

Prime Cordial Labeling for Eight Sprocket Graph

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Abstract - This papers deals with prime cordial labeling of newly introduced eight sprocket graph. This graph is already proven as cordial, Edge cordial and gracious in graph labelling. In our study we have further proved that Eight Sprocket graph related families of connected are Prime cordial graphs. Also the path union of Eight Sprocket graph, cycle of Eight Sprocket graph and star of Eight Sprocket graph are holds well with prime-cordial.

Keywords - Prime cordial, Eight- sprocket graph, Path union of graphs, Cycle of graphs, Star of a graph.

I. INTRODUCTION

The concept of cordial labeling of graph was introduced by Cahit [3] in 1987 and for numbering in graph was defined by S. W.Golomb [2,9]. It is found from Gallian [4] that many researchers have studied cordialness of several graphs. The new graph named Eight Sprocket graph was introduced by J. C. Kanani and V. J. Kaneria [5]. The graceful labeling and cordial labeling for the said graph is already proven by the same authors. Already prime cordial labeling of some wheel related graphs and for Some Cycle Related graphs were discussed by S. K. Vaidya [7,8] In this paper, the notions and definitions are followed from Harary [1]. Let us recall some basic definitions, which are used in this paper.

Definition 1.1 A prime cordial labeling of a graph G with the vertex set $V(G)$ is a bijection $f:V(G) \rightarrow \{1,2,3, \dots, |V(G)|\}$ such that each edge uv is assigned the label 1 if $\gcd(f(u), f(v)) = 1$ and 0 if, $\gcd(f(u), f(v)) > 1$ then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1.

Definition 1.2 Let G be a graph and G_1, G_2, \dots, G_n $n \geq 2$ be n copies of graph G . Let $v \in V(G)$ Then the graph obtained by joining vertex v of $G^{(i)}$ with the same vertex of $G^{(i+1)}$ by an edge, $\forall i = 1,2,3, \dots n - 1$ is called a path union of n copies of a graph G . Also if the same vertex v of $G^{(n)}$ join by an edge with v of $G^{(1)}$ then such graph is known as cycle graph of n copies of G . These are denoted by $P(n: G)$ and $C(n: G)$ respectively. Obviously $P(n: K_1) = P_n$ and $C(n: K_n) = C_n$

Definition 1.3 Let G be a graph on n vertices. The graph obtained by replacing each vertex of the star $K_{1,n}$ by a copy of G is called a star of G and is denoted by G^*

Definition 1.4 Eight Sprocket graph is an union of eight copies of C_{4n} . If $V_{i,j}$ ($\forall i = 1,2, \dots 8$; $\forall j = 1,2, \dots 4n$) be vertices of i^{th} copy of C_{4n} then we shall combine $V_{1,4n}$ and $V_{2,1}$, $V_{2,4n}$ and $V_{3,1}$, $V_{3,4n}$ and $V_{4,1}$, $V_{4,4n}$ and $V_{5,1}$, $V_{5,4n}$ and $V_{6,1}$, $V_{6,4n}$ and $V_{7,1}$, $V_{7,4n}$ and $V_{8,1}$ and $V_{8,4n}$ and $V_{1,1}$ by a single vertex. Where $n \in N - 1$. So, Graph becomes sprocket shaped, and here the number of sprockets are eight. Hence, named as Eight Sprocket. It is denoted by S_{C_n} of size n , Where $n \in N - 1$. $|V(S_{C_n})| = 16n - 8$, $|E(S_{C_n})| = 16n$. The coordinates of Eight Sprocket graph, path union of Eight Sprocket graph, cycle of Eight Sprocket graph and star of Eight Sprocket graph are already well define by J. C. Kanani and V. J. Kaneria [6]

II. MAIN RESULTS

Theorem 2.1 An Eight Sprocket graph is Prime cordial Graph, where $n \in N - \{1\}$.

Proof: let $G = S_{C_n}$ be any Eight Sprocket graph of size n , where $n \in N - \{1\}$. We mention each vertex of S_{C_n} like $V_{i,j}$ ($\forall i = 1,2,3 \dots 8$; $\forall j 1,2,3 \dots 4n$). We see the number of vertices in G is $|V(S_{C_n})| = p = 16n - 8$ and $|E(S_{C_n})| = q = 16n$.

We define labeling function $f:V(G) \rightarrow \{1,2,3 \dots |V(G)|\}$

$$\text{For } f(\vartheta_{1,j}) = \begin{cases} \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, \vartheta_6, \vartheta_7, \vartheta_8 & ; \{1,3,5,7,9,11,13,15\} \\ \vartheta_9, \vartheta_{10}, \vartheta_{11}, \vartheta_{12}, \vartheta_{13}, \vartheta_{14}, \vartheta_{15}, \vartheta_{16} & ; \{2,4,6,8,12,14,16,10\} \end{cases}$$



$$\begin{aligned}
 f(\vartheta_2, j) &= \begin{cases} \vartheta_{17}, \vartheta_{18}, \dots \vartheta_{23} & ; \{17, 19, 21, 23, 25, 27, 29\} \\ \vartheta_{24}, \vartheta_{25}, \dots \vartheta_{31} & ; \{18, 20, 22, 24, 26, 28, 30, 32\} \end{cases} \\
 f(\vartheta_3, j) &= \begin{cases} \vartheta_{31}, \vartheta_{32}, \dots \vartheta_{38} & ; \{31, 33, 35, 37, 39, 41, 43, \} \\ \vartheta_{39}, \vartheta_{40}, \dots \vartheta_{46} & ; \{34, 36, 38, 40, 42, 44, 46, 48\} \end{cases} \\
 f(\vartheta_4, j) &= \begin{cases} \vartheta_{47}, \vartheta_{48}, \dots \vartheta_{53} & ; \{45, 47, 49, 51, 53, 55, 57\} \\ \vartheta_{54}, \vartheta_{55}, \dots \vartheta_{61} & ; \{50, 52, 54, 56, 58, 60, 62, 59\} \end{cases} \\
 f(\vartheta_5, j) &= \begin{cases} \vartheta_{62}, \vartheta_{64}, \dots \vartheta_{68} & ; \{45, 47, 49, 51, 53, 55, 57\} \\ \vartheta_{69}, \vartheta_{70}, \dots \vartheta_{76} & ; \{64, 66, 68, 70, 72, 74, 76, 75\} \end{cases} \\
 f(\vartheta_6, j) &= \begin{cases} \vartheta_{77}, \vartheta_{78}, \dots \vartheta_{83} & ; \{77, 79, 81, 83, 85, 90, 87\} \\ \vartheta_{84}, \vartheta_{85}, \dots \vartheta_{91} & ; \{78, 80, 82, 84, 86, 88, 92, 89\} \end{cases} \\
 f(\vartheta_7, j) &= \begin{cases} \vartheta_{92}, \vartheta_{93}, \dots \vartheta_{98} & ; \{91, 93, 97, 99, 101, 101, 95\} \\ \vartheta_{99}, \vartheta_{100}, \dots \vartheta_{106} & ; \{94, 96, 98, 102, 104, 106, 108, 100\} \end{cases} \\
 f(\vartheta_8, j) &= \begin{cases} \vartheta_{107}, \vartheta_{108}, \dots \vartheta_{113} & ; \{110, 112, 114, 116, 118, 120, 117\} \\ \vartheta_{114}, \vartheta_{115}, \dots \vartheta_{120} & ; \{119, 107, 109, 111, 113, 105, 115\} \end{cases}
 \end{aligned}$$

The above pattern gives rise of prime cordial labeling of the graph G, Hence, G is Prime cordial graph.

Illustration 2.2: Eight Sprocket graph S_{C_n} is shown consisting $n = 8$ sprockets with Prime cordial Labeling with $p = 120$ and $q = 128$ where $vf(0) = vf(1) = 60$ and $ef(0) = ef(1) = 64$

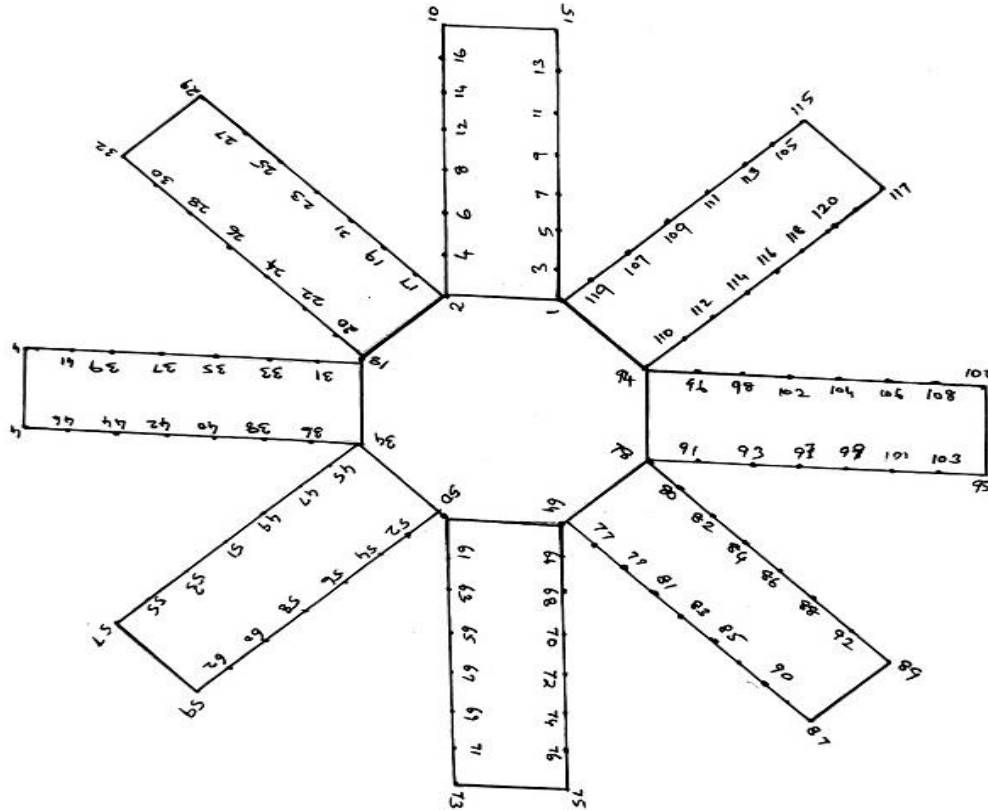


Figure 1: Prime cordial labeling of Eight sprocket graph with $p = 120$ and $q = 128$

Theorem 2.3 : Path union of finite copies of the Eight Sprocket graph S_{C_n} is a prime cordial graph ,
 where $n \in N - \{1\}$.

Proof: Let $G = P(r, S_{C_n})$ be a path union of r copies of the eight sprocket graph S_{C_n} , where $n \in N - \{1\}$.

Let f be the prime cordial labeling of S_{C_n} , as mentioned in theorem 2.1. In graph G, we observe that the number of vertices $|v(G)| = P = r(16(n) - 8)$ and the number of edges $|e(G)| = q = (r - 1) + r 16 (n)$.

Let $v_{k,i,j}$ ($\forall i = 1,2, \dots,8 ; \forall j = 1,2,3 \dots,4n$) be the vertices of k^{th} copy of S_{C_n} ($\forall k = 1,2,3 \dots, r$) where the number of vertices $p = 16(n) - 8$ and number of edges $q = 16n$. Join vertices $v_{k,1,2n+1}$ with $v_{k+1,1,2n+1}$ for $k = 1,2,3 \dots, r - 1$ by an edge to form the path union of r copies of eight Sprocket graph.

The labeling of the function $f: V(G) \rightarrow \{1,2,3 \dots, |V(G)|\}$

$$\begin{aligned}
 g(v_{1,i,j}) &= f(v_{1,i,j}) \\
 g(v_{2,i,j}) &= \begin{cases} g(v_{1,i,j}) + 1 & ; \text{ if } j = 1,2,5,6,9,10, \dots, 4n - 3, 4n - 2 \\ g(v_{1,i,j}) - 1 & ; \text{ if } j = 3,4,7,8 \dots, 4n - 1, 4n \end{cases} \\
 g(v_{3,i,j}) &= f(v_{2,i,j}) \\
 g(v_{k,i,j}) &= g(v_{k-3,i,j}) \text{ if } k = 4,5,6,7 \dots, 3n + 1, 3n + 2, 3n + 3.
 \end{aligned}$$

Where

$$\begin{aligned}
 f(v_{1,j}) &= \begin{cases} v_1, v_2, v_3, v_4, v_5, v_6 & ; \{1, 3, 5, 7, 11, 9\} \\ v_7, v_8, v_9, v_{10}, v_{11}, v_{12} & ; \{2, 4, 6, 8, 10, 12\} \end{cases} \\
 f(v_{2,j}) &= \begin{cases} v_{13}, v_{14}, v_{15}, v_{16}, v_{17} & ; \{13, 15, 17, 19, 21\} \\ v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23} & ; \{14, 16, 18, 20, 22, 24\} \end{cases} \\
 f(v_{3,j}) &= \begin{cases} v_{24}, v_{25}, v_{26}, v_{27}, v_{28} & ; \{23, 25, 27, 29, 31\} \\ v_{29}, v_{30}, v_{31}, v_{32}, v_{33}, v_{34} & ; \{26, 28, 32, 34, 30, 33\} \end{cases} \\
 f(v_{4,j}) &= \begin{cases} v_{35}, v_{36}, v_{37}, v_{38}, v_{39} & ; \{35, 37, 39, 41, 43\} \\ v_{40}, v_{41}, v_{42}, v_{43}, v_{44}, v_{45} & ; \{36, 38, 40, 42, 44, 46\} \end{cases} \\
 f(v_{5,j}) &= \begin{cases} v_{46}, v_{47}, v_{48}, v_{49}, v_{50} & ; \{45, 47, 49, 51, 53\} \\ v_{51}, v_{52}, v_{53}, v_{54}, v_{55}, v_{56} & ; \{48, 50, 52, 54, 56, 58\} \end{cases} \\
 f(v_{6,j}) &= \begin{cases} v_{57}, v_{58}, v_{59}, v_{60}, v_{61} & ; \{55, 57, 59, 61, 63\} \\ v_{62}, v_{63}, v_{64}, v_{65}, v_{66}, v_{67} & ; \{60, 62, 64, 66, 68, 65\} \end{cases} \\
 f(v_{7,j}) &= \begin{cases} v_{68}, v_{69}, v_{70}, v_{71}, v_{72} & ; \{67, 69, 71, 73, 75\} \\ v_{73}, v_{74}, v_{75}, v_{76}, v_{77}, v_{78} & ; \{70, 72, 74, 76, 78, 80\} \end{cases} \\
 f(v_{8,j}) &= \begin{cases} v_{79}, v_{80}, v_{81}, v_{82}, v_{83} & ; \{85, 77, 79, 81, 83\} \\ v_{84}, v_{85}, v_{86}, v_{87}, v_{88} & ; \{82, 84, 86, 88, 87\} \end{cases}
 \end{aligned}$$

Above labeling pattern gives rise of prime cordial labeling to graph G. Hence, the path union of finite copies of the Eight Sprocket graph is prime cordial graph.

Illustration 2.4: Path union of 3 copies of S_{C_3} and its prime cordial labeling is shown in figure 2. With $p = 3(88)$ and $q = 3(96) + 2$ where $vf(0) = vf(1) = 3(44)$ and $ef(0) = ef(1) = 3(48) + 1$

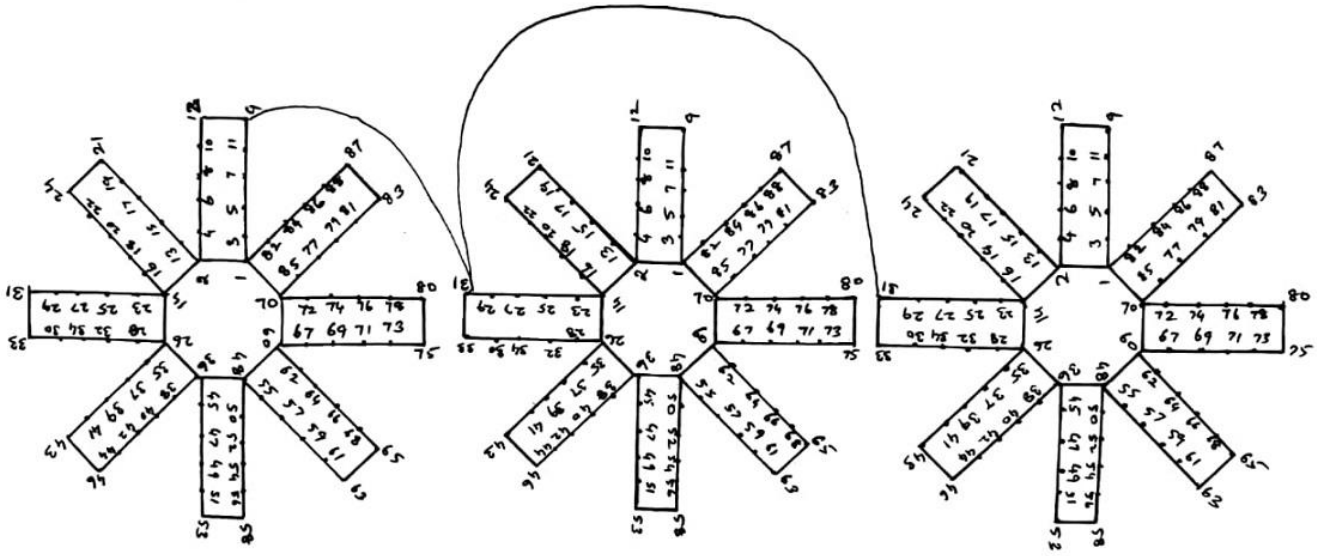


Figure 2: A Path union of 3 copies of S_{C_3} and its prime cordial labeling

Theorem 2.5 : Cycle of r copies of Eight Sprocket graph $C(r, S_{C_n})$ is a prime cordial graph , where $n \in N - \{1\}$ and $r \equiv 0,3 \pmod{4}$.

Proof: Let $G = C(r, S_{C_n})$ be a cycle of Eight Sprocket graph S_{C_n} where $n \in N - \{1\}$. Let f be the prime cordial labeling of S_{C_n} as proved in theorem 2.1 . In graph G , we observe that the number of vertices $|V(G)| = p = r(16(n) - 8)$ and the number of edges $|E(G)| = q = r(16(n) + 1)$. Let $v_{k,i,j}$ ($\forall i = 1,2, \dots, 8 ; \forall j = 1,2,3 \dots, 4n$) be the vertices of k^{th} copy of S_{C_n} ($\forall k = 1,2,3 \dots, r$), where the number of vertices $p = 16(n) - 8$ and number of edges $q = 16n$. Join the vertices $v_{k,1,2n+1}$ with $v_{k+1,1,2n+1}$ for $k = 1,2,3, \dots, r - 1$ and $v_{r,1,2n+1}$ with $v_{1,1,2n+1}$ by an edge to form $C(r, S_{C_n})$

The labeling of the function $f: V(G) \rightarrow \{1,2,3 \dots, |V(G)|\}$

$$\begin{aligned}
 g(\vartheta_{1,i,j}) &= f(\vartheta_{1,i,j}) \\
 g(\vartheta_{2,i,j}) &= \begin{cases} g(\vartheta_{1,i,j}) + 1 & ; \text{ if } j = 1,2,5,6,9,10, \dots, 4n - 3, 4n - 2 \\ g(\vartheta_{1,i,j}) - 1 & ; \text{ if } j = 3,4,7,8 \dots, 4n - 1, 4n \end{cases} \\
 g(\vartheta_{3,i,j}) &= f(\vartheta_{2,i,j}) \\
 g(\vartheta_{4,i,j}) &= f(\vartheta_{1,i,j}) \\
 g(\vartheta_{k,i,j}) &= g(\vartheta_{k-3,i,j}) ; \text{ if } k = 5,6,7, \dots, 4n + 1, 4n + 2, 4n + 3 \\
 g(\vartheta_{k,i,j}) &= g(\vartheta_{k-4,i,j}) ; \text{ if } k = 8,12,16, \dots, 4n + 4
 \end{aligned}$$

Where

$$\begin{aligned}
 f(\vartheta_{1,j}) &= \begin{cases} \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 & ; \{1, 3, 5, 7\} \\ \vartheta_5, \vartheta_6, \vartheta_7, \vartheta_8 & ; \{2, 4, 6, 8\} \end{cases} \\
 f(\vartheta_{2,j}) &= \begin{cases} \vartheta_9, \vartheta_{10}, \vartheta_{11} & ; \{9, 11, 13\} \\ \vartheta_{12}, \vartheta_{13}, \vartheta_{14}, \vartheta_{15} & ; \{10, 12, 14, 16\} \end{cases}
 \end{aligned}$$

$$f(\vartheta_3, j) = \begin{cases} \vartheta_{16}, \vartheta_{17}, \vartheta_{18} & ; \{15, 17, 19\} \\ \vartheta_{19}, \vartheta_{20}, \vartheta_{21}, \vartheta_{22} & ; \{18, 20, 22, 24\} \end{cases}$$

$$f(\vartheta_4, j) = \begin{cases} \vartheta_{23}, \vartheta_{24}, \vartheta_{25} & ; \{21, 23, 25\} \\ \vartheta_{26}, \vartheta_{27}, \vartheta_{28}, \vartheta_{29} & ; \{26, 28, 30, 27\} \end{cases}$$

$$f(\vartheta_5, j) = \begin{cases} \vartheta_{30}, \vartheta_{31}, \vartheta_{32} & ; \{29, 31, 33\} \\ \vartheta_{33}, \vartheta_{34}, \vartheta_{35}, \vartheta_{36} & ; \{32, 43, 38, 36\} \end{cases}$$

$$f(\vartheta_6, j) = \begin{cases} \vartheta_{37}, \vartheta_{38}, \vartheta_{39} & ; \{35, 37, 39\} \\ \vartheta_{40}, \vartheta_{41}, \vartheta_{42}, \vartheta_{43} & ; \{40, 42, 44, 41\} \end{cases}$$

$$f(\vartheta_7, j) = \begin{cases} \vartheta_{44}, \vartheta_{45}, \vartheta_{46} & ; \{43, 47, 49\} \\ \vartheta_{47}, \vartheta_{48}, \vartheta_{49}, \vartheta_{50} & ; \{46, 48, 50, 45\} \end{cases}$$

$$f(\vartheta_8, j) = \begin{cases} \vartheta_{51}, \vartheta_{52}, \vartheta_{53} & ; \{52, 54, 56\} \\ \vartheta_{54}, \vartheta_{55}, \vartheta_{56} & ; \{51, 53, 55\} \end{cases}$$

Above labeling Pattern proves that the cycle of r copies of eight Sprocket graph is prime cordial.

Illustration 2.6 : Cycle of 4 copies for eight Sprocket graph S_{C_4} and its prime cordial labeling is shown in figure 2. With $p = 4(56)$ and $q = 4(64) + 3$ where $vf(0) = vf(1) = 4(28)$ and $ef(0) = ef(1) = 4(32) + 2 \mp r$

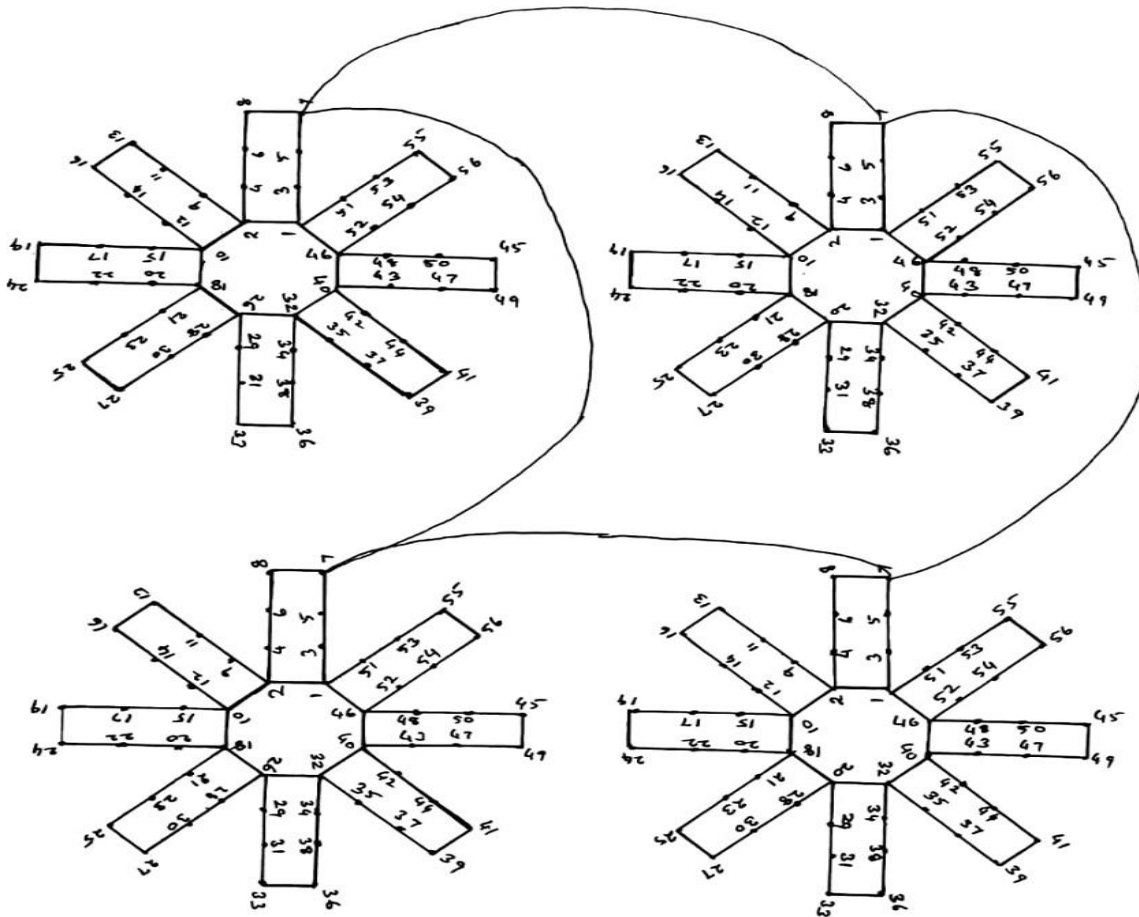


Figure 3: Cycle of 4 copies of eight sprocket graph S_{C_4} is prime cordial.

Theorem 2.7: Star of Eight Sprocket graph $(S_{Cn})^*$ is an Prime cordial graph , where $n \in N - \{1\}$.

Proof: Let $G = (S_{Cn})^*$ be a star graph of eight Sprocket graph S_{Cn} , where $n \in N - \{1\}$. Let f be the prime cordial labeling of S_{Cn} as proved in theorem 2.1

In graph G , we observe that the number of vertices $|V(G)| = P = p(p + 1)$ and the number of edges $|E(G)| = Q = q(p + 1) + p$ where $p = 16(n) - 8$ and $q = 16(n)$.

Let $v_{k,i,j}$ ($\forall i = 1,2, \dots, 8$; $\forall j = 1,2,3 \dots, 4n$) be the vertices of k^{th} copy of S_{Cn} ($\forall k = 1,2,3 \dots, p$). Where the number of vertices $p = 16(n) - 8$ and number of edges $q = 16(n)$. The central copy of $(S_{Cn})^*$ is $(S_{Cn})^0$ and other copies of $(S_{Cn})^*$ is $(S_{Cn})^{(k)}$ ($\forall k = 1,2,3, \dots, p$)

We define prime cordial labeling function $f: V(G) \rightarrow \{1,2,3 \dots, |V(G)|\}$

$$\begin{aligned}
 g(\vartheta_{1,i,j}) &= f(\vartheta_{1,i,j}) \\
 g(\vartheta_{1,i,j}) &= \begin{cases} g(\vartheta_{0,i,j}) + 1 & ; \text{ if } j = 1,2,5,6,9,10, \dots, 4n - 3, 4n - 2 \\ g(\vartheta_{0,i,j}) - 1 & ; \text{ if } j = 3,4,7,8 \dots, 4n - 1, 4n \end{cases} \\
 g(\vartheta_{2,i,j}) &= f(\vartheta_{0,i,j}) \\
 g(\vartheta_{3,i,j}) &= f(\vartheta_{1,i,j}) \\
 g(\vartheta_{k,i,j}) &= g(\vartheta_{k-2,i,j}) ; \text{ if } k = 2,3,4,5, \dots, 4n - 2, 4n - 1, 4n
 \end{aligned}$$

We observe that difference of vertices and edges for the center copy $(S_{Cn})^0$ of G and its other copies $(S_{Cn})^{(k)}$, ($1 \leq k \leq p$) is G . Using this sequence we can construct required prime cordial label by joining corresponding vertices of $(S_{Cn})^0$ with its other copies $(S_{Cn})^{(k)}$, ($1 \leq k \leq p$) in G . Thus it hold well with edge cordial labeling and total prime cordial labeling.

III. CONCLUDING

The present work contributes with some new results. We have discussed prime cordialness of Eight Sprocket graphs, path union of Eight Sprocket graph, and cycle of Eight Sprocket graph and star of Eight Sprocket graph. The labeling pattern is demonstrated by means of illustrations which provide better understanding to derived results.

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