value problem of the implicit differential equation with parameter

$$\frac{dx}{dt} = f_1(t, \int_0^t f_2(s, \frac{dx}{ds}, \mu)ds), \ t \in (0,T] \tag{1}$$

with initial data

$$x(0) = x_0. \tag{2}$$

First, we study the existence of at least one solution $x \in C^1[0,T]$. The maximal and minimal solution will be proved. Also, the sufficient condition for the uniqueness of the solution will be given. The continuous dependence of the unique solution on the parameter μ and the function f_2 will be studied.

Second, we study the existence of at least one solution $x \in AC[0,T]$. The uniqueness of the solution will be studied. The continuous dependence of the unique solution on the parameter μ and the function f_2 will be proved.

2 Existence of continuous solutions

Consider the initial value problem (1)-(2) under the following assumptions:

1. $f_1(t,y) : [0,T]\times R \longrightarrow R$ is measurable in t for all $y \in R$ and satisfies Lipschitz condition

$$|f_1(t,y) - f_1(t,z)| \leq L|y-z|.$$

where L is a positive constant. From this assumption we can deduce

$$|f_1(t,y)| - |f_1(t,0)| \leq |f_1(t,y) - f_1(t,0)| \leq L|y|,$$

and

$$|f_1(t,y)| \leq L|y| + |f_1(t,0)|.$$

2. $f_2 : [0,T] \times R \times R \longrightarrow R$ is measurable in the first argument and continuous in the other two arguments and there exist an integrable function $a \in L_1[0,T]$ and a positive constant k such that

$$|f_2(t, y(s), \mu)| \leq | a(t) | + k | y(s) | + k | \mu |$$

where

$$\sup_{t\in[0,T]} \int_0^t | a(s) | ds \leq N.$$

3. $LkT < 1$.

Now for the existence of at least one solution of the problem (1)-(2), we have the following theorem.

Theorem 1. *Let the assumptions(1)-(3) be satisfied, then problem (1)-(2) has at least one solution $x \in C[0,T]$.*

Proof. Let $\frac{dx}{dt} = y \in [0,T]$, then equation (1) will be given by

$$y(t) = f_1(t, \int_0^t f_2(s, y(s), \mu)ds) \tag{3}$$

and

$$x(t) = x_0 + \int_0^t y(s)ds. \tag{4}$$

Now, define the operator F by

$$Fy(t) = f_1(t, \int_0^t f_2(s, y(s), \mu)ds).$$

Define the set

$$Q_r = \{y \in R : \|y\|_{C[0,T]} \leq r\} \subset C[0,T], \ r = \frac{LN + LkT|\mu| + B}{(1 - LkT)}$$

where

$$B = \sup_{t\in[0,T]} |f_1(t, 0)|.$$

Now

$$\begin{aligned} |Fy(t)| &= |f_1(t, \int_0^t f_2(s, y(s), \mu)ds)| \\ &\leq L \mid \int_0^t f_2(s, y(s), \mu)ds \mid + |f_1(t, 0)| \\ &\leq L \int_0^t | f_2(s, y(s), \mu) | ds + |f_1(t, 0)| \\ &\leq L \int_0^t (| a(s) | + k | y(s) | + k | \mu |)ds + |f_1(t, 0)| \\ &\leq L \int_0^t | a(s) | ds + Lk \int_0^t |y(s)|ds + Lk \int_0^t |\mu|ds + B \\ &\leq LN + LTkr + LTk | \mu | + B = r. \end{aligned}$$

This proves that $F : Q_r \to Q_r$ and the class of functions $\{Fy\}$ is uniformly bonded on Q_r.

Now, let $y \in Q_r$. Let $t_1, t_2 \in [0,T]$ be such that $t_2 > t_1$ and $| t_1 - t_2 | \leq \delta$, then

$$\begin{aligned} | Fy(t_2) - Fy(t_1) | &= | f_1(t_2, \int_0^{t_2} f_2(s, y(s), \mu)ds) - f_1(t_1, \int_0^{t_1} f_2(s, y(s), \mu)ds) | \\ &\leq | f_1(t_2, \int_0^{t_2} f_2(s, y(s), \mu)ds) - f_1(t_2, \int_0^{t_1} f_2(s, y(s), \mu)ds) \\ &+ f_1(t_2, \int_0^{t_1} f_2(s, y(s), \mu)ds) - f_1(t_1, \int_0^{t_1} f_2(s, y(s), \mu)ds) | \\ &\leq | f_1(t_2, \int_0^{t_2} f_2(s, y(s), \mu)ds) - f_1(t_2, \int_0^{t_1} f_2(s, y(s), \mu)ds) | \\ &+ | f_1(t_2, \int_0^{t_1} f_2(s, y(s), \mu)ds) - f_1(t_1, \int_0^{t_1} f_2(s, y(s), \mu)ds) | \\ &\leq L \int_{t_1}^{t_2} | f_2(s, y(s), \mu) | ds + \epsilon_1 \\ &\leq L \int_{t_1}^{t_2} (| a(t) | + k | y(s) | + k | \mu |)ds + \epsilon_1 \\ &\leq L \int_{t_1}^{t_2} | a(t) | ds + L \ k(r + | \mu |)(t_2 - t_1) + \epsilon_1. \end{aligned}$$

This means that the class functions $\{Fy(t)\}$ is eqi-continuous on Q_r, then by Arzela Theorem [12], the operator F is relatively compact.

Now we prove that F is continuous operator. Let $\{y_n\} \subset Q_r,\ y_n \to y$, thus

$$Fy_n(t) = f_1(t, \int_0^t f_2(s, y_n(s), \mu)ds).$$

Taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} Fy_n(t) = \lim_{n\to\infty} f_1(t, \int_0^t f_2(s, y_n(s), \mu)ds)$$

and from Lebesgue dominated convergence Theorem [8], we have

$$\begin{aligned} \lim_{n\to\infty} Fy_n(t) &= f_1(t, \int_0^t f_2(s, \lim_{n\to\infty} y_n(s), \mu)ds) \\ &= f_1(t, \int_0^t f_2(s, y(s), \mu)ds) \\ &= Fy(t). \end{aligned}$$

This means that $Fy_n(t) \to Fy(t)$. Hence the operator F is continuous.

Now by Schauder fixed point Theorem [12], there exists at least one solution $y \in C[0,T]$ of the integral equation(3). Consequently, there exists at least one solution $x \in C^1[0,T]$ for the problem (1) and (2) given by (4).

3 Maximal and minimal solution

Lemma 1. *Let the assumptions of Theorem 1 be satisfied and $y(t)$ and $z(t)$ are tow continuous functions on [0,T] satisfying*

$$\begin{aligned} y(t) &\le f_1(t, \int_0^t f_2(s, y(s), \mu)ds), \\ z(t) &\ge f_1(t, \int_0^t f_2(s, z(s), \mu)ds) \end{aligned}$$

and one of them is strict. If f_1 and f_2 are monotonic nondecreasing, then

$$y(t) < z(t), \quad t > 0. \tag{5}$$

Proof. *Let the conclusion (5) be false, then there exists t_1 such that*

$$y(t_1) = z(t_1), \quad t_1 > 0$$

and

$$y(t) < z(t) \quad 0 < t < t_1.$$

From the monotonically of f_1 and f_2, we get

$$\begin{aligned} y(t_1) &\le f_1(t_1, \int_0^{t_1} f_2(s, y(s), \mu)ds) \\ &< f_1(t_1, \int_0^{t_1} f_2(s, z(s), \mu)ds) \\ &= z(t_1). \end{aligned}$$

hence $y(t_1) < z(t_1)$. This contradicts the fact that $y(t_1) = z(t_1)$, then $y(t) < z(t),\ t \in [0,T]$.

Theorem 2. *Let the assumptions (1)-(3) be satisfied. If f_1, f_2 are monotonic nondecreasing, then equations (3) has maximal and minimal solutions.*

Proof. Firstly, we prove the existence of maximal solution of (3).
Let $\epsilon > 0$, then

$$y_\epsilon(t) = \epsilon + f_1(t, \int_0^t f_2(s, y_\epsilon, \mu)ds). \tag{6}$$

It's easy to show that equation (6) has at least one solution $y_\epsilon \in [0,T]$.
Now let $\epsilon_1, \epsilon_2 > 0$ be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$, then

$$\begin{aligned} y_{\epsilon_2}(t) &= \epsilon_2 + f_1(t, \int_0^t f_2(s, y_{\epsilon_2}, \mu)ds). \\ y_{\epsilon_1}(t) &= \epsilon_1 + f_1(t, \int_0^t f_2(s, y_{\epsilon_1}, \mu)ds) \\ &> \epsilon_2 + f_1(t, \int_0^t f_2(s, y_{\epsilon_1}, \mu)ds) \end{aligned}$$

and from Lemma1, we obtain

$$y_{\epsilon_2}(t) < y_{\epsilon_1}(t),\ \ t \in [0,T].$$

Now the family $\{y_\epsilon(t)\}$ is uniformly bounded as follows:

$$\begin{aligned} |y_\epsilon(t)| &\leq \epsilon + |f_1(t, \int_0^t f_2(s, y_\epsilon, \mu)ds)| \\ &\leq \epsilon + r = r^* \end{aligned}$$

Also, the family $\{y_\epsilon(t)\}$ is equi-continuous as follows:

$$\begin{aligned} & |y_\epsilon(t_2) + y_\epsilon(t_1)| = |\epsilon + f_1(t_2, \int_0^{t_2} f_2(s, y_\epsilon, \mu)ds) \\ - \ & \epsilon + f_1(t_1, \int_0^{t_1} f_2(s, y_\epsilon, \mu)ds)| \\ & \leq |f_1(t_2, \int_0^{t_2} f_2(s, y_\epsilon, \mu)ds) - f_1(t_1, \int_0^{t_1} f_2(s, y_\epsilon, \mu)ds)| \\ & \leq |f_1(t_2, \int_0^{t_2} f_2(s, y_\epsilon, \mu)ds) - f_1(t_2, \int_0^{t_1} f_2(s, y_\epsilon, \mu)ds)| \\ + \ & |f_1(t_2, \int_0^{t_1} f_2(s, y_\epsilon, \mu)ds) - f_1(t_1, \int_0^{t_1} f_2(s, y_\epsilon, \mu)ds)| \\ \leq \ & L \int_{t_1}^{t_2} \mid f_2(s, y(s), \mu) \mid ds + \epsilon_1 \\ \leq \ & L \int_{t_1}^{t_2} (\mid a(t) \mid + k \mid y(s) \mid + k \mid \mu \mid) ds + \epsilon_1 \\ \leq \ & L \int_{t_1}^{t_2} \mid a(t) \mid ds + L[(kr^* + k \mid \mu \mid)(t_2 - t_1)] + \epsilon_1. \end{aligned}$$

Then $\{y_\epsilon(t)\}$ is equi-continuous and uniformly bounded on [0,T], then $\{y_\epsilon\}$ is relatively compact (Arzela Theorem), then there exists decreasing sequence ϵ_n such that $\varepsilon_n \to 0$, $n \to \infty$ and $\lim_{n\to\infty} y_{\epsilon_n}(t)$ exists uniformly on [0,T], let

$$\lim_{n\to\infty} y_{\epsilon_n}(t) = q(t).$$

Now form the continuity of the functions f_1 and f_2 and Lebesgue dominated convergence Theorem, we have

$$f_1(t, \int_0^t f_2(s, y_{\epsilon_n}(s), \mu)ds) \to f_1(t, \int_0^t f_2(s, q(t), \mu)ds),$$

then

$$\begin{aligned} q(t) &= \lim_{n\to\infty} y_{\epsilon_n}(t) \\ &= f_1(t, \int_0^t f_2(s, q(t), \mu)ds). \end{aligned}$$

which implies that $q(t)$ is a solution of equation (3).
Finally, $q(t)$ is the maximal solution of (3). To do this, let $y(t)$ be any solution of (3), then

$$\begin{aligned} y(t) &= f_1(t, \int_0^t f_2(s, y(s), \mu)ds) \\ y_\epsilon(t) &= \epsilon + f_1(t, \int_0^t f_2(s, y_\epsilon(s), \mu)ds) \\ &> f_1(t, \int_0^t f_2(s, y_\epsilon(s), \mu)ds) \end{aligned}$$

Applying Lemma1, we get

$$y(t) < y_\epsilon(t),\ \ t \in [0,T].$$

From the uniqueness of the maximal solutions, it is clear that $y_\epsilon(t) \to q(t)$ uniformly on [0,T] as $\epsilon \to 0$, thus q is the maximal solution of (3).
By a similar way can prove the existence of the minimal solution.

Consider now the functional integral equation (3) under the following assumption:

(4) $f_2 : [0,T] \times R \times R \to R$ is measurable in the first argument and satisfies the Lipschitz condition

$$|f_2(t,y,\mu) - f_2(t,z,\mu^*)| \leq k(|\ y - z\ | + |\ \mu - \mu^*\ |).$$

(5) $f_2(t,0) \in L_1[0,T]$.

Now for the uniqueness of the solution of the problem (1) and (2), we have the following theorem.

Theorem 3. *Let the assumption (1),(3) and (4)-(5), be satisfied, then the solution of the problem (1) and (2) is unique solution.*

Proof. From assumption (4), we have

$$|\ f_2(t,y,\mu)\ | - |\ f_2(t,0,0)\ | \leq |\ f_2(t,y,\mu) - f_2(t,0,0)\ | \leq k(|\ y\ | + |\ \mu\ |)$$

and

$$|\ f_2(t,y,\mu)\ | \leq k(|\ y\ | + |\ \mu\ |) + |\ f_2(t,0,0)\ |,\ \ a(t) = |\ f_2(t,0,0)\ |\ .$$

Thus assumption (2) is satisfied. Then all assumptions of Theorem 1 are satisfied and the solution of the functional integral equation with parameter (3) exists. Let y, z be two the solution of (3), then

$$\begin{aligned}
|\ y(t) - z(t)\ | &= |\ f_1(t, \int_0^t f_2(s,y(s),\mu)ds) - f_1(t, \int_0^t f_2(s,z(s),\mu)ds)\ | \\
&\leq L\ |\int_0^t f_2(s,y(s),\mu)ds - \int_0^t f_2(s,z(s),\mu)ds\ | \\
&\leq L\int_0^t |\ f_2(s,y(s),\mu) - f_2(s,z(s),\mu)\ |\ ds \\
&\leq kL\int_0^t |\ y - z\ |\ ds \\
&\leq kLT\ \|\ y - z\ \|\ .
\end{aligned}$$

Hence

$$\|\ y - z\ \| \quad \leq \quad kLT\ \|\ y - z\ \|\ .$$

Then

$$\|\ y - z\ \|\ (1 - kLT) \quad \leq \quad 0.$$

Since $LkT < 1$, then $\|\ y - z\ \| = 0$ and this implies that $y(t) = z(t)$ and the solution $y \in C[0,T]$ of (3) is unique. Consequently, the solution of the problem (1) and (2) is unique.

4.1 Continuous dependence

Continuous dependence on the function y

Definition 1. *The solution x of the initial value problem (1)-(2) depends continuously on the function y, if*

$$\forall \epsilon > 0, \quad \exists \quad \delta(\epsilon) \quad s.t \quad \|y - y^*\| < \delta \Rightarrow \|x - x^*\| < \epsilon,$$

where x^* is the solution of the initial value problem

$$\frac{dx^*}{dt} = f_1(t, \int_0^t f_2(s, \frac{dx^*}{dt}, \mu)ds),\ \ t \in (0,T] \tag{7}$$

$$x^*(0) = x_0. \tag{8}$$

where

$$\frac{dx^*}{dt} = y^*(t), \quad t \in (0,1]$$

Theorem 4. *Let the assumptions of Theorem 3 be satisfied, then the solution x of the initial value problem (1)-(2) depends continuously on the function y.*

Proof. Let x, x^* be the two solutions of the initial value problems (1)-(2) and (7)-(8) respectively, then

$$\begin{aligned} |x(t) - x^*(t)| &= |x_0 + \int_0^t y(s)ds - x_0 - \int_0^t y^*(s)ds| \\ &\leq |\int_0^t |y(s) - y^*(s)|ds \\ &\leq \|y - y^*\| \leq \delta. \end{aligned}$$

Hence

$$\|x - x^*\| \leq \delta = \epsilon.$$

This mean that the solution x of the initial value problems (1)-(2) depends continuously on the function y. The proof is completed. □

Continuous dependence on the parameter μ

Definition 2. *The solution $y \in C[0,T]$ of the functional integral equation (3) depends continuously on the parameter μ, if*

$$\forall \varepsilon > 0, \ \exists \ \delta(\epsilon) \ s.t \ |\mu - \mu^*| < \delta \Rightarrow \|y - y^*\| < \epsilon$$

where y^* is the unique solution of the functional equation

$$y^*(t) = f_1(t, \int_0^t f_2(s, y^*(s), \mu^*)ds) \tag{9}$$

Theorem 5. *Let the assumptions of Theorem 3 be satisfied, then the solution y of (3) depends continuously on the parameter μ.*

Proof. Let y *and* y^* be the two solutions of equations (3) corresponding to μ *and* μ^*, then

$$\begin{aligned} | \ y(t) - y^*(t) \ | &= | \ f_1(t, \int_0^t f_2(s, y(s), \mu)ds) - f_1(t, \int_0^t f_2(s, y^*(s), \mu^*)ds) \ | \\ &\leq L \int_0^t | \ f_2(s, y(s), \mu) - f_2(s, y^*(s), \mu^*) \ | \ ds \\ &\leq Lk \int_0^t (| \ y(s) - y^*(s) \ | + | \ \mu - \mu^* \ |)ds \\ &\leq Lk \int_0^t (\sup_{t \epsilon [0,T]} | \ y(s) - y^*(s) \ | \ ds) + LkT \ | \ \mu - \mu^* \ | \\ &\leq LkT \ \| \ y - y^* \ \| + LkT\delta \end{aligned}$$

and

$$\| \ y - y^* \ \| \ (1 - LkT) \quad \leq \quad LkT\delta.$$

Hence

$$\| \ y - y^* \ \| \quad \leq \quad \frac{LkT\delta}{(1 - LkT)} = \epsilon.$$

This proves that the solution y of equation (3) depends continuously on the parameter μ.

Corollary 1. *Let the assumptions of Theorem 3 be satisfied, then the solution x of the problem (1)-(2) depends continuously on the parameter μ.*

Continuous dependence on the function f_2

Definition 3. *The solution $y \in C[0,T]$ of (3) depends continuously on the function f_2, if*

$$\forall \varepsilon > 0, \ \exists \ \delta(\epsilon) \ s.t \ |f_2 - f_2^*| < \delta \Rightarrow \| \ y - y^* \ \| < \epsilon$$

where y^* is the unique solution of the functional integral equation (3) corresponding to f_2^*

Theorem 6. *Let the assumptions of theorem (3) be satisfied, then the solution y of (3) depends continuously on the function f_2.*

Proof. Let y *and* y^* be the two solutions of (3) corresponding to f_2 *and* f_2^*, then

$$\begin{aligned} \mid y(t) - y^*(t) \mid &= \mid f_1(t, \int_0^t f_2(s, y(s), \mu) ds) - f_1(t, \int_0^t f_2^*(s, y^*(s), \mu) ds) \mid \\ &\leq L \mid \int_0^t f_2(s, y(s), \mu) ds - \int_0^t f_2^*(s, y^*(s), \mu) ds \mid \\ &\leq L \int_0^t \mid f_2(s, y(s), \mu) - f_2^*(s, y^*(s), \mu) \mid ds \\ &\leq L \int_0^t \mid f_2(s, y(s), \mu) - f_2^*(s, y(s), \mu) \\ &+ f_2^*(s, y(s), \mu) - f_2^*(s, y^*(s), \mu) \mid ds \\ &\leq L \int_0^t \mid f_2(s, y(s), \mu) - f_2^*(s, y(s), \mu) \mid ds \\ &+ L \int_0^t \mid f_2^*(s, y(s), \mu) - f_2^*(s, y^*(s), \mu) \mid ds \\ &\leq L \int_0^t \mid f_2(s, y(s), \mu) - f_2^*(s, y(s), \mu) \mid \\ &+ Lk \int_0^t \mid y(s) - y^*(s) \mid ds \\ &\leq LT\delta + LkT \parallel y - y^* \parallel . \end{aligned}$$

Hence

$$\parallel y - y^* \parallel (1 - LkT) \quad \leq \quad LT\delta$$

and

$$\parallel y - y^* \parallel \quad \leq \quad \frac{LT\delta}{(1 - LkT)}.$$

This proves that the solution of (3) depends continuously on the function f_2.

Corollary 2. *Let the assumptions of Theorem 3 be satisfied, then the solution x of the problem (1) and (2) depends continuously on the function f_2.*

5 Exsistence of integrable solution

Existence of at least one integral solution

Consider the following initial value problem

$$\frac{dx}{dt} = f_1(t, \int_0^t f_2(s, \frac{dx}{ds}, \mu) ds), \quad a.e.\ t \in (0, T] \tag{10}$$

with initial data

$$x(0) = x_0. \tag{11}$$

under the following assumptions:

(*i*) $f_1 : [0, T] \times R \to R$ satisfies Carathèodory, i.e. it is measurable in $t \in [0, T]$ for every $y \in R$ and continuous in $y \in R$ for every $t \in [0, T]$ and there exist a function $a_1(t) \in L^1[0, T]$ and constant $b_1 > 0$ s.t.

$$\mid f_1(t, y) \mid \leq \mid a_1(t) \mid + b_1 \mid y \mid .$$

(*ii*) $f_1 : [0, T] \times R \times R \to R$ satisfies Carathèodory, i.e. it is measurable in $t \in [0, T]$ for every $y \in R$ and continuous in $y \in R$ for every $t \in [0, T]$ and there exist a function $a_1(t) \in L^1[0, T]$ and constant $b_1 > 0$ s.t.

$$\mid f_2(t, y, \mu) \mid \leq \mid a_2(t) \mid + b_2(\mid y \mid + \mid \mu \mid).$$

(iii) $b_1 b_2 T < 1$.

Now for existence of at least one integrable solution $y \in L_1[0,T]$ of the functional equation (3), we have the following theorem.

Theorem 7. *Let the assumptions $(i)-(iii)$ be satisfied, then the functional equation* (3) *has at least one solution $y \in L_1[0,T]$, hence the initial value problem* (10)-(11) *has at least one solution $x \in AC[0,T]$.*

Proof. Define the operator F be

$$Fy(t) = f_1(t, \int_0^t f_2(s, y(s), \mu)ds), \quad t \in [0,T].$$

Define the set $Q_r = \{y \in R : \|y\|_{L_1[0,T]} \le r\}$, where $r = \frac{\|a_1\| + b_1 T \|a_2\| + \frac{1}{2} b_1 T^2 |\mu|}{1 - b_1 b_2 T}$.

Let $x \in Q_r$, then

$$\begin{aligned} | Fy(t) | &= | f_1(t, \int_0^t (s, y(s), \mu)ds) | \\ &\le | a(t) | + b_1 | \int_0^t f_2(s, y(s), \mu)ds | \\ &\le | a_1(t) | + b_1 \int_0^t | f_2(s, y(s), \mu) | ds \\ &\le | a_1(t) | + b_1 \int_0^t | (a_2(s) + b_2 | y | + | \mu |)ds \\ &\le | a_1(t) | + b_1 \int_0^t |a_2(s)|ds + b_2 b_1 \int_0^t | y(s) | ds + b_1 \int_0^t | \mu | ds. \end{aligned}$$

Then

$$\begin{aligned} \| Fy \|_{L_1} &= \int_0^T |Fy(t)|dt \le \int_0^T |a_1(t)| + b_1 \int_0^T \int_0^t |a_2(s)|dsdt \\ &+ b_2 b_1 \int_0^T \int_0^t |y(s)|dsdt + b_1 \int_0^T \int_0^t |\mu|dsdt \\ &\quad \| a_1 \|_{L_1} + b_1 T \| a_2 \|_{L_1} + b_1 b_2 T r + \frac{1}{2} b_1 T^2 |\mu| = r. \end{aligned}$$

This prove that $F : Q_r \to Q_r$ and the class of functions $\{Fy\}$ is uniformly bonded in Q_r.
Now let $y \in Q_r$, then

$$\begin{aligned} \| (Fy)_h - (Fy) \|_{L_1} &= \int_0^T | (Fy(s))_h - (Fy(s)) | ds \\ &= \int_0^T \frac{1}{h} | \int_t^{t+h} (Fy(\theta))d\theta - (Fy(s))|ds \\ &\le \int_0^T \frac{1}{h} \int_t^{t+h} |(Fy(\theta)) - (Fy(s))|d\theta ds \\ &\le \int_0^T \frac{1}{h} \int_t^{t+h} |f_1(\theta, \int_0^\theta f_2(\tau, y(\tau), \mu)d\tau) \\ &- f_1(s, \int_0^s f_2(\tau, y(\tau), \mu)d\tau)|d\theta ds. \end{aligned}$$

Since $f_1 \in L_1[0,T]$, it follows that

$$\frac{1}{h} \int_t^{t+h} | f_1(\theta, \int_0^\theta f_2(\tau, y(\tau), \mu)d\tau) - f_1(s, \int_0^s f_2(\tau, y(\tau), \mu)d\tau) | d\theta ds \to 0, \ as h \ \to 0.$$

Hence, $Fy(t)_h \to (Fy)$ uniformly in $L_1[0,T]$. Thus the class $\{Fy\}, \ y \in Q_r$ is relatively compact. Hence F is compact operator.

Now, let $\{y_n\} \subset Q_r, \ y_n \to y$, then

$$| Fy_n(t) | = | f_1(t, \int_0^t f_2(s, y_n(s), \mu)ds) |,$$

take the limit as $n \to \infty$, then

$$\lim_{n\to\infty} Fy_n(t) = \lim_{n\to\infty} f_1(t, \int_0^t f_2(s, y_n(s), \mu)ds)$$

since f_1, f_2 are continuous in y, then

$$\lim_{n\to\infty} Fy_n(t) = f_1(t, \lim_{n\to\infty} \int_0^t f_2(s, y_n(s), \mu)ds).$$

Now, from assumptions $(i), (ii)$ and Lebesgue dominated convergence, we have

$$\lim_{n\to\infty} Fy_n(t) = f_1(t, \int_0^t f_2(s, y(s), \mu)ds) = Fy(t).$$

This means that $\{Fy\}$. Hence the operator F is continuous. Now all the conditions of Schauder fixed point Theorem are satisfied. Then the functional integral equation (3) has at lest one solution $y \in L_1[0,T]$. Consequently, there exists at least one solution $x \in AC[0,T]$ for the problem (10) and (11).

5.1 Unique integrable solution

Consider following assumption:

(i^*) $f_1 : [0,T] \times R \longrightarrow R$ is measurable in $t \in [0,T]$ and satisfies the Lipschitz condition

$$|f_1(t,y) - f_1(t,z)| \leq b_1|y - z|.$$

with Lipschitz condition $b_1 > 0$

(ii^*) $f_2 : [0,T] \times R \times R \longrightarrow R$ is measurable in $t \in [0,T]$ and satisfies the Lipschitz condition

$$|f_2(t, y(s), \mu) - f_2(t, z(s), \mu^*)| \leq b_2(|y - z| + |\mu - \mu^*|)$$

with Lipschitz condition $b_2 > 0$.

(iii^*) $f_1(t,0), \quad f_2(t,0,0) \in L_1[0,T]$.

Now, for existence of a unique integrable solution of functional integral equation (3), we have the following theorem.

Theorem 8. *Let the assumptions $(i^*) - (iii^*)$ and (iii) be satisfied, then the functional equation (3), has a unique solution $y \in L_1[0,T]$. Consequently, the solution x of the problem (10) and (11) is unique.*

Proof. From assumption i^*, we obtain

$$| f_1(t,y) | - | f_1(t,0) | \leq | f_1(t,y) - f_1(t,0) | \leq b_1|y|.$$

This implies that

$$| f_1(t,y) | \leq b_1|y| + | f_1(t,0) | = b_1|y| + a_1(t), \;\; a_1(t) = | f_1(t,0) | .$$

Similarly,

$$| f_2(t,y,\mu) | - | f_2(t,0,0) | \leq | f_2(t,y,\mu) - f_2(t,0,0) | \leq b_2|y| + | \mu | .$$

This implies that

$$| f_2(t,y,\mu) | \leq b_2|y| + | \mu | + a_2(t), \;\; a_2(t) = | f_2(t,0,0) | .$$

Then all assumptions of Theorem 7 are satisfied. Then the solution of the functional integral equation (3) exists. Now let y, z be two solutions of equation (3), then

$$\begin{aligned} |y(t) - z(t)| &= |f_1(t, \int_0^t f_2(s, y(s), \mu)ds) - f_1(t, \int_0^t f_2(s, z(s), \mu)ds)| \\ &\leq b_1 \mid \int_0^t f_2(s, y(s), \mu)ds - \int_0^t f_2(s, z(s), \mu)ds \mid \\ &\leq b_1 \int_0^t \mid f_2(s, y(s), \mu) - f_2(s, z(s), \mu) \mid ds \\ &\leq b_1 b_2 \int_0^t \mid y(s) - z(s) \mid ds. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \| y - z \|_{L_1} &\leq b_1 b_2 \int_0^T \int_0^t \mid y(s) - z(s) \mid dsdt \\ &\leq b_1 b_2 T \| y - z \|_{L_1} . \end{aligned}$$

Since $b_1 b_2 T < 1$, this implies that $y = z$, i.e. the solution of the functional integral (3) is unique. Consequently, the solution of the problem (10) and (11) is unique.

-

Continuous dependence on the parameter μ

Definition 4. *The solution $y \in L_1[0,T]$ of the functional integral equation (3) depends continuously on the parameter μ , if*

$$\forall \epsilon > 0, \quad \exists\ \delta(\epsilon)\ s.t\ |\mu - \mu^*| < \delta \Rightarrow \| y - y^* \|_{L_1} < \epsilon$$

where $y^* \in L_1[0,T]$ is the unique solution of the functional integral equation (3) corresponding to μ^*.

Theorem 9. *Let the assumptions of Theorem 8 be satisfied, then the solution of the functional integral equation (3) depends continuously on the parameter μ.*

proof. Let $y,\ y^*$ be the solutions of the functional integral equations (3) corresponding to μ and μ^* respectively, then

$$\begin{aligned} |\ y(t) - y^*(t)\ | &= |\ f_1(t, \int_0^t f_2(s, y(s), \mu)ds) - f_1(t, \int_0^t f_2(s, y^*(s), \mu^*)ds)\ | \\ &\leq b_1 \int_0^t |\ f_2(s, y(s), \mu)ds - f_2(s, y^*(s), \mu^*)\ |\ ds \\ &\leq b_1 b_2 \int_0^t (|\ y(s) - y^*(s)\ | + |\ \mu - \mu^*\ |)ds. \end{aligned}$$

Hence

$$\begin{aligned} \| y - y^* \|_{L_1} &\leq b_1 b_2 \int_0^T \int_0^t (|\ y(s) - y^*(s)\ | + |\ \mu - \mu^*\ |)dsdt \\ &\leq b_1 b_2 (T \| y - y^* \|_{L_1} + |\ \mu - \mu^*\ |\ (\frac{T^2}{2})), \end{aligned}$$

thus

$$\| y - y^* \|_{L_1}\ (1 - b_1 b_2 T) \leq \frac{1}{2} b_1 b_2 T^2 \delta$$

and

$$\| y - y^* \|_{L_1} \quad \leq \quad \frac{\frac{1}{2} b_1 b_2 T^2 \delta}{(1 - b_1 b_2 T)} \leq \epsilon.$$

This proves the continuous dependence of the solution $y \in L_1[0,T]$ of the functional integral equation (3) on the parameter μ.

Corollary 3. *The solution $x \in AC[0,T]$ of the problem (10)-(11) depends continuously on the parameter μ.*

Continuous dependence on the function f_2

Definition 5. *The solution $y \in L_1[0,T]$ of the functional integral equation (3) depends continuous on the parameter f_2, if*

$$\forall \epsilon > 0, \quad \exists\ \delta(\epsilon), \quad s.t\ \ |\ f_2 - f_2^*\ | < \delta \Rightarrow \| y - y^* \|_{L_1} < \epsilon$$

where y^* is the unique solution $y^* \in L_1[0,T]$ of the functional integral equation (3) corresponding to f_2^*

Theorem 10. *Let the assumptions of Theorem 8 be satisfied, then the solution of the functional integral equation(3) depends continuously on the function f_2.*

Proof. Let $y,\ y^*$ be the solution of the functional integral equation(3) corresponding to f_2 and f_2^* respectively, then

$$\begin{aligned} |\ y(t) - y^*(t)\ | &= |\ f_1(t, \int_0^t f_2(s, y(s), \mu)ds) - f_1(t, \int_0^t f_2^*(s, y^*(s), \mu)ds)\ | \\ &\leq b_1\ |\ \int_0^t f_2(s, y(s), \mu)ds - \int_0^t f_2^*(s, y^*(s), \mu)ds\ | \\ &\leq b_1 \int_0^t |\ f_2(s, y(s), \mu) - f_2^*(s, y^*(s), \mu)\ |\ ds \\ &\leq b_1 \int_0^t |\ f_2(s, y(s), \mu) - f_2^*(s, y(s), \mu) - f_2^*(s, y(s), \mu) - f_2^*(s, y^*(s)), \mu\ |\ ds \\ &\leq b_1 \int_0^t |\ f_2(s, y(s), \mu) - f_2^*(s, y(s), \mu)\ |\ ds \\ &+ b_1 \int_0^t |\ f_2^*(s, y(s), \mu) - f_2^*(s, y^*(s), \mu)\ |\ ds \\ &\leq b_1 \int_0^t |\ f_2(s, y(s), \mu) - f_2^*(s, y(s), \mu)\ |\ ds + b_1 b_2 \int_0^t |\ y(s) - y^*(s)\ |\ ds. \end{aligned}$$

Hence

$$\| y - y^* \|_{L_1} (1 - b_1 b_2) \leq \frac{1}{2} b_1 T^2 \delta$$

and

$$\| y - y^* \|_{L_1} \leq \frac{\frac{1}{2} b_1 T^2 \delta}{(1 - b_1 b_2 T)} = \epsilon$$

This proves the continuous dependence of the solution of the functional integral equation (3) on the function f_2.

Corollary 4. *The solution x of the problem (10)-(11) depends continuously on the function f_2.*

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