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A STUDY ON ITERATED FUNCTION SYSTEMS

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Abstract- In this paper, every metric d on a nonempty set X induces a metric h, called Hausdorff metric, on the set $\mathcal{K}(X)$ of the collection of all non-empty compact subsets of X. It is a well known fact that the induced metric preserves the completeness and compactness. In this paper, we discuss the existence of attractors, which is generally known as fractals, of iterated function systems using the Hausdorff metric. We also discuss how to construct the attractors of iterated function systems using the fixed points of contraction maps involved in the iterated function systems.

Keywords : Hausdorff metric, Compact, Complete,Contraction, Iterate Function Systems,Attractor.

I INTRODUCTION

In the modern era of mathematics the theory of classical geometry was replaced by the theory of fractal geometry. Of course the classical geometry has it own advantage and limitations too. For instance the classical geometry fails to provides effective modeling for the infinite details found in nature, there comes the birth of Fractal Geometry. The theory of fractal geometry was originated in the book "The Fractal Geometry of Nature" by the eminent mathematician Benoit Mandelbrot [6]. It was the fundamental work of Benovit Mandelbrot that opened up a new way to model natural phenomena. What is a Fractal? many definitions exists, and mathematicians have not yet agreed on one. Benovit Mandelbrot refers to the word Fractal as objects which possess Self Similarity.

Self similar objects are the objects that was made up of a number of small (in size) copies of itself. For example, lets take the galaxy which looks like a solar system but its eventually made up of number of solar system. If we accept the definition of fractals as a self similar objects, how one could arrive the fractal using mathematical modeling.

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J. Hutchinson introduced a navel method to generate self similar objects using iterated function systems [3]. Later M.F. Barnsley provided a unified theory of iterated functions systems, called Hutchinson-Barnsley theory [1]. After M.F. Barnsley's book by "Fractals Everywhere" many mathematicians shown interest towards fractal geometry, for instance see [2] and the references therein. This theory of iterated function systems applications in quantum mechanics [4], image compression [14] and many other fields. Given a metric space (X, d) and a finite collection of contraction maps on X, we can construct an attractor (which is generally a fractal [3]) using the Hausdorff metric on $\mathcal{K}(X)$, the set all compact subsets on X, and Banach contraction principle [1, 2]. N.A. Secelean [8] extended this method of constructing attractors from the finite collection of contraction mapping to the countable collection of contraction mappings and explored many metric properties (See [9, 10, 11, 12, 13]). These results further extended to collection of contractions on the product spaces [5, 7].

We attempt to provide a unified theory of iterated function systems right from finite iterated function systems to the generalized iterated function systems. This paper is organized as follows: Section 2 provides preliminary results on constructing Hausdorff metric space and explored many topological properties. In Section 3, we discussed iterated function systems(IFS) and proved the existence of unique attractor of the IFS and illustrated with some example.

II PRELIMINARY

Definition 2.1 *Let (X, d) be a metric space and $\mathcal{K}(X)$ be the class of all non-empty compact subsets of X. We define the distance function $h : \mathcal{K}(X) \times \mathcal{K}(X) \to \mathbb{R}$ as follows:*

$$h(A, B) := \max\{D(A, B), D(B, A)\} \tag{1}$$

where $D(A, B) := \sup\{r(x, B) \mid x \in A\}$ *and* $r(x, B) := \inf\{d(x, y) \mid y \in B\}$

Our ultimate aim of this chapter is to prove $(\mathcal{K}(X), h)$ is a metric space. In particular the space $(\mathcal{K}(X), h)$ is compact and complete provided (X, d) is compact and complete respectively. Most of the results in this chapter can be found in [**?**]. The following results will be useful in proving our main results.

Definition 2.2 *A set $A \subseteq X$ is* ***sequentially compact*** *in (X, d) if each sequence in A has a subsequence that converges to a point in A.*

Theorem 2.3 *Let $\{x_n\}$ and $\{y_n\}$ be sequence in a metric space (X, d). If $\{x_n\}$ converges to x and $\{y_n\}$ converges to y, then $\{d(x_n, y_n)\}$ converges to $d(x, y)$.*

Theorem 2.4 *Let $x \in X$ and let $A, B, C \in \mathcal{K}(X)$. Then*

1. *$r(x, A) = 0$ if and only if $x \in A$.*
2. *$D(A, B) = 0$ if and only if $A \subseteq B$.*
3. *There exist $a_x \in A$ such that $r(x, A) = d(x, a_x)$.*
4. *There exist $a \in A$ and $b \in B$ such that $D(A, B) = d(a, b)$.*
5. *If $A \subseteq B$, then $r(x, B) \leq r(x, A)$.*
6. *If $B \subseteq C$, then $D(A, C) \leq D(A, B)$.*
7. *$D(A \cup B, C) = \max\{D(A, C),\ D(B, C)\}$.*
8. *$D(A, B) \leq D(A, C) + D(C, B)$.*

Definition 2.5 *Given a set $A \in \mathcal{K}(X)$ and a positive number ε. We define*

$$A + \varepsilon := \{x \in X \mid r(x, A) \leq \varepsilon\}.$$

Proposition 2.6 *For each $A \in \mathcal{K}(X)$ and $\varepsilon > 0$ the set $A + \varepsilon$ is closed.*

Theorem 2.7 *Suppose that $A, B \in \mathcal{K}(X)$ and that $\varepsilon > 0$. Then $h(A, B) \leq \varepsilon$ if and only if $A \subseteq B + \varepsilon$ and $B \subseteq A + \varepsilon$.*

Theorem 2.8 *Let $\{A_n\}$ be a Cauchy sequence in $\mathcal{K}(X)$ and let $\{n_k\}$ be an increasing sequence of positive integers. If $\{x_{n_k}\}$ is a Cauchy sequence in X for which $x_{n_k} \in A_{n_k}$ for all k, then there exists a Cauchy sequence $\{y_n\}$ in X such that $y_n \in A_n$ for all n and $y_{n_k} = x_{n_k}$ for all k.*

Theorem 2.9 *Let $\{A_n\}$ be a sequence in $\mathcal{K}(X)$ and let A be the set of all points $x \in X$ such that there is a sequence $\{x_n\}$ that converges to x and satisfies $x_n \in A_n$ for all n. If $\{A_n\}$ is a Cauchy sequence, then the set A is closed and nonempty.*

Theorem 2.10 *Let $\{D_n\}$ be a sequence of totally bounded sets in X and let A be any subset of X. If for each $\varepsilon > 0$, there exists a positive integer N such that $A \subseteq D_N + \varepsilon$, then A is totally bounded.*

Theorem 2.11 *If (X, d) is complete, then $(\mathcal{K}(X), h)$ is complete.*

Theorem 2.12 *If (X, d) is compact then $(\mathcal{K}(X), h)$ is compact.*

III ITERATED FUNCTION SYSTEMS

Introduction

In this section, we discuss the theory of iterated function systems which is used to generate the attractor(fractal) using Hausdorff metric. Let (X, d) be a complete metric space and $\{\omega_1, \omega_2, \ldots, \omega_m\}$ be a collection of contractions on X. It is known from Chapter 2 that this metric d induces the Hausdorff metric h on $\mathcal{K}(X)$, which is complete with respect to h. We now induce a contraction map f on $\mathcal{K}(X)$ using the contractions $\{\omega_1, \omega_2, \ldots, \omega_m\}$ on X and applying Banach fixed point theorem to get a unique fixed point of f which is the attractor. The following results are in sequel in proving the above results.

Definition 3.13 *Let (X, d) be any metric space. A mapping $\omega : X \to X$ is called a **contraction** on X if there exists a number r with $0 < r < 1$ such that $d(\omega(x), \omega(y)) \leq rd(x, y)$ for all $x, y \in X$.*

Definition 3.14 *A finite family of contractions $\{\omega_1, \omega_2, \ldots, \omega_m\}$ on (X, d), with $m \geq 2$, is called an **iterated function system (IFS)** on X.*

Definition 3.15 *Let $\{\omega_1, ..., \omega_m\}$ be an iterated function system on X. Then a non-empty compact set $A \subseteq X$ is called an **attractor** of the IFS if $A = \bigcup\limits_{n=1}^{m} \omega_n(A)$.*

The fundamental property of an IFS is that it determines a unique attractor, which is usually a **fractal**.

Lemma 3.16 *If $A_i, B_i \in \mathcal{K}(X)$ for $1 \leq i \leq n$ for some $n \in \mathbb{N}$ then*

$$h\left(\bigcup_{i=1}^{n} A_i, \bigcup_{i=1}^{n} B_i\right) \leq \max\{h(A_i, B_i) \mid 1 \leq i \leq n\}.$$

Existence of attractor of an IFS

Theorem 3.17 *Let $\{\omega_1, \omega_2, ..., \omega_m\}$ be an IFS of contraction on a complete metric space (X, d) such that*

$$d(\omega_n(x), \omega_n(y)) \leq r_n d(x, y) \text{ for all } x, y \in X \tag{2}$$

with $r_n < 1$ for each n. Then the system has a unique attractor A, that is, there exists a unique non-empty compact set A such that $A = \bigcup\limits_{n=1}^{m} \omega_n(A)$.

Proof.

Define $f : \mathcal{K}(X) \to \mathcal{K}(X)$ by

$$f(A) := \bigcup_{n=1}^{m} \omega_n(A)$$

We prove f is a contraction on $\mathcal{K}$. If $A, B \in \mathcal{K}(X)$, then by using Lemma 3.16,

$$\begin{aligned} h(f(A), f(B)) &= h\left(\bigcup_{n=1}^{m} \omega_n(A), \bigcup_{n=1}^{m} \omega_n(B)\right) \\ &\leq \max_{1 \leq n \leq m} h(\omega_n(A), \omega_n(B)) \\ &\leq \max_{1 \leq n \leq m} r_n \; h(A, B) \\ &< r \; h(A, B), \end{aligned}$$

where $r = \max_{1 \leq n \leq m} r_n < 1$. Therefore, f is a contraction on the complete metric space $(\mathcal{K}(X), h)$. Hence by applying Banach contraction mapping theorem, f has a unique fixed point, that is, there is a unique set $A \in X$ such that $f(A) = \bigcup_{n=1}^{m} \omega_n(A) = A$.

From the Banach contraction mapping theorem we observe the following: For any set $E \in \mathcal{K}(X)$, the sequence $f^k(E) = f(f^{k-1}(E))$ converges to the unique fixed point A in $\mathcal{K}(X)$. i.e., $h(f^k(E), A) \to 0$ as $k \to \infty$. In particular, if $\omega_n(E) \subset E$ for all i, then $f(E) \subset E$, so that $f^k(E)$ is a decreasing sequence of non-empty compact sets containing A with intersection $\bigcap_{k=0}^{\infty} f^k(E)$ which must equal A.

For $i_1, i_2, \ldots, i_p \in \{1, \ldots, m\}$, $p \geq 1$, denote $\omega_{i_1 i_2 \ldots i_p} = \omega_{i_i} \circ \omega_{i_2} \circ \cdots \circ \omega_{i_p}$. In this way one obtains a contraction on X with the contraction ratio $r_{i_1 i_2 \cdots i_p} \leq r_{i_1} r_{i_2} \cdots r_{i_p}$. Then according to Banach contraction mapping theorem these maps $\omega_{i_1 i_2 \ldots i_p}$ has a unique fixed point. We set $e_{i_1 \cdots i_p}$ be the unique fixed point of $\omega_{i_1 \ldots i_p}$.

Example 3.18 *Let F be the middle third cantor set. Let $\omega_1, \omega_2 : \mathbb{R} \to \mathbb{R}$ be given by*

$$\omega_1(F) = \frac{1}{3}x \quad ; \quad \omega_2(F) = \frac{1}{3}x + \frac{2}{3}$$

Then $\omega_1(F)$ and $\omega_2(F)$ are just the left and right halves of F, so $F = \omega_1(F) \cup \omega_2(F)$. Thus F is an attractor of the IFS consisting of the contractions $\{\omega_1, \omega_2\}$, the two mappings which represent the basic self-similarities of the cantor set.

Example 3.19 *Let F be the Sierpinski triangle. Let $\omega_1, \omega_2, \omega_3 : \mathbb{R}^2 \to \mathbb{R}^2$ is given by*

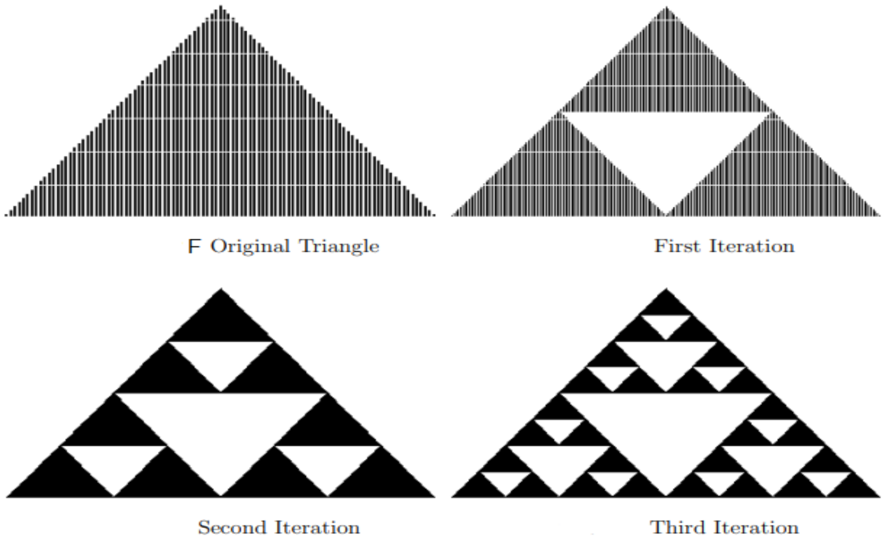

Figure 1: Sierpinski triangle

$$\omega_1(F) = \frac{1}{2}F; \quad \omega_2(F) = \frac{1}{2}\left(F + \begin{pmatrix}1\\0\end{pmatrix}\right); \quad \omega_3(F) = \frac{1}{2}F + \begin{pmatrix}\frac{1}{2}\\ \frac{\sqrt{3}}{4}\end{pmatrix}$$

So $F = \{\omega_1(F) \cup \omega_2(F) \cup \omega_3(F)\}$*. Thus* F *is an attractor of the IFS consisting of the contractions* $\{\omega_1, \omega_2, \omega_3\}$*, the three mappings which represent the basic self-similarities of the sierpinski triangle.*

Proposition 3.20 *If* $E \in \mathcal{K}(X)$ *and* $p \geq 1$ *then* $f^p(E) = \bigcup\limits_{\substack{i_j=1\\ j=1,2,\ldots,p}}^{m} \omega_{i_1 i_2 \cdots i_p}(E)$*.*

Theorem 3.21 *Let* $\{\omega_1, \omega_2, \ldots, \omega_m\}$ *be an IFS of contraction, with contraction ratio* r_n *for* $1 \leq n \leq m$*, on a complete metric space* (X, d)*. Then the unique attractor* A *of the IFS is the closure of the set of these fixed points of* $\omega_{i_1 \cdots i_p}$ *for* $p \geq 1$*.*

$$i.e., \quad A = \overline{\{e_{i_1 \cdots i_p} \mid p \in \mathbb{N}, 1 \leq i_j \leq m\}}. \tag{3}$$

Proof.

Let A be the attractor of $\{\omega_1, \omega_2, \ldots, \omega_m\}$.

Step 1: A **contains all the fixed points of** ω_n**,** $1 \leq n \leq m$**.**

As A is the attractor of the $\{\omega_1, \omega_2, \cdots, \omega_m\}$, we see that $\omega_n(A) \subseteq A$ for $1 \leq n \leq m$. Hence, for any $p \geq 1$, $\omega_n^p(a) \in A$ for all $a \in A$. Now, we infer from the Banach fixed point theorem that,

$\omega_n^p(a) \to e_n$ as $p \to \infty$ for $1 \leq n \leq m$. Since $\omega_n^p(a) \in A$ and A is closed, we see that $e_n \in A$, for $1 \leq n \leq m$

Step 2: A is the attractor of $\{\omega_{i_1 i_2 \cdots i_p} \mid 1 \leq i_1, i_2, \ldots, i_p \leq m\}$ for $p = 1, 2, \ldots$

We prove this assertion using induction on p. If $p = 1$, then clearly A is the attractor of $\{\omega_{i_1} \mid 1 \leq i_1 \leq m\} = \{\omega_1, \omega_2, \ldots, \omega_m\}$. For $p = 2$, first we observe that,

$$\{\omega_{i_1 i_2} \mid 1 \leq i_1, i_2 \leq m\} = \{\omega_{11}, \omega_{12}, \ldots, \omega_{1m}, \omega_{21}, \omega_{22}, \ldots, \omega_{2m}, \ldots, \omega_{m1}, \omega_{m2}, \ldots, \omega_{mm}\}$$

Hence,

$$\begin{aligned}
\bigcup_{\substack{i_j=1 \\ j=1,2}}^{m} \omega_{i_1 i_2}(A) &= \omega_{11}(A) \cup \omega_{12}(A) \cup \cdots \cup \omega_{1m}(A) \cup \omega_{21}(A) \cup \omega_{22}(A) \cdots \cup \omega_{2m}(A) \\
&\quad \cup \cdots \cup \omega_{m1}(A) \cup \omega_{m2}(A) \cup \cdots \cup \omega_{mm}(A) \\
&= \omega_1 \left(\bigcup_{i=1}^{m} \omega_i(A) \right) \cup \omega_2 \left(\bigcup_{i=1}^{m} \omega_i(A) \right) \cdots \cup \omega_m \left(\bigcup_{i=1}^{m} \omega_i(A) \right) \\
&= \omega_1(A) \cup \omega_2(A) \cup \cdots \cup \omega_n(A) \\
&= A.
\end{aligned}$$

Therefore the induction hypothesis satisfied for $p = 1, 2$. Now, assume the result is true for $p - 1$

$$\text{i.e.,} \quad \bigcup_{\substack{i_j=1 \\ j=1,2,\ldots,p-1}}^{m} \omega_{i_1 i_2 \cdots i_{p-1}}(A) = A$$

Claim: $\displaystyle\bigcup_{\substack{i_j=1 \\ j=1,2,\ldots,p}}^{m} \omega_{i_1 i_2 \cdots i_p}(A) = A.$

Clearly

$$\begin{aligned}
\bigcup_{\substack{i_j=1 \\ j=1,2,\ldots,p}}^{m} \omega_{i_1 i_2 \cdots i_p}(A) &= \bigcup_{\substack{i_j=1 \\ j=1,2,\ldots,p-1}}^{m} \omega_{i_1 i_2 \cdots i_{p-1}} \left(\bigcup_{i=1}^{m} \omega_i(A) \right) \\
&= \bigcup_{\substack{i_j=1 \\ j=1,2,\ldots,p-1}}^{m} \omega_{i_1 i_2 \cdots i_{p-1}}(A) \\
&= A
\end{aligned}$$

Hence, step 3 follows from induction hypothesis.

Step 3: $A \supset \omega_{i_1}(A) \supset \cdots \supset \omega_{i_1 i_2 \cdots i_p}(A) \supset \cdots$ **and** $\bigcap_{p=1}^{\infty} \omega_{i_1 i_2 \cdots i_p}(A) = \{a_{i_1 i_2 \cdots i_p \cdots}\}$**.**
where $a_{i_1 \cdots i_p \cdots} = \lim_{p \to \infty} e_{i_1 i_2 \cdots i_p}$.
The inclusion $\omega_{i_1 \cdots i_{p+1}}(A) \subset \omega_{i_1 \cdots i_p}(A)$ for any $p \geq 1$, follows by Step 2. Next for any arbitrary $p \geq 1$ and $a, b \in A$ one has

$$\begin{aligned}
d(\omega_{i_1 i_2 \cdots i_p}(a), \omega_{i_1 i_2 \cdots i_p}(b)) &\leq r_{i_p} d(\omega_{i_1 i_2 \cdots i_{p-1}}(a), \omega_{i_1 i_2 \cdots i_{p-1}}(b)) \\
&\leq r_{i_p} r_{i_{p-1}} d(\omega_{i_1 i_2 \cdots i_{p-2}}(a), \omega_{i_1 i_2 \cdots i_{p-2}}(b)) \\
&\vdots \\
&\leq r_{i_1} r_{i_2} \cdots r_{i_p} d(a, b) \\
&\leq r^p d(a, b),
\end{aligned}$$

where $r = \max\{r_{i_1}, r_{i_2}, \ldots, r_{i_p}\}$. Therefore, $d(\omega_{i_1 i_2 \cdots i_p}(a), \omega_{i_1 i_2 \cdots i_p}(b)) \to 0$ as $p \to \infty$. i.e., distance between any two points in $\omega_{i_1 i_2 \cdots i_p}(A)$ tends to zero as $p \to \infty$. Hence, it follows that

$$\text{diam } (\omega_{i_1 \ldots i_p}(A)) = \sup\{d(a, b) \mid a, b \in \omega_{i_1 \ldots i_p}(A)\} \to 0 \quad \text{as} \quad p \to \infty.$$

Therefore, intersection of all $\omega_{i_1 \ldots i_p}(A)$ is an singleton set and let $\bigcap_{p=1}^{\infty} \omega_{i_1 i_2 \cdots i_p}(A) = \{a_{i_1 i_2 \cdots i_p \cdots}\}$. Further it is an easy observation that $a_{i_1 i_2 \cdots i_p \cdots} = \lim_{p \to \infty} e_{i_1 i_2 \cdots i_p}$.
Step 4: $A = \overline{\{e_{i_1 \cdots i_p} : p \in \mathbb{N}, 1 \leq i_j \leq m\}}$.
Since A is the attractor of $\{\omega_{i_1 \cdots i_p} \mid 1 \leq i_1 \cdots i_p \leq m\}$ we see that $e_{i_1 i_2 \cdots i_p} \in A$ for $1 \leq i_1, i_2, \ldots, i_p \leq m$ and $p \geq 1$. Therefore,

$$A \supseteq \overline{\{e_{i_1 \ldots i_p} : p \in \mathbb{N}, 1 \leq i_j \leq m\}} \tag{4}$$

To prove the reverse inequality, let $a \in A$. Then for any $p \geq 1$ and $1 \leq i_1, i_2, \ldots, i_p \leq m$, as $A = \bigcup_{\substack{i_j=1 \\ j=1,2,\ldots,p}}^{m} \omega_{i_1 i_2 \cdots i_p}(A)$, we see that $a \in \omega_{i_1 i_2 \cdots i_p}(A)$ and hence $a \in \bigcap_{p=1}^{\infty} \omega_{i_1 \cdots i_p}(A) = \lim_{p \to \infty} e_{i_1 \cdots i_p}$. Hence $a \in \overline{\{e_{i_1 \cdots i_p} \mid p \in \mathbb{N}, 1 \leq i_j \leq m\}}$.

$$\text{i.e.,} \quad A \subseteq \overline{\{e_{i_1 \ldots i_p} : p \in \mathbb{N}, 1 \leq i_j \leq m\}} \tag{5}$$

Now, from equation (4) and (5), it follows that

$$A = \overline{\{e_{i_1\cdots i_p} : p \in \mathbb{N}, 1 \leq i_j \leq m\}}.$$

IV CONCLUSION

In this paper, we attempted to provide a unified theory of iterated function systems. To brief our results, given a IFS we proved the existence and uniqueness of attractor of the IFS which self similar in nature (i.e., a fractal).We also proved how to construct the attractors of iterated function systems using the fixed points of contraction maps involved in the iterated function systems.

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