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Original Article

Proposing a Novel Collatz Representation of Numbers to Establish a Different Approach in proving the Collatz Conjecture

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Abstract - *This paper proposes a novel representation of natural numbers in order to provide a framework for solving the Collatz Conjecture. The paper also through a series of direct and inductive proofs shows how these number representations called the Collatz Representation can be used to find the number of steps for a number to reach 1 according to the Collatz Conjecture while also identifying patterns.*

Keywords — *Collatz Conjecture, Hailstone Sequence, Induction.*

I. INTRODUCTION

The Collatz Conjecture is one of the most famous unsolved problems in mathematics. It was proposed by Lothar Collatz in 1937 and has since then been reputed as the simplest yet the one of the most difficult conjectures to prove in the area and field of mathematics. It is also known as the Syracuse Problem, the 3x+1 Problem and the Hailstone Problem. The Collatz funtion C(n) is defined as

$$C(n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd.} \\ n/2 & \text{If } n \text{ is even.} \end{cases}$$

The Collatz Conjecture states that repetition of this function results in a value 1 for all

$$n \in W.$$

Mathematically,

$$C^i(n) = 1 \text{ for } C^0(n) = n \in W.$$

II. DEFINITION

A. Collatz Representation: The representation of a whole number in the format x y w_1 w_2 .... w_i such that

$$n = \frac{2^x - 2^y - \sum_{i=1}^{w} 2^{z_i}(3)^i}{(3)^{(w+1)}}.$$

B. A digit of the Collatz Representation is defined as x,y,wi and so on even if they are numbers with more than one digit.

C. A pure Collatz Representation of a number is when the difference between the first and second digits is a multiple of two greater than 2.

D. Hailstone Numbers: These are numbers that appear in the sequence of iterations of a number in the Collatz Conjecture.

III. PROOF OF A LOWER SERIES

$$C(n) = \begin{cases} (2a+1)n+1 & if\ n\ is\ odd. \\ n/2 & if\ n\ is\ even. \end{cases}$$

$$n = \frac{2^x - 1}{2a+1}$$

$$n = \frac{2^x - 2^y - (2a+1)}{(2a+1)^2}$$

$$n = \frac{2^x - 2^y - \sum_{i=1}^{w} 2^{z_i}(2a+1)^i}{(2a+1)^{(w+1)}}$$

Taking a=0,

$$n = 2^x - 2^y - \sum_{i=1}^{w} 2^{z_i}$$

It can be proven that all whole numbers n can be expressed as this sum as

$$2^y + \sum_{i=1}^{w} 2^{z_i}$$

is maximum when y=x-1,

$$z_i = z_{i-1} - 1$$

. When this is so, the sum to be subtracted is a geometric series with initial term

$$2^{x-1}$$

and common ratio 1/2. Therefore, sum to be subtracted =

$$2^{w+1} - 1$$

$$x = y + 1$$

$$\Rightarrow x = w + 1$$

$$\Rightarrow Minimum\ n = 2^{w+1} - 2^{w+1} + 1 = 1$$

Any powers of 2 can be added to 1 from 0 to w+1, and if arranged in a decreasing order, this forms a binary representation. This proves that for a=0, the extended Collatz Conjecture is true. It also shows that the binary representation of numbers is an application of the extended Collatz Conjecture for a=0.

For a=1,

$$n = \frac{2^x - 2^y - \sum_{i=1}^{w} 2^{z_i}(3)^i}{(3)^{(w+1)}}$$

This is an alternative form of the Collatz Conjecture and proving that all numbers n can be expressed as above is an alternative approach to prove the
Collatz Conjecture. If the Collatz Conjecture is assumed to be true,

n	x	y	z_1
1	4	2	0
3	5	1	0
2	5	3	2
6	6	2	1
5	6	4	0
4	6	4	2
13	7	3	0

Fig 1: Example Collatz Representations

IV. COLLATZ REPRESENTATION

A few observations and proofs from the Collatz representation are:

A. It is observed that assigning

$$x = x + 2,\ y = x,\ z_1 = y \text{ and } z_i = z_{i-1}$$

, the same number is obtained for w=w+1. Proof:

$$n = \frac{2^x - 2^y - \sum_{i=1}^{w} 2^{z_i}(3)^i}{(3)^{(w+1)}}$$

$$\frac{2^{x+2} - 2^x - 2^y 3 - \sum_{i=2}^{w+1} 2^{z_i}(3)^i}{(3)^{(w+2)}}$$

$$= \frac{2^x 3 - 2^y 3 - \sum_{i=2}^{w+1} 2^{z_i}(3)^i}{(3)^{(w+2)}}$$

$$= 3\frac{2^x - 2^y - \sum_{i=1}^{w+1} 2^{z_i}(3)^i}{(3)^{(w+2)}}$$

$$= \frac{2^x - 2^y - \sum_{i=1}^{w+1} 2^{z_i}(3)^i}{(3)^{(w+1)}}$$

$$= n$$

B. Any number is multiplied by 2 when 1 is added to each digit.
Proof:

$$\frac{2^{x+1} - 2^{y+1} - \sum_{i=1}^{w} 2^{z_i+1}(3)^i}{(3)^{(w+1)}} = 2\frac{2^x - 2^y - \sum_{i=1}^{w} 2^{z_i}(3)^i}{(3)^{(w+1)}}$$

C. When 0 is removed from the last digit of a number in its Collatz representation, the next Hailstone number in the series is found.
Proof:

$$\frac{2^{x+1}-2^{y}-\sum_{i=1}^{w-1}2^{z_i}(3)^i+3^w}{(3)^{(w+1)}}=\frac{\frac{2^x-2^y-\sum_{i=1}^{w-2}2^{z_i}(3)^i}{(3)^{(w)}}-1}{3}$$

$$\frac{2^{x}-2^{y}-\sum_{i=1}^{w-2}2^{z_i}(3)^i}{(3)^{(w)}}=3\frac{2^{x+1}-2^{y}-\sum_{i=1}^{w-1}2^{z_i}(3)^i+3^w}{(3)^{(w+1)}}+1$$

D. The difference between the first and second digits is a multiple of 2. Proof: It is known that in order for the number to be divisible by the denominator, the whole numerator has to be divisible by 3. Therefore

$$(2^x - 2^y) \equiv 0(mod3)$$

$$2^y(2^{x-y} - 1) \equiv 0(mod3)$$

$$2^{x-y} - 1 \equiv 0(mod3)$$

It is observed from all Collatz Representations that x-y=2a where a belongs to N. Therefore, conjecturing that

$$2^{2a} - 1 = 3x$$

$$4^a - 1 = 3x$$

Base case: Assume a=1.

P(1) is true as 4-1=3.

Assume P(k) is true.

$$4^k - 1 = 3x$$

Verifying P(k+1)

$$4^{k+1} - 1 = 4.4^k - 1 = 4(4^k - 1) + 3 = 12x + 3 = 3(4x + 1) = 3y$$

Therefore, proven by the principle of mathematical induction. However, it also has to be proven that

$$2^{2a-1}-1\neq 3x$$

$$2^{2a-1}-1=\frac{4^a}{2}-1=\frac{4^a-2}{2}\neq 3x$$

Therefore, proven.

E. It is also observed that a number with the digits n n-1....1 0 in the end give the number preceding it when 2 is added to n and 1 is added to all following digits if the preceding digits are greater than n (by a multiple of 2 if n is the second digit.) For this statement to be true, the subtracted number has to be equal to one. Therefore, numerator is 3^{w+1} .

Therefore,

$$P(a): (2^{a+2} - 2^{a})3^{w-a} + \sum_{i=a-1}^{k} (2^{i+1} - 2^{i})3^{w-i} = 3^{w+1}$$

Base case:

$$P(0): (4-1)3^{w} = 3^{w+1}$$

Assume P(k) is true.

$$P(k): (2^{k+2} - 2^{k})3^{w-k} - \sum_{i=k-1}^{0} (2^{i+1} - 2^{i})3^{w-i} = 3^{w+1}$$

Verifying P(k+1).

$$(2^{k+3} - 2^{k+1})3^{w-k-1} + \sum_{i=k}^{0} (2^{i+1} - 2^{i})3^{w-i}$$

$$= (2^{k+2} - 2^{k})3^{w-k} + \sum_{i=k-1}^{0} (2^{i+1} - 2^{i})3^{w-i} - 2^{k+2}3^{w-k} + 2^{k}3^{w-k} + 2^{k+1}3^{w-k} - 2^{k}3^{w-k} + (2^{k+3} - 2^{k+1})3^{w-k-1}$$

$$= 3^{w+1} - 2^{k+1}3^{w-k} + 2^{k+1}3^{w-k-1+1}$$

$$= 3^{w+1} - 2^{k+1}3^{w-k} + 2^{k+1}3^{w-k}$$

$$= 3^{w+1}$$

Therefore, proven by the principle of mathematical induction.

V. PROGRAMS

A. The following represents a Python program for creating a Hailstone sequence where i is an array of the digits of the Collatz Representation of a number.

```
#Loop for checking whether or not 1 has
#been reached in the Hailstone sequence.
while i[0]!=0:
    print(i)
    #Eliminating digits to obtain
    #Collatz Representation in the pure form.
    if len(i)>1:
        if i[0]-i[1]==2:
            i.remove(i[0])
    #Loop for purest Collatz Representation.
    while i[len(i)-1]!=0:
        #Dividing by 2 until the number
        #becomes odd.
        j=0
        while j<len(i):
            i[j]=i[j]-1
            j=j+1
        print(i)
    #If last digit is 0 it removes it to
    #move to next number in Collatz or Hailstone
    #sequence. Only the last digit can possibly be 0
    #as the digits are in strictly
    #descending order.
    if i[len(i)-1]==0 and len(i)>1:
        i.remove(0)
```

B. The following Python code snippet is used in order to find the Collatz Representation of a number:

```
x=12#The number for which the
#pure Collatz Representation
#is to be found.
A=[0]
a=0
while x>1:
    if x%2==0:
        x/=2
        #Adding one to digit.
        A[a]+=1
    else:
        x=3*x+1
        #Creating 1 new digit.
        a+=1
        A.append(0)
A.reverse()
I=[]#Array of the pure Collatz Representation of the number x.
#Each digit is larger than the previous one as shown before in
#this paper.
j=sum(A)
for i in A:
    I.append(j)
    j-=i
print(I)
```

C. The following Python code snippet is used in order to find the number from its Collatz Representation (verification program):

```
a=I[0]#I[] is list of Collatz Representation digits for
#which the number is to be
#found.
x=0
for i in I.remove(I[0]):
    a-=((2**i)*(3**x))
    x+=1
a/=3**(x+1)
print(a)
```

VI. FRAMEWORK TO SOLVE THE COLLATZ CONJECTURE

A. The number of steps a number takes to reach one should be the sum of first digit of and the number of digits in the pure Collatz Representation of a number.

Proof: The algorithm is proven to work as verified by A., B. and C. from Collatz Representation. Since 1 is subtracted from all digits till each time last digit is equal to zero, the first digit represents the number of divisions by 2. Since a zero also has to be removed from the last digit, each digit adds a step.

B. Another point is that the difference between the first and last digits of the pure Collatz Representation of a number gives the greatest power of 2 in the Hailstone series or sequence of that number as after divisions by 2 and after repeated 3x+1 iterations, the last digit left is the difference between the first and last digits.

C. As can be seen, it may be easier to prove and observe patterns with respect to the Collatz Conjecture in the Collatz Representation of a number. These patterns may be used to find the Collatz Representation of a number in a more convenient way. Proofs such as E. help in showing that numbers divisible by 4 and not by 8 except 4 require the same number of steps to reach 1 as the number plus 1. Such observations may be noted and can be used to prove the Collatz Conjecture to be true if adequately sufficient proofs are found.

D. An approach to proving the Collatz Conjecture to be true could also be based upon proving that all natural numbers have a Collatz Representation.

VII. CONCLUSION

Therefore, in conclusion, the Collatz Representation of natural numbers may help in solving or proving the Collatz Conjecture by observing patterns with more ease and by providing a framework to prove these patterns by simple inductive and direct proofs.

REFERENCES

[1] Bart S., Matt T., The Collatz Problem and Analogues, Journal of Integer Sequences, 11 (2008).