International Journal of Mathematics Trends and Technology
ISSN: 2231 – 5373/doi:10.14445/22315373/IJMTT-V67I12P509

Volume 67 Issue 12, 78-81, December, 2021
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Original Article

On the Hamiltonicity of Closure of Graph

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Abstract - *In this paper we discuss about the number of spanning cycles in closure of graph. The closure of a graph G is the graph obtained by adding edges between non-adjacent vertices whose degree sum is at least $|V(G)|$, until this can no longer be done. There are countless generalizations of paths and cycles and Hamiltonian properties in graphs, and one of these generalizations is the uniquely Hamiltonian graph. A graph is uniquely Hamiltonian if it contains exactly one spanning cycle. We proved the results about the Hamiltonicity, uniquely Hamiltonicity of closure of graph.*

Keywords - *Line graph, length of path, spanning cycle, spanning path, etc.*

I. INTRODUCTION

Unless otherwise referred, throughout the paper by a graph we always mean a simple finite undirected connected graph G with vertex set $V(G)$ and edge set $E(G)$. In general, we follow the most common graph-theoretical terminology and notation. A path and a cycle in a graph G that contains every vertex of graph G is called spanning path and spanning cycle of G. A graph is Hamiltonian graph if it contains spanning cycle and a graph is traceable if it contains spanning path. The closure of a graph G, denoted by $Cl(G)$, is the graph obtained by adding edges between non-adjacent vertices whose degree sum is at least $|V(G)|$, until this can no longer be done. For two vertices u and v, let $d(u,v)$ be the length of a minimum path between vertices u and v in G, or equivalently the distance between u and v. The minimum degree of a graph G is denoted by $\delta(G)$. For a grand survey on the Hamiltonian problem see Gould[8].

II. RESULTS AND DISCUSSION

Dirac[6] obtained a nontrivial sufficient condition for a graph to be Hamiltonian, which was most likely the first achievement in the area.

Theorem 2.1. Dirac[6]. If G is a graph of order $n \geq 3$ with minimum degree $\delta(G) \geq n/2$, then G is Hamiltonian.

Ore[9] elaborate the Dirac's theorem.

Theorem 2.2. Ore[9]. If G is a graph of order $n \geq 3$ and $d(x) + d(y) \geq n$ for each pair of nonadjacent vertices $x, y \in V(G)$, then G is Hamiltonian.

Benhocine and Wojda [3] gave the following result

Theorem 2.3. Benhocine and Wojda[3]. If G is a $2-$connected graph of order $n \geq 3$ with independence number $\alpha(G) \leq n/2$, and $max\{d(x), d(y)\} \geq (n-1)/2$ for each pair of vertices x, y with distance 2, then either G is Hamiltonian or $G \in G_n$ or $G = H_9$ where G_n is a class of well characterized graphs and H_9 is a specific graph of order 9.

Chao, Song and Zhang[4] drop the independence restriction as stated in above theorem and assumed the slightly relaxed Fan type condition to give a Hamiltonian characterization,

Theorem 2.4. Chao, Song and Zhang[4]. If G is a $2-$connected graph of order $n \geq 3$, and $max\{d(x), d(y)\} \geq (n-1)/2$ for each pair of vertices x, y with distance 2, then either G is Hamiltonian or $G \in G_n$ or $G \in J_n$ or $G = H_9$, where G_n and J_n are two classes of well characterized graphs and H_9 is the graph prescribed in above theorem.

Faudree, Gould, Lesniak and Lindquester[7] gave the sufficient conditions for Hamiltonian type properties of a graph with bounded independence number and the generalized degree of vertex,

Theorem 2.5. (Faudree, Gould, Lesniak and Lindquester[7]. Let G be a graph of order n. Then for each pair of integers r and m ($1 \leq r \leq n$ and $3 \leq m < n$), and for each non-negative function $f(r,m)$ there exists a constant $C = C(r,m,f(r,m))$ such that if $\delta_r(G) \geq \frac{n}{3} + C$ and $\alpha(G) \leq f(r,m)$, then

(i) G is traceable if $\delta(G) \geq r$ and G is connected,

(ii) G is Hamiltonian if $\delta(G) \geq r+1$ and G is $2-$connected, and

(iii) G is Hamiltonian-connected if $\delta(G) \geq r+2$ and G is $3-$connected.

Our first result shows that if graph contains 2 independent path with some specific conditions then its closure is Hamiltonian.

Theorem 2.6. Let G be a connected graph of order n. If a graph G contains 2 independent path between any pair of vertices such that $max\{d(u), d(v)\} \geq n/2$ and $d(u,v) = 2$ for all $u, v \in V(G)$ then $Cl(G)$ is Hamiltonian.

Proof. Suppose on contrary $Cl(G)$ has no spanning cycle. Let $P = \{u_1\, u_2 \ldots u_{k+1}\}$ be the maximum length path in G with length k, select $d(u_1) + d(u_{k+1})$ is as large as possible. If $d(u_1) + d(u_{k+1}) \geq n$ then there exist at least two consecutive vertices u_i and u_{i+1} such that $(u_1, u_i) \in E(G)$ and $(u_{i+1}, u_{k+1}) \in E(G)$, and thus we get a cycle of length $k+1$. Since G is connected graph with above given properties we have either a spanning cycle or a path of length $k+1$. This will leads to a contradiction. Hence $d(u_1) + d(u_{k+1}) < n$. Without loss of generality assume $d(u_{k+1}) < n/2$.

Firstly suppose G has no cycle of length $k+1$. Since graph G contains 2 independent path between any pair of vertices, vertex u_1 is connected with at least 2 vertices on path P. Let $(u_1, u_m) \in E(G)$ and m is as large as possible. Obviously $m \geq 2$ and $m < k$, otherwise G has a cycle of length $k+1$. Since $(u_1, u_m) \in E(G)$, there is some other longest path $u_{m-1}u_{m-2} \ldots u_1 u_m \ldots . u_k$. By the maximality of $d(u_1) + d(u_k)$, we have $d(u_{m-1}) \leq d(u_1) < n/2$, implies $\max(d(u_{m-1}), d(u_1)) < n/2$. By the statement of theorem we have $d(u_1, u_{m-1}) \neq 2$. Although $u_1\, u_m u_{m-1}$ is a path of length 2 we must have $(u_1, u_{m-1}) \in E(G)$. By repeating above process we have $(u_1, u_i) \in E(G)\ \forall\ 1 \leq i \leq m$.

Now we claim that $d(u_i) \leq d(u_1)\ \forall\ 1 \leq i \leq m-1$. By above assertion $(u_1, u_{i+1}) \in E(G)$, $u_i u_{i-1} \ldots u_1 u_{i+1} \ldots u_m \ldots u_k$ is another maximum length path with $d(u_i) + d(u_k) > d(u_1) + d(u_k)$, which leads to contradiction. Hence $d(u_i) \leq d(u_1)\ \forall\ 1 \leq i \leq m-1$. By our choice of u_m, we have $u_1\, u_m u_{m+1}$ is a path of length 2 and $(u_1, u_{m+1}) \notin E(G)$, we have $d(u_1, u_{m+1}) = 2$. By the statement of theorem we have $\max(d(u_1), d(u_{m+1})) \geq n/2$. Since $d(u_1) < n/2$ we have $d(u_{m+1}) \geq \frac{n}{2} > d(u_1)$.

Note that for all i, $1 \leq i \leq m-1$, v_i cannot be connected to other vertex outside path P, since $u_i u_{i-1} \ldots u_1 u_{i+1} \ldots u_m \ldots u_k$ is a longest path. Since graph G contains 2 independent path between any pair of vertices, therefore there exists $(u_p, u_q) \in E(G)$ such that $p < m < q$. Consider $(u_p, u_q) \in E(G)$ such that $p < m < q$ and q is as large as possible. Now there are two cases arises.

Case 1. If $q \geq m+2$. We have $d(u_{q-1}) \leq d(u_1) < \frac{n}{2}$, since $d(u_q) + d(u_k)$ is maximum and by above assumption $(u_1, u_{p+1}) \in E(G)$, $u_{q-1}u_{q-2} \ldots u_{p+1}u_1 \ldots u_p u_q \ldots u_k$ is a longest path. Now again since $d(u_p) \leq d(u_1) < n/2$ so $\max(d(u_{q-1}),\ d(u_p)) < n/2$ then $d(u_p, u_{q-1}) \neq 2$, but $u_p u_{q-1} u_q$ is a path of length 2 and thus we have $(u_p, u_{q-1}) \in E(G)$. If $q-1 > p+1$ we have some other longest path $u_{q-2}u_{q-3} \ldots u_{p+1}u_1 \ldots u_p u_{q-1} u_q \ldots u_k$. Repeating this process again and again we have $(u_p, u_{q-2}) \in E(G)$. Consequently, we have $(u_p,\ u_i) \in E(G)\ \forall\ p+1 \leq i \leq q$.

Case 2. If $q = m+1$. Since $u_m u_{m-1} \ldots u_{q+1} u_1 \ldots u_p u_{m+1} \ldots u_k$ is longest path and $d(u_1) + d(u_k)$ is maximum we have $d(u_m) \leq d(u_1) < n/2$. If $m+1 = k$ we get a cycle of length $k+1$, which leads to contradiction. Let us assume $m+1 < k$. Since graph G contains 2 independent path between any pair of vertices we have $(u_m, u_j) \in E(G)$ such that $j \geq m+2$. This will implies there exist another longest path $u_{j-1}u_{j-2} \ldots . u_{m+1}u_p \ldots u_1 u_{p+1} \ldots u_m u_j \ldots u_k$. Since $d(u_1) + d(u_k)$ is maximum thus we have $d(u_{j-1}) \leq d(u_1) < n/2$. Repeating this process we have $(u_m, u_i) \in E(G)\ \ \forall\, i,\ m+1 \leq i \leq j$. Hence, there exist longest path since $u_{m+1}u_p \ldots u_1 u_{p+1} \ldots u_m u_{m+1} \ldots u_k$ with $d(u_{m+1}) + d(u_k) > d(u_1) + d(u_k)$ which is contradiction. Hence $Cl(G)$ contains a spanning cycle as desired. ∎

In graph theory, for measuring the connectivity of graph we can use toughness of graph. If $S \subset V(G)$, then denote $c(G-S)$, the number of components of $G-S$. We say that a graph G is $t-tough$ if for every subset $S \subset V(G)$ with $c(G-S) > 1$ we have $|S| > tc(G-S)$.

Conjecture 2.1. The closure of t −tough graph with $t > \frac{3}{2}$ is Hamiltonian.

Chvàtal[5] conjectured the relationship between toughness and Hamiltonicity. He found the necessary condition for 1-tough graph to be Hamiltonian.

Conjecture 2.2. If the closure of graph is Hamiltonian then the graph is $1-$tough.

The complete bipartite graph $K_{1,3}$ is also known as claw. Thus, a graph G is said to be claw-free if, it does not contain an induced subgraph that is isomorphic to claw.

Observation 2.1. If G is a $4-$connected claw-free graph, then its closure is Hamiltonian.

The line graph of G is obtained by associating one vertex to each edge of G, and two vertices of line graph being joined by an edge if and only if the corresponding edges in G are adjacent. A graph is uniquely Hamiltonian if it contains exactly one spanning cycle. In Beineke[2] gives a characterization of line graphs in terms of forbidden induced subgraphs. Next we proved closure of line graph contains more than one spanning cycle.

Theorem 2.7. If G is a line graph with $\delta(G) \geq 3$, then the closure of G is not uniquely Hamiltonian.

Proof. If $Cl(G)$ has no spanning cycle, then we are done. Let C be the spanning cycle in $Cl(G)$. For a vertex $v \in V(Cl(G))$, denote by v^- and v^+ the vertex preceding and succeeding v, respectively. Consider v_n^- and v_n^+ vertices at distance v along C with respect to the ordering of $V(C)$.
Notice if $x, y \in V(Cl(G))$ are vertices that are not connected by an edge of C and $xy^-, x^+y \in E(Cl(G))$, then $Cl(G)$ has a spanning cycle distinct from C, namely $C - \{xx^+, y^-y\} + \{xx^-, x^+y\}$. This will form a structure we call it cycle exchange type structure. Suppose that $Cl(G)$ has no cycle exchange type structure. In particular if $u_i, u_{i+1}, u_{i+2}, u_{i+3}$ are such that $u_j^+ = u_{j+1}$ for each $j = 0, 1, 2$, then either u_iu_{i+2} or $u_{i+1}u_{i+3}$ is not an edge.
Let $u \in Cl(G)$ such that $u^-u^+ \notin E\big(Cl(G)\big)$ and v be the neighbor of u different from u^- and u^+. Since G is a line graph this implies $\{v, u^-, u, u^+\}$ may not induce a complete bipartite graph $K_{1,3}$, v must be neighbour of at least one of u^- and u^+. Say $vu^+ \in E\big(Cl(G)\big)$. Since G is line graph, one of u^+v^+, v^-v^+ or v^-u^+ is an edge of $Cl(G)$. In this way we have $C - v^-vv^+ + v^-v^+ + uvu^+ - uu^+$ a second spanning cycle of $Cl(G)$. Assuming $v^-u_2^+ \in E(Cl(G))$. Proceeding in similar manner we have $uv, vu^+, u^+v^-v^-u_2^+, u_2^+u_2^-, \ldots$ contained in $E\big((Cl(G)\big)$, where the sequence finishing when the last edge added has endpoints which lie at distance 2 from each other along C.

Now, suppose the vertex v and its adjacent vertices $\{u, v^-, v^+\}$. Again, if uv^- or $uv^+ \in E(Cl(G))$, then $Cl(G)$ contains a second spanning cycle. Thus we shall consider that $uv^+ \in E\big(Cl(G)\big)$. A similar argument to the one above shows that we have $uv^+, v^+u^-, u^-v_2^+, v_2^+u_2^-, \ldots$ contained in $E\big((Cl(G)\big)$, where the sequence finishing when the last edge added has endpoints which lie at distance 2 from each other along C.

Let $S = \{\ldots, u_2^-v_2^+, v_2^+u^-, u^-v^+, v^+u, uv, vu^+, u^+v^-, v^-u_2^+, u_2^+v_2^-, \ldots\}$ be the set of edges added above sequence and also let a, b be two vertices not connected to any edge of S. Since $(G) \geq 3$, a has at least two neighbour. If $ab \in E(Cl(G))$, it is straightforward to see that $Cl(G)$ contains a spanning cycle consisting of the edges of S collectively with some cycle edge incident to each of a and b. To complete the proof, we relabel the vertices of $V(Cl(G)) \setminus \{a, b\}$ for ease of notation. Also note that the edges of S form a spanning path of $G - \{a, b\}$. Along this path we label the vertices in order $y_1, z_1, y_2, z_2, \ldots$ will implies $y_1a \in E(C)$.

If $ay_i \in E(Cl(G))$ for some i, then $C - y_1y_2 \ldots y_i - az_1z_2 \ldots z_{i-1} + ay_i + y_1z_1y_2z_2 \ldots z_{i-1}$ is a spanning cycle of $Cl(G)$. If $az_j \in E(Cl(G))$ for some j, then $C - y_1y_2 \ldots y_j - az_1z_2 \ldots z_j + az_j + y_1z_1y_2z_2 \ldots z_{j-1}y_i$ is a spanning cycle of $Cl(G)$. After considering all cases, we conclude that closure of G is not uniquely Hamiltonian. ■

To consider uniquely Hamiltonian graphs one natural strengthening of the problem is the graphs having fixed degree. Now suppose G be a Hamiltonian graph with some specified degree than G contains more than one spanning cycle.

Theorem 2.8. Let G be a Hamiltonian graph having one vertex of degree 2 or 4 and rest of vertex are of degree 3, then $Cl(G)$ is not uniquely Hamiltonian.

Proof. Let $C = \{u_1, u_2, \dots u_n\}$ be a spanning cycle of G and also let $deg(u_1) = 2$ and remaining vertices has degree 3. Let G' be the graph with vertex set $\{v_1, v_2, v_3, v_4\}$ and edge set $\{v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_4\}$. $(G - u_1) \cup G'$ forms a cubic Hamiltonian graph will implies it has minimum 3 spanning cycles. Since G' has only two spanning paths, $G - u_1$, has minimum two spanning paths. Therefore G has minimum two spanning cycles. Since G' has only 2 spanning path ending with v_1 and v_4, $G - u_1$ has at least 2 spanning paths ending with u_2 and u_n. Therefore, G has at least two spanning cycles and it will implies $Cl(G)$ is not uniquely Hamiltonian. Similarly if $deg(u_1) = 4$, by deleting some edges incidence on u_1 and proceeding as above we have our desired result. ■

Observation 2.2. If $N(v) - S$ is the disjoint union of two complete graphs for every minimal cut set S and neighborhood $N(v)$ of v, $\forall\, v \in S$ then connected graph G is not uniquely Hamiltonian.

III. CONCLUSION

Lots of note has been written on the hamiltonicity of graph, but in this note we obtained the results on hamiltonicity of closure of graph.

IV. ACKNOWLEDGEMENTS

The author thanks the family and colleagues for their support.

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